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DESCRIPTIVE PROPERTIES OF MAPPINGS BETWEEN NONSEPARABLE LUZIN SPACES

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Abstract. We relate some subsets G of the product $X \times Y$ of nonseparable Luzin (e.g., completely metrizable) spaces to subsets H of $\mathbb{N}^{\mathbb{N}} \times Y$ in a way which allows to deduce descriptive properties of G from corresponding theorems on H. As consequences we prove a nonseparable version of Kondô's uniformization theorem and results on sets of points y in Y with particular properties of fibres $f^{-1}(y)$ of a mapping $f\colon X \to Y$. Using these, we get descriptions of bimeasurable mappings between nonseparable Luzin spaces in terms of fibres.

Keywords: nonseparable metric spaces, Luzin spaces, σ-discrete network, uniformization, bimeasurable maps

MSC 2000: 54H05, 28A05, 54E40

1. Introduction and notation

Our main aim is to demonstrate a possibility of getting results on (index-σ-discrete) mappings f from a complete metric space X to a metric space Y by a separable reduction. It is almost standard that the study of most properties of (index-σ-discrete) mappings f can be transfered to the study of the (index-σ-discrete) projection of the graph G of f to Y. We are going to show that many questions can be further translated to the study of a projection of a subset of $\mathbb{N}^{\mathbb{N}} \times Y$ to Y. Let us recall that in [9] we described a general method how to deduce some results on projections along separable spaces to nonseparable spaces from classical results on projections to separable spaces.

The result on the transfer of projections along nonseparable spaces to projections along separable ones is formulated in Theorem 3.2 in Section 3.

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As applications of Theorem 3.2, and of the method presented in [9], we get easily a nonseparable version of Kondô's uniformization theorem in Section 4 and results on the descriptive properties of sets of points y in Y with particular properties of fibres $f^{-1}(y)$ of a mapping $f\colon X \to Y$ in Section 5.

Characterizations of bimeasurable maps of nonseparable spaces in terms of their fibres are obtained using the results of Section 5 in the last section.

In fact, our results deal also with not necessarily metrizable topological spaces. We recall a few notions of generalized analytic and Luzin topological spaces, introduced in [4] to describe the weak topologies of some nonseparable Banach spaces. We summarize and deduce several properties of them in Section 2. In particular, a fairly general Theorem 2.8 on graphs and ranges of measurable mappings is proved.

The identity mapping on the corresponding set is denoted by id. We use π_X and π_Y to denote the projection mappings of $X \times Y$ to X and Y, respectively. If $B \subset X \times Y$, we put $B_x = \{y \in Y;\ (x,y) \in B\}$ and $B^y = \{x \in X;\ (x,y) \in B\}$. If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$, then $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Given families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X and Y, respectively, we write $\mathcal{A} \times \mathcal{B}$ for the family $\{A \times B;\ A \in \mathcal{A}, B \in \mathcal{B}\}$. If moreover $f\colon X \to Y$, we write $f(\mathcal{A})$ and $f^{-1}(\mathcal{B})$ instead of $\{f(A);\ A \in \mathcal{A}\}$ and $\{f^{-1}(B);\ B \in \mathcal{B}\}$.

Given a family $\mathcal{E}$ of subsets of a set X, we use $\mathbf{S}(\mathcal{E})$ to denote the class of sets obtained from elements of $\mathcal{E}$ by the *Souslin* (or *Aleksandrov*) operation, i.e., the sets of the form $A = \bigcup\limits_{\nu \in \mathbb{N}^{\mathbb{N}}} \bigcap\limits_{k \in \mathbb{N}} A_{\nu_1,\dots,\nu_k}$, where $A_s \in \mathcal{E}$ for every finite sequence s of positive integers. If A and the complement of A are in $\mathbf{S}(\mathcal{E})$, we write $A \in \textbf{bi-S}(\mathcal{E})$. We use $\mathcal{E}_\sigma$ and $\mathcal{E}_\delta$ to denote the families of unions and intersections of all at most countable subfamilies of $\mathcal{E}$, respectively.

We say that $\mathcal{N}$ is a network for a family $\mathcal{E}$ of subsets of a set X if $E = \bigcup\{N \in \mathcal{N};\ N \subset E\}$ for every $E \in \mathcal{E}$. We say that $\mathcal{N}$ is a network (a base) of a topological space X if $\mathcal{N}$ is a network (a network consisting of open sets) for the family of all open subsets of X.

All topological spaces are supposed to be Hausdorff and regular. If X is a topological space, we denote by $\mathbf{F}(X)$, $\mathbf{G}(X)$, $\mathbf{K}(X)$, and $\mathbf{B}(X)$ the classes of all closed, open, compact, and Borel subsets of X, respectively. The symbol $(\mathbf{F} \wedge \mathbf{G})(X)$ stands for the family of sets of the form $F \cap G$ with $F \in \mathbf{F}(X)$ and $G \in \mathbf{G}(X)$. Similar notation is used if other families stand in the place of $\mathbf{F}$ and $\mathbf{G}$.

A collection $\mathcal{E}$ of subsets of a topological space X is said to be *discrete* if each point of X belongs to an open set which meets at most one element of $\mathcal{E}$. A collection $\mathcal{E}$ is *relatively discrete* (or, equivalently, *isolated*) if $\mathcal{E}$ is discrete in $\bigcup \mathcal{E}$. A collection

$\mathcal{E}$ is said to be *scattered* if $\mathcal{E}$ is disjoint and there is a well-ordering $\leqslant$ of $\mathcal{E}$ such that, for each $E \in \mathcal{E}$, the set $\bigcup\{F \in \mathcal{E}\,;\ F \leqslant E\}$ is open relative to $\bigcup \mathcal{E}$.

It is clear that any discrete collection is isolated, and it is also not difficult to show that any isolated collection is scattered [4, Lemma 2.2(e)].

In what follows $\mathcal{D} = \mathcal{D}(X)$ stands for the family of all discrete, $\mathcal{I} = \mathcal{I}(X)$ for the family of all relatively discrete, and $\mathcal{S} = \mathcal{S}(X)$ for the family of all scattered families in the corresponding topological space X. We use the symbols $\mathcal{D}$, $\mathcal{I}$, and $\mathcal{S}$ sometimes also as an abbreviation for the words discrete, isolated, and scattered, respectively, as this should not lead to any confusion.

We use $\mathcal{Q}$ for any of the symbols $\mathcal{D}$, $\mathcal{I}$, or $\mathcal{S}$ until Theorem 2.1.

By saying that an indexed family $(D_a\,;\ a \in A)$ is in $\mathcal{Q}$ (or is $\mathcal{Q}$) we mean that the set $\{D_a\,;\ a \in A\}$ is in $\mathcal{Q}$ and $D_a \cap D_b = \emptyset$ (equivalently, $D_a \neq D_b$ if they are nonempty) if $a \neq b$, $a, b \in A$.

By $\mathcal{E}_{\mathcal{Q}}$ we denote the collection of all sets that are unions of $\mathcal{Q}$ families of elements of $\mathcal{E}$. A family is σ-$\mathcal{Q}$ if it is the union of countably many $\mathcal{Q}$ families.

We shall use without further reference the easy fact that $\bigcup\limits_{a\in A} \mathcal{E}_a$ is in $\mathcal{Q}(X)$ if all $\mathcal{E}_a$'s and the family $\{\bigcup \mathcal{E}_a\,;\ a \in A\}$ are in $\mathcal{Q}$ (see [4, Lemma 2.2] for the most difficult case of scattered families).

Clearly, if the family $\mathcal{E}$ is relatively discrete, then there are *$\mathcal{I}$-associated open sets* $U(E)$, $E \in \mathcal{E}$, such that $U(E) \cap \bigcup \mathcal{E} = E$, and if $\mathcal{E}$ is scattered, then there is a well-ordering $\leqslant$ of $\mathcal{E}$ and *$\mathcal{S}$-associated open sets* $U(E)$ such that $U(E) \cap \bigcup \mathcal{E} = \bigcup\{F \in \mathcal{E}\,;\ F \leqslant E\}$. It can be easily verified that the existence of the correspondingly associated sets $U(E)$, $E \in \mathcal{E}$, implies that $\mathcal{E}$ is relatively discrete or scattered, respectively (see [4, Lemma 2.1 and the remark after it] for the scattered families). We put $H(E) = \overline{E} \cap U(E)$ if $\mathcal{E}$ is in $\mathcal{I}$ or $\mathcal{S}$. We see in each of these cases that the family $(H(E)\,;\ E \in \mathcal{E})$ is in $\mathcal{I}$ or $\mathcal{S}$, respectively, and each $H(E)$ is in $(\mathbf{F} \wedge \mathbf{G})(X)$ (cf. also [4, Lemmas 2.3 and 3.2]).

An indexed family $\mathcal{E} = (E_a\,;\ a \in A)$ of subsets of a topological space X is called *σ-$\mathcal{Q}$ resolvable* if every $E_a \in \mathcal{E}$ is the union of a family of sets $\{E_a(n, l)\,;\ n \in \mathbb{N}, l \in \Lambda(n, a)\}$ such that the indexed families $(E_a(n, l)\,;\ a \in A,\ l \in \Lambda(n, a))$, $n \in \mathbb{N}$, belong to $\mathcal{Q}$. For our convenience, we may and do suppose that all the index sets $\Lambda(n, a)$ are equal to one fixed set Λ. Indeed, putting $\Lambda = \bigcup\limits_{n\in\mathbb{N}, a\in A} \Lambda(n, a)$ and $E(n, l) = \emptyset$ if $l \notin \Lambda(n, a)$, the modified decomposition has the required properties, too. The notion of σ-$\mathcal{Q}$-resolvable families is equivalent with σ-$\mathcal{Q}$-decomposable families defined by Hansell in [4, p. 6] if $\mathcal{Q}$ stands for discrete or isolated families. However, the notion of σ-scattered resolvable families does not coincide with that of σ-scattered decomposable ones. This is related to Hansell's example [4, Example 2.9].

Let us note that the existence of a σ-discrete basis of metric spaces implies that scattered families of subsets of a metric space are σ-discrete resolvable (in fact σ-discrete decomposable). This shows that the notions of σ-discrete resolvability, σ-isolated resolvability, and σ-scattered resolvability coincide in metric spaces.

It is not difficult to check that an indexed family $(E_a\,;\ a \in A)$ of subsets of a topological space X is σ-$\mathcal{Q}$-resolvable if and only if it is point-countable (as an indexed family, i.e., for each $x \in X$ there are at most countably many $a \in A$ such that $x \in E_a$) and has a σ-$\mathcal{Q}$ network. Point-countable families with a σ-$\mathcal{Q}$ network were used by Hansell in his definition of $\mathcal{Q}$-(K-)analytic spaces in [4, pp. 7 and 11]. In [5] Hansell used a modified definition of σ-$\mathcal{Q}$-decomposable families which is equivalent with that of our σ-$\mathcal{Q}$-resolvable families, although it is formally different.

Let $\mathbf{H}^{\mathcal{Q}}(X) \subset \mathbf{B}^{\mathcal{Q}}(X)$ denote the smallest algebra of subsets of X which contains $(\mathbf{F} \wedge \mathbf{G})(X)$ and which is closed under unions of $\mathcal{Q}$ families of its elements. Let us recall (see, e.g., [10]) that $\mathbf{H}^{\mathcal{S}}(X) = (\mathbf{F} \wedge \mathbf{G})_{\mathcal{S}}(X)$.

Finally, by $\mathbf{B}^{\mathcal{Q}}(X)$ we denote the smallest σ-algebra of subsets of X which contains all Borel sets of X and is closed under unions of $\mathcal{Q}$ families of its elements. Its elements are called *$\mathcal{Q}$-Borel sets*.

It follows easily from [4, Lemma 3.3(b)] that $\mathbf{S}(\mathbf{B}^{\mathcal{I}}(X)) = \mathbf{S}(\mathbf{B}(X))$ for every topological space X.

The classes of $\mathbf{B}^{\mathcal{D}}(X)$, $\mathbf{B}^{\mathcal{I}}(X)$, and $\mathbf{B}^{\mathcal{S}}(X)$ sets coincide if X is metrizable (their elements are called *extended Borel sets*) due to the existence of a σ-discrete basis for X. Consequently, in all the three cases of $\mathcal{Q}$ the class $\mathbf{S}(\mathbf{B}^{\mathcal{Q}}(X))$ coincides with the class $\mathbf{S}(\mathbf{F}(X))$ of all Souslin sets if X is metrizable. In separable metrizable spaces the families of Borel and extended Borel sets coincide, while this is not the case in general nonseparable metric spaces.

2. Generalized analytic and Luzin topological spaces

We are going to present results on mappings between (**bi**-)$\mathbf{S}(\mathbf{F})$ subsets of complete metric spaces and also between their generalizations to $\mathcal{Q}$-analytic and $\mathcal{Q}$-Luzin topological spaces introduced by Hansell in [4] as mentioned in the introduction. A mapping $f\colon X \to Y$ is said to be *index-σ-$\mathcal{Q}$* if $(f(E_a)\,;\ a \in A)$ is σ-$\mathcal{Q}$-resolvable, whenever $(E_a\,;\ a \in A)$ is σ-$\mathcal{Q}$-resolvable in X, i.e., f preserves σ-$\mathcal{Q}$-resolvable indexed families. It is not difficult to check that f is index-σ-$\mathcal{Q}$ if it maps indexed families from $\mathcal{Q}(X)$ to σ-$\mathcal{Q}(X)$-resolvable indexed families.

A topological space X is called *$\mathcal{Q}$-analytic* if there exists a continuous index-σ-$\mathcal{Q}$ mapping f of a complete metric space M onto X. A topological space X is *$\mathcal{Q}$-Luzin*, if it is $\mathcal{Q}$-analytic, and the mapping f in the definition can be taken one-to-one. The mapping f in the above definitions is called a *$\mathcal{Q}$-analytic* or a *$\mathcal{Q}$-Luzin* *parametrization* of X, respectively. Every complete metric space M is an injective image of a closed subset F of $D^{\mathbb{N}}$ for some discrete space D under a continuous and index-σ-discrete mapping $\varphi\colon F \to M$ by [3, Theorem 5.6]. Due to this fact and the fact that every scattered family in a metric space is σ-discretely decomposable as mentioned above, we may replace the complete metric space M in the preceding definitions by a closed subset $F \subset D^{\mathbb{N}}$ for some discrete D. If X has a countable network, then the notions of $\mathcal{D}$-analytic, $\mathcal{I}$-analytic, and $\mathcal{S}$-analytic spaces coincide and we speak about *analytic spaces*. Similarly we define *Luzin spaces*.

Isolated-analytic spaces were introduced by Hansell under the name descriptive spaces and scattered-analytic spaces under the name almost descriptive spaces in [4]. We however follow the terminology used later in [5]. The basic properties of these spaces can be found in [4]. Let us point out that, e.g., every $\mathcal{Q}$-analytic space has a σ-$\mathcal{Q}$ network and that the classes of $\mathcal{Q}$-analytic spaces (sets) are closed under countable products, countable unions and intersections, and unions of $\mathcal{Q}$ families. Let us point out that, e.g., all Banach spaces which admit an equivalent norm having the Kadec property, are isolated-analytic with respect to the weak topology. This is one of the main reasons why we are interested also in nonmetrizable spaces here. We should keep in mind that a metrizable space X is $\mathcal{D}$-analytic if and only if it is $\mathcal{I}$-analytic, and also if and only if it is $\mathcal{S}$-analytic as the three notions of σ-$\mathcal{Q}$-resolvability coincide in metrizable spaces. Similar claim is true concerning $\mathcal{Q}$-Luzin metrizable spaces. However, in the case of general topological spaces, the fact that discrete families in $X \subset Y$ need not be discrete in Y makes the notions with $\mathcal{Q} = \mathcal{D}$ less natural. Therefore, we use $\mathcal{P} = \mathcal{P}(X)$ *which stands just for* $\mathcal{I}$ *or* $\mathcal{S}$ rather than $\mathcal{Q}$, and we should realize that the above remarks give a possibility to replace $\mathcal{I}$ or $\mathcal{S}$ by $\mathcal{D}$ in what follows if we limit ourselves just to metrizable spaces.

We will need later a few facts on $\mathcal{P}$-analytic and $\mathcal{P}$-Luzin spaces that either follow from known results easily or can be proved by straightforward modification of standard methods. We need the following version of the "perfect set theorem". Since we did not find any reference for it, we indicate a proof.

Theorem 2.1. *Let A be a scattered-analytic space which is not σ-scattered. Then there is a homeomorphic copy of the Cantor set in A.*

Proof. Let $f\colon M \to A$ be a scattered-analytic parametrization. As f maps, in particular, σ-discrete sets in M to σ-scattered subsets of A and M is paracompact, subtracting from M all its σ-discrete open subsets (i.e., equivalently, the open sets with σ-scattered image), we get a nonempty closed $F \subset M$ with $f(U)$ not σ-scattered for every nonempty open subset U of F. Thus we may suppose that $F = M$ further on. Proceeding inductively in $n \in \mathbb{N}$ in an almost obvious way, we find nonempty open sets U_ι, $\iota \in \{0,1\}^n$, such that

$$U_{i_1,\dots,i_{n+1}} \subset U_{i_1,\dots,i_n},\ f(\overline{U_\iota}) \cap f(\overline{U_{\iota'}}) = \emptyset \text{ if } \iota \neq \iota',\ \text{and}\ \operatorname{diam} U_{i_1,\dots,i_n} \leqslant \frac{1}{n}.$$

Finally, it is not difficult to check that the mapping that takes each $(i_1, i_2, \dots) \in \{0,1\}^{\mathbb{N}}$ to the only element of $f\Big(\bigcap\limits_{n\in\mathbb{N}} \overline{U_{i_1,\dots,i_n}}\Big)$ is a homeomorphism. $\square$

Theorem 2.2. *Let A be a subset of a $\mathcal{P}$-analytic space X. Then*

(a) *A is $\mathcal{P}$-analytic if and only if A is in $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$;*
(b) *B is in $\textbf{bi-S}(\mathbf{B}^{\mathcal{S}}(X))$ if and only if $B \in \mathbf{B}^{\mathcal{S}}(X)$.*

Moreover, if A is $\mathcal{I}$-analytic, then $A \in \mathbf{S}(\mathbf{B}(X))$.

Proof. (a) If $A \subset X$ is $\mathcal{P}$-analytic, it is clearly scattered-(K-)analytic. According to [6, Proposition 2], see also [7, Theorem 2], it is in $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$.

Conversely, let $A \in \mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$. According to [6, Proposition 2] or [7, Theorem 2], it is scattered-K-analytic. Since X is $\mathcal{P}$-analytic, it has a σ-$\mathcal{P}$ network [8, Theorem 5], thus also A has a σ-$\mathcal{P}$ network. Using [8, Theorem 5] again, we get that A is $\mathcal{P}$-analytic.

(b) Every scattered-Borel set is in $\textbf{bi-S}(\mathbf{B}^{\mathcal{S}}(X))$ in any space. According to (a), $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$ coincides with the collection of all $\mathcal{P}$-analytic subspaces of X. So the elements of $\textbf{bi-S}(\mathbf{B}^{\mathcal{S}}(X))$ are bi-scattered-analytic, and they coincide with $\mathbf{B}^{\mathcal{S}}(X)$ sets according to [5, Theorem 6.28] (or [7, Theorem 5]).

The last assertion follows from [4, Theorem 4.1]. $\square$

Theorem 2.3. *Let B be a subset of a $\mathcal{P}$-Luzin space X. Then the following are equivalent.*

(a) *B is $\mathcal{P}$-Luzin,*
(b) *$B \in \textbf{bi-S}(\mathbf{B}^{\mathcal{S}}(X))$, and*
(c) *$B \in \mathbf{B}^{\mathcal{S}}(X)$.*

If $\mathcal{P} = \mathcal{I}$, then $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$ can be replaced by $\mathbf{S}(\mathbf{B}(X))$ in (b) and $\mathbf{B}^{\mathcal{S}}(X)$ by $\mathbf{B}^{\mathcal{I}}(X)$ in (c).

Proof. The equivalence of (b) and (c) holds by Theorem 2.2(b).

To prove (c) implies (a), let $B \in \mathbf{B}^{\mathcal{S}}(X)$. If $f\colon M \to X$ is a $\mathcal{P}$-Luzin parametrization of X, then $f^{-1}(B)$ is in $\mathbf{B}^{\mathcal{S}}(M)$. In metrizable spaces, $\mathbf{B}^{\mathcal{S}}(M) = \mathbf{B}^{\mathcal{D}}(M)$, and in complete metric spaces the sets from $\mathbf{B}^{\mathcal{D}}$ are $\mathcal{D}$-Luzin by [3, Theorem 5.6]. Let $\varphi\colon L \to M$ be a $\mathcal{D}$-Luzin parametrization of $f^{-1}(B)$. Then $f \circ \varphi$ is a $\mathcal{P}$-Luzin parametrization of B.

To prove (a) implies (c) let B be $\mathcal{P}$-Luzin. Then B is scattered-(K-)Luzin, and consequently, $B \in \mathbf{B}^{\mathcal{S}}(X)$ due to [7, Theorem 7].

In the particular case of an isolated-Luzin space X, $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X))$ in (b) can be replaced by $\mathbf{S}(\mathbf{B}(X))$ according to Theorem 2.2. To improve (c) as required in this case, note that both B and $X \setminus B$ are isolated-analytic by Theorem 2.2. Using the separation principle [5, Theorem 6.28], we get that $B \in \mathbf{B}^{\mathcal{I}}(X)$. $\square$

Corollary 2.4. *Let X, Y be $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin) spaces and $f\colon X \to Y$ be such that $f^{-1}(\mathbf{B}^{\mathcal{P}}(Y)) \subset \mathbf{B}^{\mathcal{P}}(X)$. Then the preimages of $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin) subsets of Y are $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin) subsets of X.*

Proof. According to Theorem 2.2(a), $\mathcal{P}$-analytic sets in Y are $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$, their preimages are $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(X))$ by our assumption on f, hence $\mathcal{P}$-analytic by Theorem 2.2(a) again. If Y is $\mathcal{P}$-Luzin, the $\mathcal{P}$-Luzin sets in Y are **bi-**$\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ by Theorem 2.3, their preimages are **bi-**$\mathbf{S}(\mathbf{B}^{\mathcal{P}}(X))$, thus $\mathcal{P}$-Luzin by Theorem 2.3 again. $\square$

We need some almost standard results on product spaces and mappings. The next two assertions are slight modifications of those which can be found, e.g., in [5].

Lemma 2.5. *Let X and Y be topological spaces and X have a σ-$\mathcal{P}$ network $\mathcal{N}$. For every $\mathcal{P}$ family $\mathcal{T}$ of sets in $X \times Y$ there are open sets $U_T^N \subset Y$, for $N \in \mathcal{N}$ and $T \in \mathcal{T}$, so that the sets $T_N = T \cap (N \times U_T^N)$, $N \in \mathcal{N}$, satisfy*

$T = \bigcup_{N \in \mathcal{N}} T_N$ and

$(\pi_Y(T_N);\ T \in \mathcal{T})$ is in $\mathcal{P}(Y)$ with $\mathcal{P}$-associated open sets $U_T^N = U(\pi_Y(T_N))$, $T \in \mathcal{T}$, for every $N \in \mathcal{N}$.

Proof. Let $(U(T);\ T \in \mathcal{T})$ be a collection of $\mathcal{P}$-associated open sets for $\mathcal{T}$. For each $T \in \mathcal{T}$ and $N \in \mathcal{N}$ we denote by U_T^N the maximal open set in Y which satisfies $N \times U_T^N \subset U(T)$. Put $T_N = T \cap (N \times U_T^N)$. Then $\bigcup_{N \in \mathcal{N}} T_N = T$. To show this, consider a $t \in T$. Since $T \subset U(T)$, there are open sets $U_1 \subset X$ and $U_2 \subset Y$ such that $t \in U_1 \times U_2 \subset U(T)$. As $\mathcal{N}$ is a network of X, there is an $N \in \mathcal{N}$ such that $\pi_X(t) \in N \subset U_1$. Then $N \times U_2 \subset U(T)$, and thus $U_2 \subset U_T^N$, $t \in N \times U_T^N$, and consequently, $t \in T_N$.

Finally, for a given $N \in \mathcal{N}$, the sets U_T^N, $T \in \mathcal{T}$, are obviously $\mathcal{P}$-associated open sets for the family $(\pi_Y(T_N);\ T \in \mathcal{T})$, which is thus in $\mathcal{P}$. $\square$

Lemma 2.6.

(a) *Let $f_i\colon X_i \to Y_i$, $i = 1, 2, \ldots$, be index-σ-$\mathcal{P}$ mappings between topological spaces and X_i, $i = 2, 3, \ldots$, have a σ-$\mathcal{P}$ network. Then the product mapping $\prod_{n \in \mathbb{N}} f_n\colon (x_1, x_2, \ldots) \mapsto (f_1(x_1), f_2(x_2), \ldots)$ is also index-σ-$\mathcal{P}$.*

(b) *Let $f\colon X \to Y$ be an index-σ-$\mathcal{P}$ mapping, where either X or Y has a σ-$\mathcal{P}$ network. Let G be the graph of f. Then the restriction to G of the projection $\pi_Y\colon X \times Y \to Y$ is also index-σ-$\mathcal{P}$.*

Proof. (a) Let $\mathcal{N}_n$ be a σ-$\mathcal{P}$ network for X_n, $n = 2, \ldots$ The collection $\mathcal{N} = \{\pi_2^{-1}(N_2) \cap \ldots \cap \pi_k^{-1}(N_k);\ k \geqslant 2, N_i \in \mathcal{N}_i\}$, where π_i stands for the projection of $\prod_{n\geqslant 2} X_n$ to X_i, is a σ-$\mathcal{P}$ network for $\prod_{n\geqslant 2} X_n$.

Let $\mathcal{T}$ be in $\mathcal{P}\bigl(\prod_{n\in\mathbb{N}} X_n\bigr)$. We are going to prove that $\bigl((\prod_{n\in\mathbb{N}} f_n)(T);\ T \in \mathcal{T}\bigr)$ is σ-$\mathcal{P}$ resolvable.

Let $(U(T);\ T \in \mathcal{T})$ be an indexed family of $\mathcal{P}$-associated open sets for $\mathcal{T}$.

Using Lemma 2.5, we get open sets U_T^N in X_1 such that $T = \bigcup_{N\in\mathcal{N}} T^N$, where $T^N = T \cap (U_T^N \times N)$. Moreover, given an $N \in \mathcal{N}$, the family $\{\pi_1(T^N);\ T \in \mathcal{T}\}$ is in $\mathcal{P}(X_1)$ with the $\mathcal{P}$-associated open sets U_T^N, $T \in \mathcal{T}$.

Now

$$\Bigl(\prod_{n\in\mathbb{N}} f_n\Bigr)(T) \subset \bigcup_{N\in\mathcal{N}} \Bigl(\prod_{n\in\mathbb{N}} f_n\Bigr)(T^N) \subset \bigcup_{N\in\mathcal{N}} \Bigl(f_1(\pi_1(T^N)) \times \Bigl(\prod_{n\geqslant 2} f_n\Bigr)(N)\Bigr).$$

The mapping $\prod_{n\geqslant 2} f_n$ is index-σ-$\mathcal{P}$ by [5, Lemma 6.9 (d)]. So the family $\bigl((\prod_{n\geqslant 2} f_n)(N);\ N \in \mathcal{N}\bigr)$ is σ-$\mathcal{P}$-resolvable, and thus the family

$$\Bigl(\Bigl(\bigcup_{T\in\mathcal{T}} f_1(\pi_1(T^N))\Bigr) \times \Bigl(\prod_{n\geqslant 2} f_n\Bigr)(N);\ N \in \mathcal{N}\Bigr)$$

is also σ-$\mathcal{P}$-resolvable.

Since f_1 is an index-σ-$\mathcal{P}$ mapping, $(f_1(\pi_1(T^N));\ T \in \mathcal{T})$ is σ-$\mathcal{P}$-resolvable for each N, and clearly also $\bigl(f_1(\pi_1(T^N)) \times (\prod_{n\geqslant 2} f_n)(N);\ T \in \mathcal{T}\bigr)$ is σ-$\mathcal{P}$-resolvable. Using [4, Lemma 2.2 (c)], we get that

$$\Bigl(f_1(\pi_1(T^N)) \times \Bigl(\prod_{n\geqslant 2} f_n\Bigr)(N);\ T \in \mathcal{T}, N \in \mathcal{N}\Bigr)$$

is σ-$\mathcal{P}$-resolvable, and by [4, Lemma 2.7 (c)], we get that

$$\Bigl(\bigcup_{N\in\mathcal{N}} (f_1(\pi_1(T^N)) \times \Bigl(\prod_{n\geqslant 2} f_n\Bigr)(N));\ T \in \mathcal{T}\Bigr)$$

is σ-$\mathcal{P}$-resolvable. Thus also $\bigl((\prod_{n\in\mathcal{N}} f_n)(T);\ T \in \mathcal{T}\bigr)$ is σ-$\mathcal{P}$-resolvable which concludes the proof of (a).

(b) Let us denote the restriction $\pi_Y|_G$ by p. Then $p = h \circ g$, where $g\colon G \to Y \times Y$ maps (x, y) to $(f(x), y)$ and $h\colon D = \{(y, y);\ y \in Y\} \to Y$ maps (y, y) to y. The inclusion $g(G) \subset D$ holds because $(x, y) \in G$ implies $y = f(x)$. Now, g is index-σ-$\mathcal{P}$ according to the part (a), and h is a homeomorphism, so p is index-σ-$\mathcal{P}$. $\square$

We derive a quite general theorem on graphs and ranges of measurable mappings (cf. [1, Theorem 1]) in the next two assertions.

Lemma 2.7. *Let X and Y be topological spaces, $f\colon X \to Y$ be a mapping with graph G, and $\mathcal{A}$ be a family of subsets of X. Let M be a metrizable space with a σ-discrete basis $\mathcal{B}$ and $\varphi\colon M \to Y$ be continuous such that $f^{-1}(\varphi(I)) \in \mathcal{A}$ for each $I \in \mathcal{B}$. Then the set $H = \{(x,m) \in X \times M;\ (x,\varphi(m)) \in G\}$ is in $(\mathcal{A} \times \mathcal{B})_{\mathcal{D}\sigma\delta}$.*

P r o o f. Let us fix a compatible metric d on M. Since $\mathcal{B}$ is a σ-discrete basis of M, we can write $\mathcal{B} = \bigcup\limits_n \mathcal{B}_n$, where $\mathcal{B}_n$, $n \in \mathcal{N}$, are σ-discrete covers of M by open sets of diameter at most $1/n$, and $\mathcal{B}_n = \bigcup\limits_m \mathcal{B}_n^m$, where the families $\mathcal{B}_n^m$, $m \in \mathcal{N}$, are discrete. We prove that $H = \bigcap\limits_n \bigcup\limits_m \bigcup\limits_{I \in \mathcal{B}_n^m} (X_I \times I)$, where $X_I = \{x \in X;\ H_x \cap I \neq \emptyset\}$. The inclusion $\subset$ is obvious. To prove the other one, suppose that $(x,\iota) \notin H$. Since $H_x = \varphi^{-1}(f(x))$, it is closed, and there exists an $n \in \mathbb{N}$ such that $d(\iota, H_x) > 1/n$. If $\iota \in I$ and $I \in \mathcal{B}_n$, then $I \cap H_x = \emptyset$, thus $x \notin X_I$. So $(x,\iota) \notin X_J \times J$ for all $J \in \mathcal{B}_n$, and also the other inclusion is proved.

For each $I \in \mathcal{B}$, $X_I = f^{-1}(\varphi(I))$ is in $\mathcal{A}$, so $X_I \times I$ is in $\mathcal{A} \times \mathcal{B}$. Each family $\mathcal{B}_n^m$ is discrete, hence $\bigcup\limits_{I \in \mathcal{B}_n^m} (X_I \times I)$ is in $(\mathcal{A} \times \mathcal{B})_{\mathcal{D}}$ and H is in $(\mathcal{A} \times \mathcal{B})_{\mathcal{D}\sigma\delta}$. $\square$

Theorem 2.8.

(a) *Let Y be a $\mathcal{P}$-analytic space (or a $\mathcal{P}$-Luzin space) and $f\colon X \to Y$ be such that the preimages of $\mathcal{P}$-analytic subspaces are $\mathcal{P}$-analytic (or such that the preimages of $\mathcal{P}$-Luzin subspaces are $\mathcal{P}$-Luzin). Then the graph G of f is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin, respectively).*

(b) *If moreover f is index-σ-$\mathcal{P}$ and Y is $\mathcal{P}$-analytic (or injective and index-σ-$\mathcal{P}$, and Y is $\mathcal{P}$-Luzin), then $f(X)$ is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin, respectively).*

P r o o f. (a) Since Y is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin), there exists a complete metric space M and a continuous (or continuous and injective) index-σ-$\mathcal{P}$ parametrization $\varphi\colon M \to Y$. Let $\mathcal{B}$ be a σ-discrete basis of M. Then, for each $B \in \mathcal{B}$, $\varphi(B)$ is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin), and $f^{-1}(\varphi(B))$ is of the same type. According to the preceding Lemma 2.7, $H = \{(x,m) \in X \times M;\ (x,\varphi(m)) \in G\}$ is in $(\mathcal{A} \times \mathcal{B})_{\mathcal{D}\sigma\delta}$, where $\mathcal{A}$ is the class of $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin) subsets of X. So H is $\mathcal{P}$-analytic ($\mathcal{P}$-Luzin) by Theorems 2.2 and 2.3. Since $G = (\mathrm{id} \times \varphi)(H)$ and the mapping $\mathrm{id} \times \varphi$ is index-σ-$\mathcal{P}$ (Lemma 2.6(a)) and continuous (or continuous and injective), it follows from the definition that G is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin).

(b) According to part (a), the graph G of f is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin). The restriction p to G of the projection $\pi_Y\colon X \times Y \to Y$ is index-σ-$\mathcal{P}$ according to

Lemma 2.6(b). If f is injective, then the same holds for p. The projection is also continuous, hence it follows from the definition that $f(X)$, which is equal to $\pi_Y(G)$, is $\mathcal{P}$-analytic (or $\mathcal{P}$-Luzin). $\square$

As we are going to get results on subsets of products of topological spaces as applications of the reduction described in the next section and of the corresponding results from [9], we need the following two lemmas which improve [9, Lemma 4], where scattered-Borel sets were not investigated. We prove first a result on generation of scattered-Borel sets.

Lemma 2.9. *Let $\mathcal{C}$ be a family of subsets of a topological space X which contains all Borel sets and which is closed under intersections of countable subfamilies and under unions of σ-scattered subfamilies. Then $\mathcal{C}$ contains all scattered-Borel sets.*

Proof. Put $\mathcal{C}_0 = \{C \in \mathcal{C};\ X \setminus C \in \mathcal{C}\}$. Obviously, $\mathcal{C}_0$ contains all Borel sets, is closed under unions and intersections of countable subfamilies, and is closed under the operation of taking complements. Thus it is sufficient to prove that unions of scattered subfamilies of $\mathcal{C}_0$ are in $\mathcal{C}_0$ since then $\mathbf{B}^{\mathcal{S}}(X) \subset \mathcal{C}_0 \subset \mathcal{C}$. Let $\mathcal{E} \subset \mathcal{C}_0$ be scattered. Then $\bigcup \mathcal{E} \in \mathcal{C}$ by our assumptions. Let $H(E)$ be as in the introduction, i.e., such that $E \subset H(E) \in (\mathbf{F} \wedge \mathbf{G})(X)$ for $E \in \mathcal{E}$ and $(H(E);\ E \in \mathcal{E})$ is scattered. Then $X \setminus \bigcup \mathcal{E} = (X \setminus \bigcup\{H(E);\ E \in \mathcal{E}\}) \cup \bigcup\{H(E) \setminus E;\ E \in \mathcal{E}\}$. The latter union is in $\mathcal{C}$ as a scattered union of elements of $\mathcal{C}$. Realizing that $\bigcup\{H(E);\ E \in \mathcal{E}\} \in \mathbf{H}^{\mathcal{S}}(X) \subset \mathcal{C}$, we get that $X \setminus \bigcup\{H(E);\ E \in \mathcal{E}\}$ is in $\mathbf{H}^{\mathcal{S}}(X) \subset \mathcal{C}$, too. So the set $X \setminus \bigcup \mathcal{E}$ belongs to $\mathcal{C}$ being the union of two elements of $\mathcal{C}$. We conclude that $\bigcup \mathcal{E} \in \mathcal{C}_0$, and the proof is finished. $\square$

Lemma 2.10. *Let X, Y be topological spaces and X have a countable basis. Then*

$$\mathbf{S}\big(\mathbf{B}^{\mathcal{P}}(X \times Y)\big) = \mathbf{S}\big(\mathbf{B}^{\mathcal{P}}(X) \times \mathbf{B}^{\mathcal{P}}(Y)\big).$$

Proof. As we have already mentioned, the case $\mathcal{P} = \mathcal{I}$ was proved in [9, Lemma 4]. So it remains to prove the case $\mathcal{P} = \mathcal{S}$. The inclusion $\mathcal{S}(\mathbf{B}^{\mathcal{S}}(X) \times \mathbf{B}^{\mathcal{S}}(Y)) \subset \mathcal{S}(\mathbf{B}^{\mathcal{S}}(X \times Y))$ is obvious. As in [9, Lemma 4] we can prove easily that

$$\mathbf{G}(X \times Y) \subset (\mathbf{G}(X) \times \mathbf{G}(Y))_{\sigma} \subset \mathbf{S}(\mathbf{B}^{\mathcal{S}}(X) \times \mathbf{B}^{\mathcal{S}}(Y))$$

and

$$\mathbf{F}(X \times Y) \subset (\mathbf{F}(X) \times \mathbf{F}(Y))_{\sigma\delta} \subset \mathbf{S}(\mathbf{B}^{\mathcal{S}}(X) \times \mathbf{B}^{\mathcal{S}}(Y)).$$

The family $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X) \times \mathbf{B}^{\mathcal{S}}(Y))$ is closed under Souslin operation and thus also under unions and intersections of countable subfamilies. Hence $\mathbf{B}(X \times Y) \subset \mathbf{S}(\mathbf{B}^{\mathcal{S}}(X) \times$

$\mathbf{B}^{\mathcal{S}}(Y))$. We are going to prove that $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X)\times\mathbf{B}^{\mathcal{S}}(Y))$ is closed under unions of scattered subfamilies, which according to Lemma 2.9 gives that

$$\mathbf{B}^{\mathcal{S}}(X\times Y)\subset\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X)\times\mathbf{B}^{\mathcal{S}}(Y)).$$

Then $\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X\times Y))\subset\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X)\times\mathbf{B}^{\mathcal{S}}(Y))$ and this will conclude the proof of the lemma.

Let $\mathcal{U}$ be a countable basis of X such that $\mathcal{U}\supset\{\emptyset,X\}$ and put $\mathcal{U}^c=\{X\setminus U;\ U\in\mathcal{U}\}$. We have that $\mathbf{S}((\mathcal{U}\wedge\mathcal{U}^c)\times\mathbf{B}^{\mathcal{S}}(Y))=\mathbf{S}(\mathbf{B}(X)\times\mathbf{B}^{\mathcal{S}}(Y))=\mathbf{S}(\mathbf{B}^{\mathcal{S}}(X)\times\mathbf{B}^{\mathcal{S}}(Y))$ since $\mathcal{U}\wedge\mathcal{U}^c$ contains $\mathcal{U}\cup\mathcal{U}^c$ and $\mathcal{U}$ is a countable basis of X and $\mathbf{B}^{\mathcal{S}}(X)=\mathbf{B}(X)$ for X with a countable basis. So it is sufficient to prove that $\mathbf{S}((\mathcal{U}\wedge\mathcal{U}^c)\times\mathbf{B}^{\mathcal{S}}(Y))$ contains the unions of its scattered subfamilies.

Let $\mathcal{T}\subset\mathbf{S}((\mathcal{U}\wedge\mathcal{U}^c)\times\mathbf{B}^{\mathcal{S}}(Y))$ be scattered. Following Lemma 2.5 we may write each $T\in\mathcal{T}$ as $T=\bigcup\limits_{V\in\mathcal{U}}T_V$, where $(\pi_Y(T_V);\ T\in\mathcal{T})$ is scattered for $V\in\mathcal{U}$. Moreover, T_V being of the form $T\cap(V\times U_T^V)$ with U_T^V open belongs to $\mathbf{S}((\mathcal{U}\wedge\mathcal{U}^c)\times\mathbf{B}^{\mathcal{S}}(Y))$ again. We may thus suppose without loss of generality that $\mathcal{T}=(T_a;\ a\in A)\subset\mathbf{S}((\mathcal{U}\wedge\mathcal{U}^c)\times\mathbf{B}^{\mathcal{S}}(Y))$ is such that $(\pi_Y(T_a);\ a\in A)$ is scattered. For each $a\in A$ we find some $V^a_{n_1,\dots,n_k}\in\mathcal{U}\wedge\mathcal{U}^c$ and $B^a_{n_1,\dots,n_k}\in\mathbf{B}^{\mathcal{S}}(Y)$ such that $T_a=\bigcup\limits_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}}\bigcap\limits_{k\in\mathbb{N}}(V^a_{n_1,\dots,n_k}\times B^a_{n_1,\dots,n_k})$. Replacing, if necessary, $V^a_{n_1,\dots,n_k}$ by $\bigcap\limits_{i=1}^{k}V^a_{n_1,\dots,n_i}$ and $B^a_{n_1,\dots,n_k}$ by $\bigcap\limits_{i=1}^{k}B^a_{n_1,\dots,n_i}$, we may and do suppose that the scheme is regular, i.e.,

$$V^a_{n_1,\dots,n_{k+1}}\times B^a_{n_1,\dots,n_{k+1}}\subset V^a_{n_1,\dots,n_k}\times B^a_{n_1,\dots,n_k}.$$

We may and do suppose that the sets $B^a_{n_1,\dots,n_k}$, $a\in A$, are pairwise disjoint for each $(n_1,\dots,n_k)\in\mathbb{N}^k$ by intersecting them with the $(\mathbf{F}\wedge\mathbf{G})(Y)$ sets $H(\pi_Y(T_a))$, $a\in A$, from the introduction, which form a scattered family. It follows that

$$\begin{aligned}\bigcup_{a\in A}T_a&=\bigcup_{a\in A}\ \bigcup_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}}\ \bigcap_{k\in\mathbb{N}}(V^a_{n_1,\dots,n_k}\times B^a_{n_1,\dots,n_k})\\&=\bigcup_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}}\ \bigcap_{k\in\mathbb{N}}\ \bigcup_{a\in A}(V^a_{n_1,\dots,n_k}\times B^a_{n_1,\dots,n_k}).\end{aligned}$$

Using further the fact that $\mathcal{U}$ is countable, we may write $\mathcal{U}\wedge\mathcal{U}^c$ as $\{V(i);\ i\in\mathbb{N}\}$ to get

$$\bigcup_{a\in A}T_a=\bigcup_{(n_1,n_2,\dots)\in\mathbb{N}^{\mathbb{N}}}\ \bigcap_{k\in\mathbb{N}}\ \bigcup_{i\in\mathbb{N}}\ \bigcup_{\{a;\ V^a_{n_1,\dots,n_k}=V(i)\}}(V(i)\times B^a_{n_1,\dots,n_k}).$$

The last union over a's can be replaced by $V(i) \times \bigcup_{\{a;\, V^a_{n_1,\dots,n_k}=V(i)\}} B^a_{n_1,\dots,n_k}$, which is in $(\mathcal{U} \wedge \mathcal{U}^c) \times \mathbf{B}^{\mathcal{S}}(Y)$. Their countable unions over i's are in $\mathbf{S}((\mathcal{U} \wedge \mathcal{U}^c) \times \mathbf{B}^{\mathcal{S}}(Y))$ and applying to them the Souslin operation, we remain in the latter class. This concludes the proof. $\square$

3. A reduction of projections along nonseparable spaces to projections along separable spaces

Let $\xi = (\xi_1, \dots, \xi_k)$ or $\xi = (\xi_1, \dots)$. If $m \in \{1, \dots, k\}$, we use $\xi|m$ to denote the sequence $(\xi_1, \dots, \xi_m)$. The symbol $\xi|0$ stands for the empty sequence. If $\eta = (\eta_1, \dots, \eta_n)$, then $\xi\hat{}\eta$ stands for the concatenation $(\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_n)$ and the symbol η_- is an abbreviation for $\eta|(n-1)$ in what follows.

Lemma 3.1. *Let D be an arbitrary set and let $(Y_\alpha;\ k \in \mathbb{N},\ \alpha \in D^k)$ be an indexed σ-$\mathcal{P}$-resolvable family in a topological space Y such that $Y_\alpha \subset Y_{\alpha_-}$ if $k > 1$ and $\alpha \in D^k$.*

Then there is an index set Λ and, for every $k \in \mathbb{N}$, $\alpha \in D^k$, $\nu \in \mathbb{N}^k$ and $\lambda \in \Lambda^k$, there is a set $E^{\nu,\lambda}_\alpha$ in $\mathbf{H}^{\mathcal{P}}(Y)$ such that the following properties hold true for every $k \in \mathbb{N}$:

(1_k) $Y_\alpha \subset \bigcup_{\nu \in \mathbb{N}^k} \bigcup_{\lambda \in \Lambda^k} E^{\nu,\lambda}_\alpha$ *for every* $\alpha \in D^k$.

(2_k) $E^{\nu,\lambda}_\alpha \subset E^{\nu_-,\lambda_-}_{\alpha_-}$ *if* $k > 1$, $\alpha \in D^k$, $\nu \in \mathbb{N}^k$, *and* $\lambda \in \Lambda^k$.

(3_k) *The family*
$$\mathcal{E}^\nu = (E^{\nu,\lambda}_\alpha;\ \alpha \in D^k, \lambda \in \Lambda^k)$$
belongs to $\mathcal{P}$ for every $\nu \in \mathbb{N}^k$.

(4_k) *The family*
$$\mathcal{E}_\alpha = (E^{\nu,\lambda}_\alpha; \nu \in \mathbb{N}^k, \lambda \in \Lambda^k)$$
is disjoint for every $\alpha \in D^k$.

Proof. Let $(Y_\alpha(n,l); k \in \mathbb{N},\ \alpha \in D^k,\ n \in \mathbb{N},\ l \in \Lambda)$ be the corresponding decomposition of the σ-$\mathcal{P}$-resolvable family $(Y_\alpha; \alpha \in D^k,\ k \in \mathbb{N})$.

To avoid repetition of the construction needed both in the case $k = 1$ and in the general induction step, we extend our statement a bit artificially to the case $k = 0$ by putting $\mathbb{N}^0 = D^0 = \{\emptyset\}$, $\Lambda^0 = \{\emptyset\}$, $Y_\emptyset = Y$, $E^{\emptyset,\emptyset}_\emptyset = Y$. Let the families $(E^{\nu,\lambda}_\alpha\ ;\ \alpha \in D^k,\ \nu \in \mathbb{N}^k,\ \lambda \in \Lambda^k)$ of $\mathbf{H}^{\mathcal{P}}(Y)$-sets fulfilling the conditions (1_k), (2_k), (3_k), and (4_k) be already defined for some $k \in \{0, 1, 2, \dots\}$.

Then the family

$$(Y_{\alpha\hat{}a} \cap E_\alpha^{\nu,\lambda}; a \in D)$$

is σ-$\mathcal{P}$-resolvable with the corresponding decomposition $(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l) = Y_{\alpha\hat{}a}(n,l) \cap E_\alpha^{\nu,\lambda};\ a \in D, n \in \mathbb{N}, l \in \Lambda)$ for fixed $\alpha \in D^k$, $\nu \in \mathbb{N}^k$, and $\lambda \in \Lambda^k$ by our assumptions. For every $n \in \mathbb{N}$ let

$$\text{(D)} \qquad (H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l));\ a \in D, l \in \Lambda)$$

be the $\mathcal{P}$ family of $(\mathbf{F} \wedge \mathbf{G})$-sets from the introduction.

Finally, for every $\nu \in \mathbb{N}^k$, $\lambda \in \Lambda^k$, and $\alpha \in D^k$ we put

$$\text{(E)} \qquad E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l} = E_\alpha^{\nu,\lambda} \cap \Bigg(H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l)) \setminus \bigcup_{n'<n} \bigcup_{l' \in \Lambda} H(F_{\alpha\hat{}a}^{\nu,\lambda}(n',l')) \Bigg).$$

We proceed inductively in k to prove that $E_\alpha^{\nu,\lambda} \in \mathbf{H}^{\mathcal{P}}(Y)$ for $\nu \in \mathbb{N}^k$, $\alpha \in D^k$, $\lambda \in \Lambda^k$.

Clearly, $E_\emptyset^{\emptyset,\emptyset} = Y$ is in $\mathbf{H}^{\mathcal{P}}(Y)$ for $\nu \in \mathbb{N}^k$, $\alpha \in D^k$, $\lambda \in \Lambda^k$. Suppose that $E_\alpha^{\nu,\lambda} \in \mathbf{H}^{\mathcal{P}}(Y)$. Let us recall that $\mathbf{H}^{\mathcal{P}}(Y)$ is an algebra closed under unions of $\mathcal{P}$ families of its elements. Since the sets $H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l))$ belong to $\mathbf{H}^{\mathcal{P}}(Y)$ by their definition and the families $\{H(F_{\alpha\hat{}a}^{\nu,\lambda}(n',l'));\ l' \in \Lambda\}$ are in $\mathcal{P}$, we see from (E) that $E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l} \in \mathbf{H}^{\mathcal{P}}(Y)$.

We can proceed inductively again to prove (1_k). We have $Y_\emptyset = E_\emptyset^{\emptyset,\emptyset} = Y$. Let $Y_\alpha \subset \bigcup_{\nu \in \mathbb{N}^k} \bigcup_{\lambda \in \Lambda^k} E_\alpha^{\nu,\lambda}$ for $\alpha \in \Lambda^k$. By the assumptions $Y_{\alpha\hat{}a} \subset Y_\alpha$ for $a \in \Lambda$. So $Y_{\alpha\hat{}a} \subset \bigcup_{\nu \in \mathbb{N}^k} \bigcup_{\lambda \in \Lambda^k} Y_{\alpha\hat{}a} \cap E_\alpha^{\nu,\lambda}$. Further, $Y_{\alpha\hat{}a} \cap E_\alpha^{\nu,\lambda} = \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \Lambda} F_{\alpha\hat{}a}^{\nu,\lambda}(n,l) \subset \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \Lambda} H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l)) \cap E_\alpha^{\nu,\lambda}$. It is now obvious from (E) that

$$Y_{\alpha\hat{}a} \cap E_\alpha^{\nu,\lambda} \subset \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \Lambda} H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l)) \cap E_\alpha^{\nu,\lambda} = \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \Lambda} E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l},$$

which proves (1_{k+1}).

(2_k) follows directly from the definition of $E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l}$ in (E).

We show, by induction in $k = 0, 1, \ldots$, that (3_k) is satisfied. The case $k = 0$ is trivial. Using (D) and the induction assumption that $(E_\alpha^{\nu,\lambda};\ \alpha \in D^k, \lambda \in \Lambda^k)$ is in $\mathcal{P}$ for every $\nu \in \mathbb{N}^k$, we get by [4, Lemma 2.2 (a) and (b)] that also $(E_\alpha^{\nu,\lambda} \cap H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l));\ a \in D, l \in \Lambda)$ is in $\mathcal{P}$, and it remains to note that, by (E), also each family $(E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l};\ a \in D, l \in \Lambda)$ is in $\mathcal{P}$ for $\nu \in \mathbb{N}^k$, $n \in \mathbb{N}$, $\alpha \in D^k$, $\lambda \in \Lambda^k$.

To prove (4_k) note that if $E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l}$, $E_{\alpha\hat{}a}^{\nu\hat{}n',\lambda\hat{}l'}$ and $n < n'$, then, by (E), $E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l} \subset H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l))$ and $E_{\alpha\hat{}a}^{\nu\hat{}n',\lambda\hat{}l'} \cap H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l)) = \emptyset$. If $n = n'$ and $l \neq l'$, then, by (D), $E_{\alpha\hat{}a}^{\nu\hat{}n,\lambda\hat{}l} \cap E_{\alpha\hat{}a}^{\nu\hat{}n',\lambda\hat{}l'} \subset H(F_{\alpha\hat{}a}^{\nu,\lambda}(n,l)) \cap H(F_{\alpha\hat{}a}^{\nu,\lambda}(n',l')) = \emptyset$. $\square$

Theorem 3.2. *Let $A \subset D^{\mathbb{N}} \times Y$, where Y is an arbitrary topological space and D a discrete topological space. Suppose that $\pi_Y|_A$, the projection of $D^{\mathbb{N}} \times Y$ to Y restricted to A, is index-σ-$\mathcal{P}$.*

Then there exists a set $B \in \mathbf{H}^{\mathcal{P}}_{\sigma\delta}(D^{\mathbb{N}} \times Y)$ such that $A \subset B$, and a mapping $\Psi\colon B \to \mathbb{N}^{\mathbb{N}} \times Y$ such that

(1) *$\Psi^{-1}(G)$ is in $\mathbf{H}^{\mathcal{P}}_{\sigma}(B)$ for every open subset G of $\mathbb{N}^{\mathbb{N}} \times Y$, and $\Psi^{-1}(\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y)) \subset \mathbf{B}^{\mathcal{P}}(B)$.*
(2) *Ψ is index-σ-$\mathcal{P}$,*
(3) *$\pi_Y\big(\Psi(x,y)\big) = \{y\}$ for every $(x,y) \in B$, and*
(4) *$\Psi|_{B^y}$ is a homeomorphism for every $y \in Y$.*

P r o o f. Let $I_\alpha = \{\beta \in D^{\mathbb{N}};\ \beta|k = \alpha\}$ for $k \in \mathbb{N}$ and $\alpha \in D^k$ be the *Baire intervals* in the space $D^{\mathbb{N}}$. Let us recall that the families $\mathcal{I}_k = \big\{I_\alpha;\ \alpha \in D^k\big\}$ are discrete for $k \in \mathbb{N}$ and that the Baire intervals form a σ-discrete basis of the topology of $D^{\mathbb{N}}$. Let us consider the sets $Y_\alpha = \pi_Y(A \cap (I_\alpha \times Y))$ for $\alpha \in D^k$, $k \in \mathbb{N}$. As $\pi_Y|_A$ preserves indexed σ-$\mathcal{P}$-resolvable families and the family $(A \cap (I_\alpha \times Y);\ \alpha \in D^k, k \in \mathbb{N})$ is σ-$\mathcal{P}$-resolvable, even σ-discrete, in A, the family $(Y_\alpha;\ \alpha \in D^k, k \in \mathbb{N})$ is σ-$\mathcal{P}$-resolvable, and so it easily follows that it satisfies the assumptions of Lemma 3.1.

Let

$$E^{\nu,\lambda}_\alpha \text{ with } \alpha \in D^k,\ \nu \in \mathbb{N}^k,\ \lambda \in \Lambda^k,\ k \in \mathbb{N},$$

be the $\mathbf{H}^{\mathcal{P}}(Y)$ sets obtained using Lemma 3.1.

We define the set B by

$$B = \bigcap_{k\in\mathbb{N}} \bigcup_{\nu\in\mathbb{N}^k} \bigcup_{\alpha\in D^k} \bigcup_{\lambda\in\Lambda^k} (I_\alpha \times E^{\nu,\lambda}_\alpha).$$

It follows immediately that $A \subset B$ due to property (1_k) of Lemma 3.1, since $A \subset \bigcup_{\alpha\in D^k} I_\alpha \times Y_\alpha$, and that $B \in \mathbf{H}^{\mathcal{P}}_{\sigma\delta}(D^{\mathbb{N}} \times Y)$, since the unions over $\alpha \in D^k$ and over $\lambda \in \Lambda^k$ are unions of families from $\mathcal{P}$ due to the property (3_k) of Lemma 3.1.

For a fixed pair $(\alpha, y) \in B$ we obtain by the property (4_k) of Lemma 3.1 uniquely determined sequences $\nu \in \mathbb{N}^{\mathbb{N}}$ and $\lambda \in \Lambda^{\mathbb{N}}$ such that $y \in \bigcap_{i=1}^{\infty} E^{\nu|i,\lambda|i}_{\alpha|i}$. We put $\Psi(\alpha, y) = (\nu, y)$ in such a case.

The property (3) of Ψ is obvious.

Let us notice that every open subset of $\mathbb{N}^{\mathbb{N}} \times Y$ is the union of countably many sets of the form $J_\mu \times U$, where U is an open subset of Y and $J_\mu = \{\nu \in \mathbb{N}^{\mathbb{N}};\ \nu|k = \mu\}$ for $\mu \in \mathbb{N}^k$ and $k \in \mathbb{N}$. As

$$\Psi^{-1}(J_\mu \times U) = \left\{(\alpha, y) \in B;\ y \in U \cap \bigcup_{\lambda\in\Lambda^k} E^{\mu,\lambda}_{\alpha|k}\right\},$$

the preimages of open sets by Ψ are in $\mathbf{H}_\sigma^{\mathcal{P}}(B)$ which is the first claim from property (1).

Consider the family $\mathcal{B} = \{E \subset \mathbb{N}^\mathbb{N} \times Y;\ \Psi^{-1}(E) \in \mathbf{B}^{\mathcal{P}}(B)\}$. By the already proved part of (1), $\mathcal{B} \supset \mathbf{G}(\mathbb{N}^\mathbb{N} \times Y)$. Since $\mathcal{B}$ is closed under unions and intersections of countable subfamilies as well as under complements, it is sufficient to prove that it is closed under the unions of $\mathcal{P}$ families to prove the remaining part of (1).

Let $\mathcal{E} \subset \mathcal{B}$ be a $\mathcal{P}$ family consisting of sets from $\mathbf{B}^{\mathcal{P}}(\mathbb{N}^\mathbb{N} \times Y)$. Assume that $\mathcal{N}$ is a countable basis of $\mathbb{N}^\mathbb{N}$. By Lemma 2.5 each $E \in \mathcal{E}$ can be decomposed as $E = \bigcup_{N \in \mathcal{N}} E_N$, where $E_N = E \cap (N \times U_E^N)$ for some open sets U_E^N, such that $(\pi_Y(E_N);\ E \in \mathcal{E})$ is a $\mathcal{P}$ family for every $N \in \mathcal{N}$. By the properties of Ψ we have $\pi_Y(E_N) = \pi_Y(\Psi^{-1}(E_N))$ for every E_N, and so $(\Psi^{-1}(E_N);\ E \in \mathcal{E})$ is a $\mathcal{P}(B)$ family for $N \in \mathcal{N}$.

Now $\Psi^{-1}(E_N) = \Psi^{-1}(E) \cap \Psi^{-1}(N \times U_E^N)$, where $\Psi^{-1}(E) \in \mathbf{B}^{\mathcal{P}}(B)$ since $\mathcal{E} \subset \mathcal{B}$ and $\Psi^{-1}(N \times U_E^N) \in \mathbf{B}^{\mathcal{P}}(B)$ according to the already proved part of (1). Thus $\Psi^{-1}(\bigcup \mathcal{E}) = \bigcup_{N \in \mathcal{N}} \bigcup_{E \in \mathcal{E}} \Psi^{-1}(E_N)$ is a union of a σ-$\mathcal{P}$ family of sets from $\mathbf{B}^{\mathcal{P}}(B)$. Thus the union $\bigcup \mathcal{E}$ belongs to $\mathcal{B}$.

To prove (4), we show first that $\Psi|_{B^y}$ is injective. Let (α, y) and (β, y) be distinct elements of B and let $\Psi(\alpha, y) = \Psi(\beta, y) = (\nu, y)$. Then $\alpha|k \neq \beta|k$ for some $k \in \mathbb{N}$ and $y \in E_{\alpha|k}^{\nu|k,\lambda} \cap E_{\beta|k}^{\nu|k,\iota}$ for some $\lambda, \iota \in \Lambda^k$. However, the family $\mathcal{E}^{\nu|k}$ is in $\mathcal{P}$ and thus it is disjoint. This is a contradiction.

Now, fix $y \in Y$ and $\nu \in \mathbb{N}^k$. Then $\Psi^{-1}\big(J_\nu \times \{y\}\big) = B \cap \big(\bigcup\{I_\alpha;\ y \in \bigcup_{\lambda \in \Lambda^k} E_\alpha^{\nu,\lambda}\} \times \{y\}\big)$, which is open in $B^y \times \{y\}$. This proves that $\Psi|_{B^y}$ is continuous.

Conversely,

$$\Psi(I_\alpha \times \{y\}) = \Psi(B) \cap \left(\bigcup \left\{ J_\nu; y \in \bigcup_{\lambda \in \Lambda^k} E_\alpha^{\nu,\lambda} \right\} \times \{y\} \right),$$

and so $\Psi^{-1}|_{\Psi(B)^y}$ is also continuous and (4) is verified.

Finally, we prove (2), i.e., that Ψ preserves indexed σ-$\mathcal{P}$-resolvable families. Let $\mathcal{R} \in \mathcal{P}(B)$. It suffices to show that $(\Psi(Q) \cap (J_\nu \times Y);\ Q \in \mathcal{R})$ is σ-$\mathcal{P}$-resolvable for every fixed $k \in \mathbb{N}$ and $\nu \in \mathbb{N}^k$. Let $k \in \mathbb{N}$ and $\nu \in \mathbb{N}^k$ be fixed.

Since $\{I_\alpha\colon \alpha \in D^k, k \in \mathbb{N}\}$ is a σ-$\mathcal{D}$ network for $D^\mathbb{N}$, using Lemma 2.5 we find Q_α, $\alpha \in D^k$, for $Q \in \mathcal{R}$, such that $Q = \bigcup_\alpha Q_\alpha$ and $(\pi_Y(Q_\alpha);\ Q \in \mathcal{R})$ is in $\mathcal{P}(Y)$ for each $\alpha \in D^k$.

Since $\Psi(S) \subset \bigcup_{\nu \in \mathbb{N}^k} \bigcup_{\lambda \in \Lambda^k} \big(J_\nu \times (E_\alpha^{\nu,\lambda} \cap \pi_Y(S))\big)$ for each $\alpha \in D^k$ and each $S \subset B \cap (I_\alpha \times Y)$, also

$$\Psi(Q_\alpha) \subset \bigcup_{\nu \in \mathbb{N}^k} \bigcup_{\lambda \in \Lambda^k} \big(J_\nu \times (E_\alpha^{\nu,\lambda} \cap \pi_Y(Q_\alpha))\big).$$

It suffices to check that the family of the right-hand sides indexed by $\alpha \in D^k$ and $Q \in \mathcal{R}$ is σ-$\mathcal{P}$-resolvable. This follows from the facts that the family $(J_\nu \times \pi_Y(Q_\alpha);\ Q \in \mathcal{R})$ is in $\mathcal{P}(\mathbb{N}^\mathbb{N} \times Y)$ and the family $(J_\nu \times E_\alpha^{\nu,\lambda};\ \alpha \in D^k, \lambda \in \Lambda^k)$ is also in $\mathcal{P}(\mathbb{N}^\mathbb{N} \times Y)$. Thus the mapping Ψ preserves indexed σ-$\mathcal{P}$-resolvable families. $\square$

4. Uniformization

Let X, Y be topological spaces and $C \subset X \times Y$. Recall that $U \subset C$ is a *uniformization* of C (over Y) if for each $y \in \pi_Y(C)$, the section U^y is a singleton.

We are going to improve the classical Kondô uniformization theorem ([14, Theorem 36.14]). We achieve it using [9, Theorem 7], or the classical claim, and our Theorem 3.2. Let us note that we get in this way also improvements of [17, Theorem 17] and [15, Theorem 5.5].

Theorem 4.1. *Let X, Y be $\mathcal{P}$-Luzin spaces. Let the complement of $C \subset X \times Y$ be in $\mathbf{S}(\mathbf{B}^\mathcal{S}(X \times Y))$, and let the projection π_Y of $X \times Y$ onto Y restricted to C be index-σ-$\mathcal{S}$.*

Then there exists a set $U \subset C$ which is a uniformization of C whose complement is in $\mathbf{S}(\mathbf{B}^\mathcal{S}(X \times Y))$. If $\mathcal{P} = \mathcal{I}$, the complement of U is in $\mathbf{S}(\mathbf{B}(X \times Y))$.

Proof. Suppose for a while that the claim holds if $D^\mathbb{N}$, with D being a discrete space, stands in the place of X. Let $\varphi\colon F \subset D^\mathbb{N} \to X$ be a continuous index-σ-$\mathcal{P}$ bijection of a closed subset F of $D^\mathbb{N}$ onto X for some discrete space D. Let C_0 be the preimage of C under the continuous bijective and index-σ-$\mathcal{P}$ mapping $\varphi \times \mathrm{id}\colon F \times Y \to X \times Y$. Its complement is in $\mathbf{S}(\mathbf{B}^\mathcal{P}(D^\mathbb{N} \times Y))$ as the preimage of the complement of C under $\varphi \times \mathrm{id}$. The projection to Y restricted to C_0 is index-σ-$\mathcal{P}$ being a composition of index-σ-$\mathcal{P}$ mappings $\varphi \times \mathrm{id}$ (defined on C_0) (see Lemma 2.6(a)) and $\pi_Y|_C$. Let U_0 be a uniformization of C_0 such that its complement is in $\mathbf{S}(\mathbf{B}^\mathcal{S}(D^\mathbb{N} \times Y))$. So the complement is $\mathcal{P}$-analytic by Theorem 2.2. Then the complement of $U = (\varphi \times \mathrm{id})(U_0)$, as the image of the complement of U_0 by $\varphi \times \mathrm{id}$, is also $\mathcal{P}$-analytic, or equivalently, $\mathbf{S}(\mathbf{B}^\mathcal{P}(X \times Y))$, and U is a uniformization of C.

Thus we may and do suppose further on that $C = C_0$ and $X = D^\mathbb{N}$. Using Theorem 3.2, we get a $\mathcal{P}$-Borel set $B \supset C$ and an injective $\mathcal{P}$-Borel measurable index-σ-$\mathcal{P}$ mapping Ψ of B to $\mathbb{N}^\mathbb{N} \times Y$ that preserves the second coordinate. Put $B_1 = \Psi(B)$ and $C_1 = \Psi(C)$. The complement of C_1 is the union of $B_1 \setminus C_1$ and of the complement of B_1. The set B is $\mathcal{P}$-Luzin (as a $\mathcal{P}$-Borel subset of the $\mathcal{P}$-Luzin space $D^\mathbb{N} \times Y$ by Theorem 2.3) and the same holds for its image B_1 (Theorem 2.8). It follows that B_1 is $\mathbf{bi}$-$\mathbf{S}(\mathbf{B}^\mathcal{P}(\mathbb{N}^\mathbb{N} \times Y))$ (Theorem 2.3), so in particular its complement is in $\mathbf{S}(\mathbf{B}^\mathcal{P}(\mathbb{N}^\mathbb{N} \times Y))$. The set $B_1 \setminus C_1$ is the image of the $\mathcal{P}$-analytic set $B \setminus C$ by Ψ, so it is also $\mathcal{P}$-analytic, and thus it is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y))$ again, thus the complement of C_1 is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y))$.

Applying Lemma 2.10 and [9, Lemma 1] with $\mathcal{M} = \mathbf{B}^{\mathcal{P}}(Y)$ and $\mathcal{H} = \mathbf{B}(\mathbb{N}^{\mathbb{N}})$ to the complement S_1 of C_1, we get a $\mathbf{B}^{\mathcal{P}}(Y)$ measurable mapping $f\colon Y \to f(Y) \subset \{0,1\}^{\mathbb{N}}$ and a set $S_2 \in \mathbf{S}(\mathbf{B}(\mathbb{N}^{\mathbb{N}}) \times \mathbf{F}(f(Y)))$ with $S_{1y} = S_{2f(y)}$ as in [9, Lemma 1]. Note that consequently the sets $(\mathrm{id} \times f)(S_1) = S_2$ and $(\mathrm{id} \times f)(C_1) = C_2$ form a partition of $\mathbb{N}^{\mathbb{N}} \times f(Y)$. Applying Kondô's theorem to C_2, we get a co-Souslin uniformization U_2 of C_2. Its preimage $U_1 = (\mathrm{id} \times f)^{-1}(U_2)$ is the complement of $(\mathrm{id} \times f)^{-1}((\mathbb{N}^{\mathbb{N}} \times f(Y)) \setminus U_2)$ which is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y))$ since the mapping $\mathrm{id} \times f$ is $\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y)$ measurable and U_1 is a uniformization of C_1.

Finally, putting $U = \Psi^{-1}(U_1)$, we get the required uniformization of C, which concludes the proof. □

5. Generalized projections along nonseparable spaces

In the next proposition, we use the notion of hereditary co-Souslin families of subsets of separable metric spaces as in [9]. We recall the needed definitions.

Let Z be a topological space. A collection of sets $\mathcal{C} \subset \mathbf{F}(Z)$ is called *a hereditary family* if every $H \in \mathbf{F}(Z)$, such that $H \subset F$ for some $F \in \mathcal{C}$, is in $\mathcal{C}$.

If Z is a separable metric space, we say that $\mathcal{C} \subset \mathbf{F}(Z)$ is a *co-Souslin family* if there exists a metric completion $\widehat{Z}$ of Z such that $\{\overline{S}^{\widehat{Z}};\ S \in \mathcal{C}\}$ is a co-Souslin subset of the Effros Borel structure on $\mathbf{F}(\widehat{Z})$.

Finally, let $\mathcal{C}$ be a family of closed sets in some space. We denote by $\mathcal{C}^*$ the class of all sets whose closures are in $\mathcal{C}$.

Proposition 5.1. *Let Y be a $\mathcal{P}$-analytic space, D be a discrete space, $S \subset D^{\mathbb{N}} \times Y$ be $\mathcal{S}$-analytic, and suppose that the projection $\pi_Y|_S$ is index-σ-$\mathcal{S}$. Let $\mathcal{E}$ be a hereditary coanalytic family in $\mathbf{F}(\mathbb{N}^{\mathbb{N}})$.*

(a) *Suppose that for every $C \in \mathcal{E}^*$, each homeomorphic copy of C in $\mathbb{N}^{\mathbb{N}}$ is in $\mathcal{E}^*$. Put*

$$\mathcal{C}^* = \{E \subset D^{\mathbb{N}};\ E \text{ is homeomorphic to some } H \in \mathcal{E}^*\}.$$

Then the sets

$$C_1 = \{y \in Y;\ S^y \in \mathcal{C}^*\}, \quad C_2 = \{y \in Y;\ S^y \in \mathcal{C}^*_\sigma\}$$

are complements of $\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ sets in Y.

(b) *Let Y and S be $\mathcal{P}$-Luzin. Suppose that for every $F \in \mathcal{E}$, each homeomorphic copy of F in $\mathbb{N}^{\mathbb{N}}$ is in $\mathcal{E}$. Put*

$$\mathcal{C} = \{E \in \mathbf{F}(D^{\mathbb{N}});\ E \text{ is homeomorphic to some } H \in \mathcal{E}\}.$$

Then the sets

$$C_3 = \{y \in Y;\ S^y \in \mathcal{C}\}, \quad C_4 = \{y \in Y;\ S^y \in \mathcal{C}_\sigma\}$$

are complements of $\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ sets in Y.

(c) *Let the assumptions of (b) be fulfilled, and moreover let each element of $\mathcal{E}$ be σ-compact.*

Then the sets

$$C_5 = \{y \in Y;\ \emptyset \neq S^y \in \mathcal{C}\}, \quad C_6 = \{y \in Y;\ \emptyset \neq S^y \in \mathcal{C}_\sigma\}$$

are complements of $\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ sets in Y.

Proof. Using Theorem 3.2 we find a $\mathcal{P}$-Borel set $B \subset D^{\mathbb{N}} \times Y$ containing S and a one-to-one index-σ-$\mathcal{P}$ mapping $\Psi\colon\ B \to \mathbb{N}^{\mathbb{N}} \times Y$ such that $\Psi^{-1}(\mathbf{B}^{\mathcal{P}}(\mathbb{N}^{\mathbb{N}} \times Y)) \subset \mathbf{B}^{\mathcal{P}}(B)$, the mapping Ψ preserves the second coordinate, and $\Psi|S^y$ is a homeomorphism for each $y \in \pi_Y(B)$.

The set $\Psi(S)$ is $\mathcal{P}$-analytic in $\mathbb{N}^{\mathbb{N}} \times Y$, being the image of $\mathcal{P}$-analytic S under a $\mathcal{P}$-Borel measurable index-σ-$\mathcal{P}$ mapping (Corollary 2.4 and Theorem 2.8). So $\Psi(S)$ is $\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ in $\mathbb{N}^{\mathbb{N}} \times Y$ by Theorem 2.2, $C_1 = \{y \in Y;\ \Psi(S)^y \in \mathcal{E}^*\}$, and $C_2 = \{y \in Y;\ \Psi(S)^y \in \mathcal{E}^*_\sigma\}$.

Now we use Lemma 2.10 and [9, Theorem 3] with $\mathcal{M} = \mathbf{B}^{\mathcal{P}}(Y)$ and $\mathcal{H} = \mathbf{B}(\mathbb{N}^{\mathbb{N}})$.

In the cases (b) and (c), both $\Psi(B)$ and $\Psi(S)$ are $\mathcal{P}$-Luzin in $\mathbb{N}^{\mathbb{N}} \times Y$, being one-to-one images of $\mathcal{P}$-Luzin sets under a $\mathcal{P}$-Borel measurable index-σ-$\mathcal{P}$ mapping (Theorem 2.8). Thus $\Psi(B)$ is **bi-**$\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ in $\mathbb{N}^{\mathbb{N}} \times Y$ due to Theorem 2.3, $C_3 = \{y \in Y;\ \Psi(S)^y \in \mathcal{E}\}$, and similar equalities hold also for C_4, C_5, and C_6. Using Lemma 2.10 and [9, Theorem 3], we finish the proof. $\square$

Remark. Let us mention that families of subsets of $D^{\mathbb{N}}$ with at most k elements, $k \geqslant 0$, are examples of families $\mathcal{C}$ from (a) and that the family of all compact subsets of $D^{\mathbb{N}}$ is an example of a family from (b) and (c).

In Theorem 5.2 we use a parametrization of X by a continuous index-σ-$\mathcal{P}$ injective mapping, which clearly preserves the cardinality of sets, but it is difficult to say anything about its behaviour to others properties, e.g., to compactness. So we obtain in this way results analogous to the previous ones for classes $\mathcal{C}$ defined in terms of cardinality of their elements only. For a metric space X, we find, in Lemma 5.3, another parametrization, which is not, in general, injective, but is continuous and preimages of compact sets are compact (in other words, it is perfect), and then we apply Proposition 5.1 to the class $\mathcal{C}$ of compact sets in Theorem 5.4.

Theorem 5.2. *Let X be a $\mathcal{P}$-Luzin space, the projection $\pi_Y|_S$ be index-σ-$\mathcal{S}$, $\kappa \in [1, \aleph_0] \cup \{\aleph_1\}$, and $\mathcal{C} = \{F \subset X\,;\ \operatorname{card} F < \kappa\}$.*

(a) *If Y is a $\mathcal{P}$-analytic space and S is a $\mathcal{P}$-analytic subset of $X \times Y$, then $P_1 = \{y \in Y\,;\ S^y \notin \mathcal{C}\}$ is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$.*

(b) *If both Y and $S \subset X \times Y$ are $\mathcal{P}$-Luzin, then the complement of the set $P_2 = \{y \in Y\,;\ \emptyset \neq S^y \in \mathcal{C}\}$ is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$.*

Proof. There exists a discrete set D and a continuous index-σ-$\mathcal{P}$ injective mapping φ of a closed subset of $D^{\mathbb{N}}$ onto X. Then $S_0 = (\varphi \times \mathrm{id})^{-1}(S)$ is $\mathcal{P}$-analytic (even $\mathcal{P}$-Luzin in the case (b)) by Corollary 2.4, the projection $\pi_Y|_{S_0}$ is index-σ-$\mathcal{P}$, being the composition of $\varphi \times \mathrm{id}$ (which is index-σ-$\mathcal{P}$ due to Lemma 2.6) and of $\pi_Y|_S$, and the cardinality of $(S_0)^y$ is equal to the cardinality of S^y for each $y \in Y$.

Let $\kappa < \aleph_1$ and $\mathcal{E} = \{F \in \mathbf{F}(\mathbb{N}^{\mathbb{N}})\,;\ \operatorname{card} F < \kappa\}$. Then $\mathcal{E}$ is a hereditary co-analytic family of closed sets stable under homeomorphisms in $\mathbb{N}^{\mathbb{N}}$ (cf. [14]). Let $\mathcal{C}_0 = \{F \in \mathbf{F}(D^{\mathbb{N}})\,;\ \operatorname{card} F < \kappa\}$. Then $\mathcal{C}_0 = \mathcal{C}_0{}^*$, and $P_1 = \{y \in Y\,;\ (S_0)^y \in \mathcal{C}^*\}$. So we can use Proposition 5.1(a).

For $\kappa = \aleph_1$, we put $\mathcal{E} = \{F \in \mathbf{F}(\mathbb{N}^{\mathbb{N}})\,;\ \operatorname{card} F \leqslant 1\}$. This is a hereditary co-analytic family of closed sets stable under homeomorphisms again, and if we put $\mathcal{C}_0 = \{F \in \mathbf{F}(D^{\mathbb{N}})\,;\ \operatorname{card} F \leqslant 1\}$, then $\mathcal{C}_0 = \mathcal{C}_0{}^*$, and $P_1 = \{y \in Y\,;\ (S_0)^y \in (\mathcal{C}_0^*)_\sigma\}$. Now we use Proposition 5.1(a) again.

To prove (b), we proceed similarly, using the fact that S_0 is $\mathcal{P}$-Luzin in this case, and applying Proposition 5.1(c). □

Remark. Notice first that as examples of a family $\mathcal{C}$ from Theorem 5.2 may serve the family $\mathcal{C} = \{\emptyset\}$ (i.e., P_1 is the complement of the projection of S), the families of all sets having at most k points for $k \in \mathbb{N}$, the family of finite sets, and the family of all at most countable (not necessarily closed) sets.

The statements for the classes of singletons and countable sets were proved in [2, Lemma in Section 5.2] for complete metric spaces X and Y by a different method.

The existence of a perfect parametrization $f\colon D^{\mathbb{N}} \to M$ of a complete metric space M was proved in [12, Lemma 9]. Using a result of [13] we might deduce that such a mapping is index-σ-discrete and apply this to prove Theorem 5.4. We get an index-σ-discrete perfect parametrization in a more straightforward and elementary way.

Lemma 5.3. *Let M be a complete metric space. Then there exist a discrete metric space D, a closed set $F \subset D^{\mathbb{N}}$, and a continuous index-σ-discrete mapping $f\colon F \to M$ such that $f(F) = M$ and $f^{-1}(K)$ is compact for compact K.*

Proof. Let us choose for each $n \in \mathbb{N}$ a cover $\mathcal{P}_n$ of M consisting of open sets of diameter smaller than $1/n$, which is σ-discrete and locally finite (the existence of such a cover follows from the paracompactness of M). Let $\prod \mathcal{P}_n$ be the space of sequences $(P_n)_{n=1}^{\infty}$ of subsets of M, with $P_n \in \mathcal{P}_n$ for each n. We consider $\mathcal{P}_n$ endowed with the discrete topology and the product space $\prod\limits_{n\in\mathbb{N}} \mathcal{P}_n$ embedded into $D^{\mathbb{N}}$ for some, sufficiently large, discrete space D. Finally, let $F = \{(P_n) \in \prod \mathcal{P}_n;\ \{P_n\}$ is a centered system$\}$.

Then F is closed in $\prod \mathcal{P}_n$: if $(P_n)_{n=1}^{\infty}$ is not centered, then there is $k \in \mathbb{N}$ such that $\bigcap\limits_{i=1}^{k} \overline{P_i} = \emptyset$. Then $\{(Q_n)_{n=1}^{\infty};\ Q_i = P_i, i = 1, \ldots, k\}$ is an open neighbourhood of $(P_n)_{n=1}^{\infty}$, which does not contain any centered sequence.

We define a mapping f on F so that $\{f((P_n)_{n=1}^{\infty})\} = \bigcap\limits_{i=1}^{\infty} \overline{P_i}$.

Then $f(F) = M$ since for each $y \in M$ and each $n \in \mathbb{N}$ there is $P_n \in \mathcal{P}_n$ such that $y \in P_n$.

Further, f is continuous. Let $(P_n)_{n=1}^{\infty} \in F$, $f((P_n)) = y \in M$. Let U be open in M, containing y. Then there is $k \in \mathbb{N}$ such that for each $n \geqslant k$, $\overline{P_n} \subset U$. But then $\{(Q_n)_{n=1}^{\infty} \in F;\ Q_i = P_i,\ i = 1, \ldots, k\}$ is an open neighbourhood of $(P_n)_{n=1}^{\infty}$, which is mapped to U.

Let K be compact in M. We prove that for each $n \in \mathbb{N}$, K meets only finitely many sets from $\mathcal{P}_n$: for each $y \in K$ we find U^y containing y, which meets only finitely many sets from $\mathcal{P}_n$. We choose a finite subcover from the cover $\{U^y;\ y \in K\}$ of K, and so we obtain a finite system $\mathcal{S}_n$ of all sets from $\mathcal{P}_n$ that meet K. Now $\{(P_n)_{n=1}^{\infty} \in F;\ \forall i\ P_i \in \mathcal{S}_i\}$ is a compact set in F, which is the preimage of K.

Finally, we prove that f is index-σ-discrete. To see this, it suffices to prove for one σ-discrete basis of F, that its image is σ-discrete. We choose the basis of Baire intervals $\{I_{P_1,\ldots,P_k} = \{(Q_n)_{n=1}^{\infty};\ Q_i = P_i, i = 1, \ldots, k\};\ k \in \mathbb{N}, P_i \in \mathcal{P}_i\}$. Here $f(I_{P_1,\ldots,P_k}) = \overline{P_1} \cap \ldots \cap \overline{P_k}$. Since $\mathcal{P}_i$ is σ-discrete for each $i \in \mathbb{N}$, also $\{\overline{P};\ P \in \mathcal{P}_i\}$ is σ-discrete for each i, and the system of all finite intersections of closures of sets from $\{\mathcal{P}_i;\ i \in \mathbb{N}\}$ is also σ-discrete. □

Theorem 5.4. *Let S be a $\mathcal{P}$-Luzin subset of $M \times Y$, where Y is a $\mathcal{P}$-Luzin space and M is a complete metric space. Let the projection $\pi_Y|_S$ be index-σ-$\mathcal{P}$. Then*

(a) $P_3 = \{y \in Y;\ S^y$ *is compact*$\}$,

(b) $P_4 = \{y \in Y;\ S^y$ *is σ-compact*$\}$,

(c) $P_5 = \{y \in Y\,;\ S^y$ *is nonempty and compact*$\}$,

(d) $P_6 = \{y \in Y\,;\ S^y$ *is nonempty and* σ*-compact*$\}$

are complements of $\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ *sets in* Y.

P r o o f. There exists a closed set $F \subset D^{\mathbb{N}}$ and a continuous $f\colon F \to M$ such that $f(F) = M$, $f^{-1}(K)$ is compact for compact K, and f is index-σ-discrete (Lemma 5.3). Then $S_0 = (\mathrm{id} \times f)^{-1}(S)$ is $\mathcal{P}$-Luzin by Corollary 2.4, the projection $\pi_Y|_{S_0}$ is index-σ-$\mathcal{P}$, being the composition of $\mathrm{id} \times f$ (which is index-σ-$\mathcal{P}$ due to Lemma 2.6) and of $\pi_Y|_S$, and S^y is compact, σ-compact, nonempty compact, or nonempty σ-compact, respectively, if and only if ${S_0}^y$ has the same property.

Applying Proposition 5.1(b) and (c) to S_0, we obtain the respective claims. $\square$

6. Bimeasurable mappings

We use here Theorem 3.2 and theorems from Section 5 to deduce generalizations of characterizations of two types of Borel bimeasurable mappings. The first concerns a combination of theorems of Luzin (see, e.g., [14, Theorem 15.1]) and Purves [16, Theorem] giving a characterization of all Borel measurable mappings between Polish spaces that map Borel sets to Borel sets. The other concerns a combination of the classical theorem of Arsenin and Kunugui and its counterpart proved in [11] characterizing those Borel measurable mappings that map closed sets to Borel sets in the classical setting.

Theorem 6.1. *Let* X *and* Y *be* $\mathcal{P}$*-Luzin spaces and* $f\colon X \to Y$ *be an index-*σ*-*$\mathcal{P}$ *mapping such that* $f^{-1}(\mathbf{B}^{\mathcal{P}}(Y)) \subset \mathbf{B}^{\mathcal{P}}(X)$.

Then $f(\mathbf{B}^{\mathcal{P}}(X)) \subset \mathbf{B}^{\mathcal{P}}(Y)$ *if and only if the set* $\{y \in Y\,;\ f^{-1}(y)$ *is uncountable*$\}$ *is* σ*-*$\mathcal{P}$.

P r o o f. Suppose that the set $\{y \in Y\,;\ f^{-1}(y)$ is uncountable$\}$ is σ-$\mathcal{P}$ and let $B \in \mathbf{B}^{\mathcal{P}}(X)$. Then $f(B)$ is $\mathcal{P}$-analytic according to Theorem 2.8(b), thus $f(B) \in \mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$. Further,

$$f(B) = \{y \in Y\,;\ f^{-1}(y) \cap B \text{ is uncountable}\} \\ \cup \{y \in Y\,;\ f^{-1}(y) \cap B \text{ is nonempty and countable}\}.$$

The first set is σ-$\mathcal{P}$, thus an element of $\mathbf{B}^{\mathcal{P}}(Y)$. The other set equals to

$$\{y\,;\ G^y \text{ is nonempty and countable}\},$$

where $G \subset B \times Y$ is the graph of f. The projection π_Y restricted to G is index-σ-$\mathcal{P}$ according to Lemma 2.6, thus we can use Theorem 5.2(b) to show that the second set is a complement of a $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ set, thus $f(B) \in \mathbf{bi}\text{-}\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$. Using Theorem 2.3, we have that $f(B) \in \mathbf{B}^{\mathcal{P}}(Y)$ and one implication is proved.

Suppose that the set $N = \{y \in Y;\ f^{-1}(y) \text{ is uncountable}\}$ is not σ-$\mathcal{P}$. According to Theorem 5.2, it is $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$, and so scattered-analytic by Theorem 2.2. Using Theorem 2.1, we find a homeomorphic copy C of the Cantor set in N. The preimage of C under f is in $\mathbf{bi}\text{-}\mathbf{S}(\mathbf{B}^{\mathcal{P}})(X)$ and thus $\mathcal{P}$-Luzin by Theorem 2.3. As f is an index-σ-$\mathcal{P}$ mapping and C is separable metric, $f^{-1}(C)$ is Luzin ($\mathcal{P}$ families in $f^{-1}(C)$ are countable). So there is a one-to-one continuous mapping $\varphi\colon F \to f^{-1}(C)$ of a Polish space F onto $f^{-1}(C)$ by definition and $(f \circ \varphi)^{-1}(y)$ is not countable for any $y \in C$. Thus we can find a Borel set B_0 in F, such that its image is not Borel in C by [11, Luzin-Purves Theorem]. Hence, putting $B = \varphi(B_0)$, we get that $f(B)$ is not $\mathcal{P}$-Luzin. $\square$

Theorem 6.2. *Let Y be a $\mathcal{P}$-Luzin space and X be a metrizable $\mathcal{P}$-Luzin space. Let $f\colon X \to Y$ be an index-σ-$\mathcal{P}$ mapping such that $f^{-1}(\mathbf{B}^{\mathcal{P}}(Y)) \subset \mathbf{B}^{\mathcal{P}}(X)$. Then $f(F) \in \mathbf{B}^{\mathcal{P}}(Y)$ for every closed $F \subset X$ if and only if the set $\{y \in Y;\ f^{-1}(y) \notin \mathbf{K}_\sigma\}$ is σ-$\mathcal{P}$.*

Proof. Let M be a complete metric space containing X. Let $G \subset M \times Y$ be the graph of f. Due to Corollary 2.4 and Theorem 2.8 (a), G is $\mathcal{P}$-Luzin. According to Lemma 2.6 (b), the projection $\pi_Y|_G$ is index-σ-$\mathcal{P}$.

Suppose that the set $\{y \in Y;\ f^{-1}(y) \notin \mathbf{K}_\sigma\}$ is σ-$\mathcal{P}$. Then each its subset is $(\mathbf{H}^{\mathcal{P}})_\sigma$. Fix a closed set $F \subset X$ and put $H = (F \times Y) \cap G$. Since $f(F) = \{y \in Y;\ \emptyset \neq H^y \in \mathbf{K}_\sigma\} \cup \{y \in Y;\ H^y \notin \mathbf{K}_\sigma\}$, the first set in the union is the complement of a set from $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ according to Theorem 5.4 (d), and the second set is a subset of $\{y \in Y;\ f^{-1}(y) \notin \mathbf{K}_\sigma\}$, it follows that the complement of $f(F)$ is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$. On the other hand, $f(F) = \{y;\ G^y \neq \emptyset\}$ is in $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ according to Theorem 5.2 (a) (with $\kappa = 1$, i.e., $\mathcal{C} = \{\emptyset\}$).

Conversely, suppose that $N = \{y \in Y;\ f^{-1}(y) \text{ is not } \mathbf{K}_\sigma\}$ is not σ-$\mathcal{P}$. According to Theorems 5.4 (b) and 2.2, N is $\mathcal{P}$-analytic. So it contains a copy C of the Cantor set by Theorem 2.1. The preimage of C under f is $\mathcal{P}$-Luzin as a $\mathbf{bi}\text{-}\mathbf{S}(\mathbf{B}^{\mathcal{P}})$ subset of the $\mathcal{P}$-Luzin space X by Theorem 2.2, and separable, since f is index-σ-$\mathcal{P}$. So it is a metrizable Luzin space and we may use [11, Main Theorem] applied to $f|_{f^{-1}(C)}$ to find a closed set F in $f^{-1}(C)$, such that $f(F)$ is not Borel in C. It follows that $f(\overline{F}^X)$ is not $\mathbf{bi}\text{-}\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ because $f(\overline{F}^X) \cap C = f(F)$ and in complete separable metric spaces (even in analytic spaces) ($\mathcal{P}$-)Borel sets are just the bi-Souslin subsets (e.g., by Theorem 2.3). $\square$

Considering just projections, we get a variant which does not follow from the preceding theorem since the set B in the statement of the following theorem need not be metrizable.

Proposition 6.3. *Let Y be a $\mathcal{P}$-Luzin space and M be a complete metric space. Let $B \subset M \times Y$ be $\mathcal{P}$-Luzin and such that the projection π_Y is index-σ-$\mathcal{P}$ when restricted to B. Then $\pi_Y(F) \in \mathbf{B}^{\mathcal{P}}(Y)$ for every closed $F \subset B$ if and only if the set $\{y \in Y;\ B^y$ is not $\mathbf{K}_\sigma\}$ is σ-$\mathcal{P}$.*

Proof. If the set $\{y \in \pi_Y(B);\ B^y$ is not $\mathbf{K}_\sigma\}$ is σ-$\mathcal{P}$, then each its subset is $(\mathbf{H}^{\mathcal{S}})_\sigma$. Since, for every closed $F \subset B$, $\pi_Y(F) = \{y \in Y;\ \emptyset \neq F^y \in \mathbf{K}_\sigma\} \cup \{y \in \pi_Y(F);\ F^y$ is not $\mathbf{K}_\sigma\}$, the first set in the union is the complement of an $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ set according to Theorem 5.4 (d), and the second set is a subset of $\{y \in \pi_Y(B);\ B^y \notin \mathbf{K}_\sigma\}$, it follows that $\pi_Y(F)$ is the complement of an $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ set. On the other hand, $\pi_Y(F) = \{y;\ F^y \neq \emptyset\}$, which is $\mathbf{S}(\mathbf{B}^{\mathcal{P}}(Y))$ according to Theorem 5.2(a) used with $\kappa = 1$.

Conversely, suppose that $\{y \in Y;\ B^y$ is not $\mathbf{K}_\sigma\}$ is not σ-$\mathcal{P}$. According to Theorem 5.4 and Theorem 2.2, it is $\mathcal{P}$-analytic. So it contains a copy C of the Cantor set by Theorem 2.1. The preimage of C by $\pi_Y|_B$ is $\mathcal{P}$-Luzin as a closed subset of the $\mathcal{P}$-Luzin set B. Since the restricted projection is index-σ-$\mathcal{P}$ and C is separable, $(M \times C) \cap B$ is metrizable and Luzin. Thus we can use [11, Main Theorem] to find a closed set $F \subset B \cap (M \times C)$ such that $\pi_Y(F)$ is not Borel. Now $\pi_Y(F)$ is not in $\textbf{bi-S}(\mathbf{B}^{\mathcal{P}}(C))$, because in separable complete metric spaces the classes of bi-Souslin sets and of Borel sets coincide. So $\pi_Y(F)$ is not in $\textbf{bi-S}(\mathbf{B}^{\mathcal{P}}(Y))$. $\square$

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