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ON HONG’S CONJECTURE FOR POWER LCM MATRICES

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Abstract. A set $S = \{x_1, \ldots, x_n\}$ of $n$ distinct positive integers is said to be gcd-closed if $(x_i, x_j) \in S$ for all $1 \leq i, j \leq n$. Shaofang Hong conjectured in 2002 that for a given positive integer $t$ there is a positive integer $k(t)$ depending only on $t$, such that if $n \leq k(t)$, then the power LCM matrix $([x_i, x_j]^t)$ defined on any gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular, but for $n > k(t) + 1$, there exists a gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ on $S$ is singular. In 1996, Hong proved $k(1) = 7$ and noted $k(t) > 7$ for all $t > 2$. This paper develops Hong’s method and provides a new idea to calculate the determinant of the LCM matrix on a gcd-closed set and proves that $k(t) \geq 8$ for all $t \geq 2$. We further prove that $k(t) \geq 9$ if a special Diophantine equation, which we call the LCM equation, has no $t$-th power solution and conjecture that $k(t) = 8$ for all $t \geq 2$, namely, the LCM equation has $t$-th power solution for all $t \geq 2$.

Keywords: gcd-closed set, greatest-type divisor (GTD), maximal gcd-fixed set (MGFS), least common multiple matrix, power LCM matrix, nonsingularity

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1. Introduction

Let $S = \{x_1, \ldots, x_n\}$ be a set of $n$ distinct positive integers. For any $x_i, x_j \in S$, we use $(x_i, x_j)$ and $[x_i, x_j]$ to denote their greatest common divisor and least common multiple respectively. If $(x_i, x_j) \in S$ for all $1 \leq i, j \leq n$, the set $S$ is said to be gcd-closed. There is a special case for gcd-closed set $S$ when it contains every divisor of $x$ for any $x \in S$, in which case we say it is factor-closed. The matrix $((x_i, x_j))$, whose $i, j$-entry is $(x_i, x_j)$, is called the greatest common divisor (GCD) matrix and denoted by $(S)_n$. Similarly, the matrix $([x_i, x_j])$, whose $i, j$-entry is $[x_i, x_j]$, is called the least common multiple (LCM) matrix and denoted by $[S]_n$.

Smith [17] obtained the formulae for the determinants of those two matrices on a factor-closed set $S = \{x_1, \ldots, x_n\}$: $\det(S)_n = \prod_{i=1}^{n} \varphi(x_i)$ where $\varphi$ is Euler’s totient
function and $\det[S]_n = \prod_{i=1}^{n} \varphi(x_i)\pi(x_i)$ where $\pi$ is the multiplicative function which is defined for the prime power $p^r$ by $\pi(p^r) = -p$. Bourque and Ligh [4] generalized Smith’s result to the LCM matrix $[S]_n$ on a gcd-closed set $S = \{x_1, \ldots, x_n\}$ by showing that

\[(1) \quad \det[S]_n = \prod_{k=1}^{n} x_k^{2\alpha_k} \text{ where } \alpha_k = \alpha_k(x_1, \ldots, x_k) = \sum_{d|x_k \atop d|\pi(x_1, \ldots, x_k)} g(d)\]

with the arithmetical function $g$ defined by $g(m) = \frac{1}{m} \sum_{d|m} d \cdot \mu(d)$ and the function $\mu$ is the Möbius function.

What interests us is the nonsingularity of those matrices. From Beslin and Ligh’s result [2], one knows that the GCD matrix $([S]_n)$ on any set $S = \{x_1, \ldots, x_n\}$ of $n$ distinct integers is always nonsingular. However, this is not true for LCM matrices in general [1]. From Smith’s result [17], one also knows that the LCM matrix on any factor-closed set is nonsingular. Further, it has been conjectured by Bourque and Ligh [4] that the LCM matrix $[S]_n$ on any gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular. In [8]–[11], Hong systematically investigated the Bourque-Ligh conjecture. In fact, Hong [8] found a simple formula of the determinant of LCM matrix on a gcd-closed set. Using this reduced formula, Hong [8] confirmed the Bourque-Ligh conjecture when $n \leq 5$ while Hong [10] showed that the Bourque-Ligh conjecture holds for a certain class of gcd-closed sets. In [9], [11], Hong introduced the concept of greatest-type divisor to reduce greatly the formula of the determinant of LCM matrices on a gcd-closed set. Based on this new reduced formula, Hong [9], [11] showed that the Bourque-Ligh conjecture is true if $n \leq 7$, but not true if $n \geq 8$. Note that Haukkanen et al. [7] also found a counterexample to the Bourque-Ligh conjecture when $n = 9$. We also remark that according to the method found in [9], [11], Hong [16] confirmed Sun’s conjecture which claims that the LCM matrix defined on any gcd-closed set such that each of this set has no more than two distinct prime factors is nonsingular. In [13]–[15], Hong further developed his method.

For any given integer $t \geq 2$ and any set $S = \{x_1, \ldots, x_n\}$ of $n$ distinct positive integers, it follows from Bourque and Ligh’s result [3] that the power GCD matrix $\left(\left[x_i, x_j\right]^t\right)$ on $S$ is nonsingular. But it is not clear that the power LCM matrix $\left(\left[x_i, x_j\right]^t\right)$ on $S$ is also nonsingular. For the factor-closed case, one knows by [5] that the answer to this question is affirmative. For the gcd-closed case, Hong [12] raised the following conjecture which can be viewed as the generalization of Hong’s solution [9], [11] to the Bourque-Ligh conjecture:
**Conjecture 1.1** [(Hong, [12]). Let $t$ be a given positive integer and $n$ any positive integer. Then there is a positive integer $k(t)$, depending only on $t$, such that if $n \leq k(t)$, then the power LCM matrix $([x_i, x_j]^t)$ defined on any gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular. But for $n \geq k(t) + 1$, there exists a gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ is singular.

By [9], [11], we know $k(1) = 7$. In [12], Hong noted that $k(t) \geq 7$ for all $t \geq 2$. We note that Chun [6] guessed that $k(t) = \infty$ for all $t \geq 1$. The current paper follows and develops Hong’s method by providing a new idea to calculate the determinant of LCM matrix on a gcd-closed set and proves that $k(t) \geq 8, t \geq 2$. We further prove that $k(t) \geq 9$ iff a special Diophantine equation, which we call the LCM equation, has no $t$-th power solution and conjecture that $k(t) = 8$ for all $t \geq 2$, namely, the LCM equation has $t$-th power solution for all $t \geq 2$. The paper is organized as follows: Section 2 introduces the notations, conceptions and lemmas used in this paper and meanwhile discusses a few special cases. Some more complicated cases are discussed in Section 3 and Section 4. The last section gives the main results of this paper.

### 2. Preparations and some special cases

Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set and $1 \leq x_1 < \ldots < x_n$. Since $(x_i, x_j)^t = (x_i^t, x_j^t)$ and $[x_i, x_j]^t = [x_i^t, x_j^t]$, we can regard the $t$-th power LCM matrix $([x_i, x_j]^t)$ on $S = \{x_1, \ldots, x_n\}$ as the LCM matrix $([x_i^t, x_j^t])$ on a gcd-closed set $S^t := \{x_1^t, \ldots, x_n^t\}$. Since the case $t = 1$ of the nonsingularity problem of the power LCM matrices has been solved by Hong [8]–[11], throughout this paper we always suppose $t \geq 2$ and any $x \in S^t$ is the $t$-th power of some positive integer. Let $|\mathcal{A}|$ denote the cardinality of a finite set $\mathcal{A}$.

**Definition 2.1** (see [9], [11]). For $a, b \in S$, we say that $a$ is a greatest-type divisor (GTD) of $b$ in $S$, if $a | b, a < b$ and it can be deduced that $c = a$ from $a | c, c | b, c < b$ and $c \in S$.

Note that the concept of greatest-type divisor played key roles in Hong’s solution [9], [11] to the Bourque-Ligh conjecture [4] and in Hong’s solution [16] to Sun’s conjecture. As in [9], [11], let $\mathcal{R}_k = \{y_1, \ldots, y_m\}$ be the set of GTDs of $x_k$ ($1 \leq k \leq n$) in $S^t$. Clearly, $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_k \neq \emptyset$ for $k \geq 2$. Suppose $(y_1, \ldots, y_m) = G$ and hence $y_i = Gy_i'$ for $1 \leq i \leq m$ where $(y_1', \ldots, y_m') = 1$. Define $\mathcal{M}^{(m)} := \bigcup_{r=2}^{m} \mathcal{M}_r^{(m)}$ where $\mathcal{M}_r^{(m)} = \{(y_{i_1}, \ldots, y_{i_r}) : 1 \leq i_1 < \ldots < i_r \leq m\}$ ($2 \leq r \leq m$). Suppose $\mathcal{M}^{(m)} = \{a_0 = G, a_1, \ldots, a_s\}$. It is easy to see that $G \mid a$ for any $a \in \mathcal{M}^{(m)}$ and
\[ s \leq 2^m - m - 2 \] since
\[ (2) \quad |M^{(m)}| \leq \binom{m}{2} + \binom{m}{3} + \ldots + \binom{m}{m} = 2^m - m - 1. \]

**Lemma 2.2.** If \( n = |S^t| \geq 2 \), we have
\[ \sum_{x \in S^t \setminus \{1\}} \frac{1}{x} < 1. \]
In particular, for \( m = |R_k| \geq 2 \), we have
\[ (3) \quad \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{G}. \]

**Proof.** Noting that any \( x \in S^t \) is the \( t \)-th power \((t \geq 2)\) of some positive integer and that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645 \), we have
\[ \sum_{x \in S^t \setminus \{1\}} \frac{1}{x} < \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 \approx 0.645 < 1. \]
Multiplying both sides of (3) by \( G \), we get
\[ (3') \quad \frac{1}{x_k/G} + \sum_{i=1}^{m} \frac{1}{y_i/G} + \sum_{j=1}^{s} \frac{1}{a_j/G} < 1. \]
It is easy to see that \( x_k/G, y_1/G, \ldots, y_m/G, a_1/G, \ldots, a_s/G \) are all \((t \geq 2)\) \( t \)-th powers of positive integers. So we only need to prove that they are distinct and none of them is equal to 1. It is equivalent to prove that \( x_k, y_1, \ldots, y_m, a_1, \ldots, a_s \) are distinct and none of them is equal to \( G \). Obviously, \( x_k > y \) for any \( y \in R_k \), and hence \( x_k > a \geq G \) for any \( a \in M^{(m)} \). We claim that \( R_k \cap M^{(m)} = \emptyset \) for \( m \geq 2 \). If not, assuming \( y \in R_k \cap M^{(m)} \), there exist \( y_i, \ldots, y_r \in R_k \) such that \((y_i, \ldots, y_r) = y \) which contradicts the fact that \( y \) is a GTD in \( R_k \). The proof is complete. \( \square \)

**Remark 2.3.** It is well known that the Riemann zeta function \( \zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t} \) converges rapidly as \( t \) grows: \( \zeta(3) \approx 1.202, \zeta(4) \approx 1.082, \ldots. \) Similarly, we can show that:
\[ \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{4G} \quad \text{for} \quad t \geq 3 \quad \text{and} \]
\[ \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{12G} \quad \text{for} \quad t \geq 4 \quad \ldots. \]
Lemma 2.4. For any distinct $y_{i_1}, \ldots, y_{i_r} \in \mathcal{R}_k$ where $r \geq 2$, we have
\[
\frac{1}{y_{i_1}} + \ldots + \frac{1}{y_{i_r}} < \frac{1}{(y_{i_1}, \ldots, y_{i_r})}.
\]
In particular, for $r = 2$ and $r = m$, we have
\[
\frac{1}{y_i} + \frac{1}{y_j} < \frac{1}{(y_i, y_j)} \quad \text{and} \quad \sum_{i=1}^m \frac{1}{y_i} < \frac{1}{G}.
\]

Proof. Let $(y_{i_1}, \ldots, y_{i_r}) = a$. Note that $y_{i_1}/a, \ldots, y_{i_r}/a$ are distinct $t$-th integer powers. For the same reason as in the above lemma, we have
\[
\frac{1}{y_{i_1}/a} + \ldots + \frac{1}{y_{i_r}/a} < 1.
\]
The desired result follows by letting $a$ divide both sides of the inequality above. □

Definition 2.5. For any finite set $T$ in $\mathbb{Z}$ and $r, a \in \mathbb{N}$, define
\[
\mathcal{L}_{T,r}(a) := \{\{z_1, \ldots, z_r\} : z_1, \ldots, z_r \in T \text{ are distinct, and } (z_1, \ldots, z_r) = a\},
\]
\[
\mathcal{G}_{T,r}(a) := \{z : \exists w \in \mathcal{L}_{T,r}(a) \text{ such that } z \in w\}, \quad \mathcal{G}_T(a) := \bigcup_{r=2}^{\lfloor |T| \rfloor} \mathcal{G}_{T,r}(a),
\]
\[
g_{T,r}(a) := |\mathcal{G}_{T,r}(a)|, \quad l_{T,r}(a) := |\mathcal{L}_{T,r}(a)|, \quad l_T(a) := \sum_{r=2}^{\lfloor |T| \rfloor} (-1)^r l_{T,r}(a).
\]

If $T = \mathcal{R}_k$, we omit the subscript $\mathcal{R}_k$ and simply denote $\mathcal{L}_{\mathcal{R}_k,r}(a)$ by $\mathcal{L}_r(a)$, $l_{\mathcal{R}_k,r}(a)$ by $l_r(a)$ and $l_{\mathcal{R}_k}(a)$ by $l(a)$, etc.

Proposition 2.6. For $\mathcal{M}^{(m)} = \{a_0 = G, a_1, \ldots, a_s\}$ and $G < a \in \mathcal{M}^{(m)}$, we have:
(a) $\sum_{j=0}^s l_r(a_j) = \binom{m}{r}$.
(b) $a \mid y$ for any $y \in \mathcal{G}(a)$.
(c) $l_r(a) \leq \binom{g_r(a)}{r}$ and $g_r(a) \leq |\mathcal{G}(a)|$.
(d) $|\mathcal{G}(a)| \leq m - 1$.

Proof. (a), (b) and (c) are trivial by definitions. To prove (d), assuming $|\mathcal{G}(a)| = m$, then by (b) we have $G < a \mid (y_1, \ldots, y_m)$ which contradicts the fact that $(y_1, \ldots, y_m) = G$. □

Now we need Hong’s formula for $\alpha_k$: 257
Lemma 2.7 ([14], Lemma 2.6). For $1 \leq k \leq n$, we have

\[ \alpha_k = \frac{1}{x_k} + \sum_{r=1}^{m} (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq m} \frac{1}{(y_{i_1}, \ldots, y_{i_r})}. \]

Using $l(a)$, $\alpha_k$ can be rewritten as follows:

Lemma 2.8.

(4) \[ \alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=0}^{s} \frac{l(a_j)}{a_j}, \quad \text{where} \quad \sum_{j=0}^{s} l(a_j) = m - 1. \]

Proof. Using $l_r(a)$ and $l(a)$, $\alpha_k$ can be expressed as

\[ \alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{r=2}^{m} (-1)^r \sum_{j=0}^{s} \frac{l_r(a_j)}{a_j} = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=0}^{s} \frac{l(a_j)}{a_j}. \]

By Proposition 2.6 (a), we have

\[ \sum_{j=0}^{s} l(a_j) = \sum_{j=0}^{s} \sum_{r=2}^{m} (-1)^r l_r(a_j) \]
\[ = \sum_{r=2}^{m} (-1)^r \sum_{j=0}^{s} l_r(a_j) \]
\[ = \sum_{r=2}^{m} (-1)^r \binom{m}{r} = m - 1. \]

The result follows.

\[ \square \]

Lemma 2.9. If $l(G) \geq 1$ and $l(a_j) \geq 0$ for all $1 \leq j \leq s$ then $\alpha_k > 0$.

Proof. This follows immediately from (4) and Lemma 2.4.

\[ \square \]

Corollary 2.10. If $|M^{(m)}| = 1$, then $\alpha_k > 0$.

Proof. $|M^{(m)}| = 1$ means $M^{(m)} = \{G\}$. By (4), $l(G) = m - 1$. Since $m \geq 2$, we have $l(G) \geq 1$. The result follows by Lemma 2.9.

\[ \square \]
Lemma 2.11. If \( l(a_j) \geq 0 \) for all \( 0 \leq j \leq s \) and \( \left| \bigcup_{l(a_j) > 0} G(a_j) \right| = m \), then \( \alpha_k > 0 \).

Proof. \( \left| \bigcup_{l(a_j) > 0} G(a_j) \right| = m \) implies that \( \bigcup_{l(a_j) > 0} G(a_j) = \mathcal{R}_k \). Thus for any \( y \in \mathcal{R}_k \) there must exist \( 1 \leq j \leq s \), \( 2 \leq r \leq m \) and \( y_{i_1}, \ldots, y_{i_{r-1}} \in G_r(a) \), such that \( l(a_j) > 0 \) and \( (y_{i_1}, \ldots, y_{i_{r-1}}, y) = a_j \). By Lemma 2.4, we have

\[
\frac{1}{y_{i_1}} + \ldots + \frac{1}{y_{i_{r-1}}} + \frac{1}{y} < \frac{1}{(y_{i_1}, \ldots, y_{i_{r-1}}, y)} = \frac{1}{a_j}.
\]

Repeat the similar step for \( y' \in \mathcal{R}_k \setminus \{y_{i_1}, \ldots, y_{i_{r-1}}, y\} \), .... Finally, we will get

\[
\sum_{i=1}^{m} \frac{1}{y_i} < \sum_{l(a_j) > 0} \frac{1}{a_j} = \alpha_k - \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} - l(G).
\]

This implies that \( \alpha_k > 0 \). This completes the proof. \( \square \)

Lemma 2.12. If \( l(G) \neq 0 \) and \( |l(a_j)| \leq G \) for all \( 1 \leq j \leq s \) then \( \alpha_k \neq 0 \).

Proof. By Lemma 2.8, we have

\[
|\alpha_k - l(G)| = \left| \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{l(a_j)}{a_j} \right| \\
\leq \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{G}{a_j} \\
\leq G \left( \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} \right) < 1.
\]

The last inequality follows from Lemma 2.2. So we have

\[
l(G) - 1 < \alpha_k < l(G) + 1,
\]

which implies \( \alpha_k > 0 \) if \( l(G) \geq 1 \) and \( \alpha_k < 0 \) if \( l(G) \leq -1 \). \( \square \)

Remark 2.13. By Remark 2.3, we can relax the condition on \( |l(a_j)| \) (\( 1 \leq j \leq s \)) in the above lemma as \( t \) grows: \( |l(a_j)| \leq 4G \) for \( t \geq 3 \), and \( |l(a_j)| \leq 12G \) for \( t \geq 4 \), etc. This will be very useful in the proof of \( \alpha_k \neq 0 \) for \( t \geq 3 \), because we can just estimate the bound on \( l(a) \) instead of calculating its exact value. This method is also effective for some special cases when \( t = 2 \) which we will see later on.
Corollary 2.14. If $|\mathcal{M}(m)| = 2^m - m - 1$, then $\alpha_k \neq 0$.

Proof. By (2), $|\mathcal{M}(m)| = 2^m - m - 1$ means $|l(a_j)| = 1$ for $1 \leq j \leq 2^m - m - 2$ and $l(G) = (-1)^m$. By the proof of Lemma 2.12, we have $\alpha_k > 0$ if $2 \mid m$ and $\alpha_k < 0$ if $2 \nmid m$. □

3. MGFS and the case of $|\mathcal{M}(m)| \leq 3$

In this section, we first introduce the concept of the so-called “MGFS”, which will play an important role in the proof of our main lemmas.

Definition 3.1. Let $G < a \in \mathcal{M}(m)$. Suppose that a set $\mathcal{F}$ in $\mathcal{G}(a)$ satisfies:

(a) For any $y_{i_1}, \ldots, y_{i_r} \in \mathcal{F}$ where $r \geq 2$, we have $(y_{i_1}, \ldots, y_{i_r}) = a$.

(b) For any $y \in \mathcal{G}(a) \setminus \mathcal{F}$, $\exists y' \in \mathcal{F}$ such that $(y, y') \neq a$.

We call $\mathcal{F}$ a maximal gcd-fixed set (MGFS) of $a$ in $\mathcal{G}(a)$, and denote it by $\mathcal{F}(a)$.

Proposition 3.2. For $G < a, b \in \mathcal{M}(m)$, we have:

(a) If $\mathcal{F}(a) \neq \emptyset$, then $2 \leq |\mathcal{F}(a)| \leq m - 1$.

(b) If $a \neq b$, then $|\mathcal{F}(a) \cup \mathcal{F}(b)| \leq m$ and $|\mathcal{F}(a) \cap \mathcal{F}(b)| \leq 1$.

(c) If $\mathcal{F}(a) = \mathcal{G}(a)$, then $l(a) = |\mathcal{F}(a)| - 1$.

Proof. (a) Suppose $\mathcal{F}(a) \neq \emptyset$. It is easy to see that $2 \leq |\mathcal{F}(a)| \leq m - 1$ follows from $\mathcal{F}(a) \subset \mathcal{G}(a)$ and $|\mathcal{G}(a)| \leq m - 1$ by Proposition 2.6 (d).

(b) Clearly, $(\mathcal{F}(a) \cup \mathcal{F}(b)) \subset (\mathcal{G}(a) \cup \mathcal{G}(b)) \subset \mathcal{R}_k$. It follows that $|\mathcal{F}(a) \cup \mathcal{F}(b)| \leq |\mathcal{R}_k| \leq m$. If $|\mathcal{F}(a) \cap \mathcal{F}(b)| \geq 2$, there exist at least two distinct $y, y' \in \mathcal{F}(a) \cap \mathcal{F}(b)$. So we get $a = (y, y') = b$. This is a contradiction.

(c) Let $r \geq 2$ and $|\mathcal{F}(a)| = n'$. By the definition of MGFS, it is clear that $\mathcal{F}(a) \subset \mathcal{G}_r(a)$. On the other hand, for any $y_{i_1}, \ldots, y_{i_r} \in \mathcal{G}_r(a)$, since $\mathcal{G}_r(a) \subset \mathcal{G}(a)$, it follows that $(y_{i_1}, \ldots, y_{i_r}) \subset \mathcal{G}(a) = \mathcal{F}(a)$. This means that $\mathcal{G}_r(a) \subset \mathcal{F}(a)$. So we get $\mathcal{G}_r(a) = \mathcal{F}(a)$. Thus $l_r(a) = \binom{n'}{r}$, and hence

\[ l(a) = \sum_{r=2}^{m} (-1)^r l_r(a) = \sum_{r=2}^{m} (-1)^r \binom{n'}{r} = n' - 1. \]

The proof is complete. □

As seen from above, $l(a)$ is easy to calculate if $\mathcal{F}(a) = \mathcal{G}(a)$. Naturally, we want to know when this condition is satisfied? The following proposition gives us an equivalent statement.
Proposition 3.3. Let \( a \in M(m) \). \( F(a) = G(a) \) iff \( a \) is a GTD of \( x_k \) in \( M(m) \).

Proof. “⇒” Assume \( a \) is not a GTD of \( x_k \) in \( M(m) \). Then there exists \( b \in M(m) \) such that \( a < b \) and \( a \mid b \). Since \( a, b \in M(m) \), we must have \( y_{i_1}, \ldots, y_{i_r} \in R_k \) such that \( (y_{i_1}, \ldots, y_{i_r}) = a \) and \( y_{j_1}, \ldots, y_{j_r} \in R_k \) such that \( (y_{j_1}, \ldots, y_{j_r}) = b \). It follows that \( (y_{i_1}, \ldots, y_{i_r}, y_{j_1}, \ldots, y_{j_r}) = (a, b) = a \). So we get \( y_{j_1}, \ldots, y_{j_r} \in G(a) = F(a) \) which implies \( (y_{j_1}, \ldots, y_{j_r}) = a \). This is a contradiction.

“⇐” Assume \( F(a) \neq G(a) \). Since \( F(a) \subset G(a) \), there must exist \( y_{i_1}, \ldots, y_{i_r} \in G(a) \) such that \( (y_{i_1}, \ldots, y_{i_r}) \neq a \). By Proposition 2.6 (b) we have \( a \mid y_{i_1}, \ldots, a \mid y_{i_r} \). It follows that \( a \mid (y_{i_1}, \ldots, y_{i_r}) \) which contradicts that \( a \) is a GTD of \( x_k \) in \( M(m) \). □

For convenience, if \( a \in M(m) \) is a GTD of \( x_k \) in \( M(m) \), we just say \( a \) is a GTD.

Corollary 3.4. Let \( M(m) = \{a_0 = G, a_1, \ldots, a_s\} \) with \( G < a_1 < \ldots < a_s \).
(a) If \( a_1, \ldots, a_s \) are all GTDs in \( M(m) \), suppose \( n_j = |F(a_j)| \), then
\[
\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{n_j - 1}{a_j} + \frac{m + s - 1 - \sum_{j=1}^{s} n_j}{G}.
\]

(b) \( l(a_s) = |G(a_s)| - 1 \).

Proof. (a) This follows immediately from Proposition 3.3, 3.2 (c) and (4).
(b) Note that since \( a_s \) is the greatest in \( M(m) \) it must be a GTD in \( M(m) \). The proof is complete. □

Remark 3.5. As seen from above, GTDs are “good” elements. Unfortunately as \( |M(m)| \) grows, the number of non-GTDs in \( M(m) \) may also increase. This makes the discussion of \( \alpha_k \) more complicated. However, it is enough for this paper to consider the cases when \( s \) is very small.

Corollary 3.6. If \( |M(m)| = 2 \), then \( \alpha_k > 0 \).

Proof. Let \( M(m) = \{G, a_1\} \). Obviously \( M(m) \) has only one GTD, i.e. \( a_1 \). Suppose \( |F(a_1)| = n_1 \), by Proposition 3.2 (a) and (c) we have that \( 2 \leq n_1 \leq m - 1 \) and \( l(a_1) = n_1 - 1 \). So by (4) and Lemma 2.9 it follows that \( \alpha_k > 0 \). □

There is a special case of the so-called divisor chain (see [10]), in which \( a_{i-1} \mid a_i \) for all \( 1 \leq i \leq s \). We can obtain the general formula for \( \alpha_k \) in this case and hence show that \( \alpha_k > 0 \). To do this, we first need:
Lemma 3.7. For $G < a' \in M^{(m)}$, define
\[
\begin{align*}
M' &:= \{a \in M^{(m)} : a' \mid a\}, \\
G' &= \bigcup_{a \in M'} G(a), \\
m' &= |G'|, \\
L' &= \{a \in M^{(m)} : a' \mid a\}, \\
L_{r}(a) &= L_{G', r}(a), \\
l_{r}(a) &= l_{G', r}(a).
\end{align*}
\]

We have: (a) $m' < m$. (b) $l(a) = l'(a)$ for any $a \in M'$.

Proof. (a) Obviously, $m' \leq m$. We claim that $m' = m$ is impossible. Otherwise $G' = R_k$. For any $a \in M'$, by Proposition 2.6 (b), we have $a \mid y$ for all $y \in G(a)$. Therefore $a' \mid y$ for all $y \in G(a)$ and hence $a' \mid y$ for all $y \in \bigcup_{a \in M'} G(a) = G' = R_k$. It follows that $G < a' \mid (y_1, \ldots, y_m)$ which contradicts the fact that $(y_1, \ldots, y_m) = G$.

(b) It is sufficient to show that $L_r(a) = L'_r(a)$ for $a \in M'$. Obviously, $L_r(a) \supset L'_r(a)$. We show that $L_r(a) \cap L'_r(a)$ is also true. Otherwise there exist $y_{i_1}, \ldots, y_{i_r} \in R_k$ where $y_{i_j} \notin G'$ $(1 \leq j \leq r)$ such that $(y_{i_1}, \ldots, y_{i_r}) = a$. So we have $y_{i_j} \notin G(a) \subset G'$. This is a contradiction.

Lemma 3.8. Suppose that $M^{(m)}$ is a divisor chain, that is, $a_{i-1} \mid a_i$ for all $1 \leq i \leq s$. If $m_i = \left| \bigcup_{j=i}^{s} G(a_j) \right|$, then we have

\[
\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \frac{m_s - 1}{a_s} + \sum_{j=0}^{s-1} \frac{m_j - m_{j+1}}{a_j} > 0.
\]

Proof. For $a_i \in M^{(m)}$ define $G^{(i)} := \bigcup_{j=i}^{s} G(a_j)$ and for $a \in M^{(m)}$ define $l^{(i)}(a) := l_{G^{(i)}}(a)$.

If $G^{(s)} = G(a_s)$, we have $l(a_s) = l^{(s)}(a_s) = m_s - 1$ by Lemma 3.7.

If $G^{(s-1)} = G(a_s) \cup G(a_{s-1})$, we have $l^{(s-1)}(a_s) + l^{(s-1)}(a_{s-1}) = m_{s-1} - 1$ by (4) and $l^{(s-1)}(a_s) = l(a_s) = m_s - 1$ by Lemma 3.7. Therefore $l(a_{s-1}) = l^{(s-1)}(a_{s-1}) = m_{s-1} - m_s$ and $m_s < m_{s-1}$ by Lemma 3.7 again.

Repeat the similar step in $G^{(s-2)}, \ldots, G^{(0)} = R_k$. Finally we get $l(a_s) = m_s - 1$, and $l(a_j) = m_j - m_{j+1}, m_{j+1} < m_j$ for $s - 1 \geq j \geq 0$. The result follows by (4) and Lemma 2.9.

Remark 3.9. Corollary 3.6 can also obtained as be a corollary of Lemma 3.8, since if $|M^{(m)}| = 2$ it is certainly a divisor chain. In fact, $M^{(m)}$ is a divisor chain satisfying in addition that all $a_j$ $(1 \leq j \leq s)$ are GTDs iff $s = 1$, i.e. $|M^{(m)}| = 2$. 262
Corollary 3.10. If $|\mathcal{M}^{(m)}| = 3$, then $\alpha_k > 0$.

Proof. Let $\mathcal{M}^{(m)} = \{G, a_1, a_2\}$ with $G < a_1 < a_2$. According as $a_1$ divides $a_2$, there are two cases to deal with:

Case 1. $a_1 \nmid a_2$. It is clear that $a_1, a_2$ are both GTDs in $\mathcal{M}^{(m)}$. Suppose $|\mathcal{F}(a_i)| = n_i$ for $i = 1, 2$, then by Proposition 3.2 (a) and (c) we have $l(a_i) = n_i - 1$ ($i = 1, 2$) and $l(G) = m + 1 - (n_1 + n_2)$. By Proposition 3.2 (b) we have

$$n_1 + n_2 = |\mathcal{F}(a_1)| + |\mathcal{F}(a_2)| = |\mathcal{F}(a_1) \cup \mathcal{F}(a_2)| + |\mathcal{F}(a_1) \cap \mathcal{F}(a_2)| \leq m + 1.$$ 

It follows that $l(G) \geq 0$ and $l(G) = 0$ iff $|\mathcal{F}(a_1) \cup \mathcal{F}(a_2)| = m$ and $|\mathcal{F}(a_1) \cap \mathcal{F}(a_2)| = 1$. If $l(G) \geq 1$ then $\alpha_k > 0$ by Lemma 2.9; if $l(G) = 0$ then $\alpha_k > 0$ by Lemma 2.11.

Case 2. $a_1 \mid a_2$. Clearly, $\mathcal{M}^{(m)}$ is a divisor chain, so by Lemma 3.8 we have $\alpha_k > 0$.

The proof is complete. \(\Box\)

To better understand the role of MGFS in $\mathcal{R}_k$, we can imagine them as a family of circles in a plane. In general, those circles may have different centers and meet each other. Corollary 3.4 and Lemma 3.8 just deal with two extreme cases: isolated circles and concentric circles.

We integrate Corollary 2.10, 3.6 and 3.10 into the following corollary:

Corollary 3.11. If $|\mathcal{M}^{(m)}| \leq 3$, then $\alpha_k > 0$.

4. THE CASE OF $|\mathcal{M}^{(4)}| = 4$ AND THE LCM EQUATION

For the case of $|\mathcal{M}^{(4)}| = 4$, there are two methods to examine whether $\alpha_k = 0$: by estimating the bound on $l(a)$, or by discussing the distribution of GTDs in $\mathcal{M}^{(4)}$.

Here we use the former method, which will yield the same result as the latter. In analysis, we naturally introduce a special Diophantine equation that we call the LCM equation. The solvability of the LCM equation is vital to deciding whether $k(t) \geq 9$.

Lemma 4.1. Let $G < a \in \mathcal{M}^{(4)}$. We have $l(a) \in \{-1, 0, 1, 2\}$, and if $l(a) = 2$ there cannot exist $G < b \in \mathcal{M}^{(4)}$ such that $b \neq a$ and $l(b) = 2$.

Proof. Since $l_4(G) = 1$ and $l_4(a) = 0$ for $G < a \in \mathcal{M}^{(4)}$, we have $l(a) = l_2(a) - l_3(a)$. First, it follows from Proposition 2.6 (c) that $l_2(a) \leq \binom{3}{2} = 3$ and $l_3(a) \leq \binom{3}{3} = 1$. Second, if $l_2(a) \geq 2$ then there must be three (four is impossible, since $g_2(a) \leq 3$ by Proposition 2.6 (c)) distinct $y_a, y_b, y_c \in \mathcal{R}_k$ such that $(y_a, y_b) = (y_a, y_c) = a$ which implies $(y_b, y_c) = a$. Thus $l(a) \leq 3 - 1 = 2$. Moreover, if $l_2(a) = 3$ we must have $(y_b, y_c) = a$. And we claim that there cannot exist another $b \in \mathcal{M}^{(4)}$.
such that \( l_3(b) = 3 \). Otherwise, we must have \((y_a,y_d) = (y_b,y_d) = (y_c,y_d) = b\). This contradicts the fact that \( g_2(b) \leq 3 \) by Proposition 2.6 (c). Hence we conclude that the possible values of \( l(a) \) are \(-1, 0, 1\) and \(2\), and there is at most one element \( G < a \in \mathcal{M}^{(4)} \) such that \( l(a) = 2 \). This is just what is desired. \( \square \)

**Lemma 4.2.** For \( \mathcal{M}^{(4)} \), if \( l(G) \neq 0 \) then \( \alpha_k \neq 0 \).

**Proof.** By Lemmas 2.8 and 4.1 and the similar analysis as in Lemma 2.2, we have

\[
G|\alpha_k - l(G)| \leq \frac{1}{x_k/G} + \sum_{i=1}^{4} \frac{1}{y_i/G} + \sum_{j=1}^{\infty} \frac{|l(a_j)|}{a_j/G} \\
\leq \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4} + \frac{1}{2} \\
= \frac{\pi^2}{6} - \frac{3}{4} \approx 0.895 < 1.
\]

By the similar discussion as in Lemma 2.12, the result follows. \( \square \)

From Lemma 4.2 above, we know that to examine whether \( \alpha_k = 0 \) for the case of \( |\mathcal{M}^{(4)}| = 4 \), we only need to consider the case of \( l(G) = 0 \). Let \( \mathcal{M}^{(4)} = \{G, a_1, a_2, a_3\} \). By (4) and Lemma 4.1, we need to solve a simple Diophantine equation: \( l(a_1) + l(a_2) + l(a_3) = 3 \) in \((-1, 0, 1, 2)\) with the constraint that there is at most one \( l(a_j) (1 \leq j \leq 3) \) equal to 2. Without loss of generality, let \( l(a_1) \geq l(a_2) \geq l(a_3) \). Easily, we get two solutions: \((l(a_1), l(a_2), l(a_3)) = (2, 1, 0)\) or \((1, 1, 1)\).

For the case of \((l(a_1), l(a_2), l(a_3)) = (2, 1, 0)\), we claim that \( |G(a_1) \cup G(a_2)| = 4 \). Since \( l(a_1) = 2 \), there must exist \( y_a, y_b, y_c \in \mathcal{R}_k \) such that \((y_a, y_b) = (y_a, y_c) = (y_b, y_c) = a_1\) by Proposition 2.6 (c). Since \( l(a_2) = 1 \), we must have \((y_c, y_d) = a_2\) where \( e \in \{a, b, c\} \).

Thus the claim is true. By Lemma 2.11, we have \( \alpha_k > 0 \). So there remains only one case to deal with, namely, \( l(a_1) = l(a_2) = l(a_3) = 1 \). Without loss of generality, let \((y_1, y_2) = a_1\). If \((y_3, y_4) = a_2\), then we again get \( |G(a_1) \cup G(a_2)| = 4 \) and hence \( \alpha_k > 0 \) by Lemma 2.11. Thus without loss of generality, suppose \((y_1, y_3) = a_2\). Consider \( G_2(a_3) \). If \( y_4 \in G_2(a_3) \), then again we get \( |G(a_1) \cup G(a_2) \cup G(a_3)| = 4 \) and hence \( \alpha_k > 0 \) by Lemma 2.11. So there remains only one case deserving our consideration: \((y_1, y_2) = a_1, (y_1, y_3) = a_2\) and \((y_2, y_3) = a_3\). Note that since \( F(a_i) = G(a_i) \) for \( 1 \leq i \leq 3 \), by Proposition 3.3 they are all GTDs in \( \mathcal{M}^{(4)} \), namely, they cannot be divided by each other. By (4) we have

\[
\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}.
\]
From (6), we see that \( y_4 \) is a “free” element that has no relation with \( a_i \). By Lemma 2.4 we have \( \alpha_k < 0 \) if \( a_i \gg y_4 \); and \( \alpha_k > 0 \) if \( a_i \ll y_4 \). Thus there may exist a set \( \{ x_k, y_1, y_2, y_3, y_4 \} \) such that \( \alpha_k = 0 \). In fact, if such a set exists we must have \( x_k = [y_1, y_2, y_3, y_4] \). Suppose \( x_k = [y_1, y_2, y_3, y_4]g \) with \( g \geq 1 \) and let \( x_k \) multiply both sides of (6), then we get that \( 1/g \) is an integer implying that \( g = 1 \). In detail, we wonder whether the following Diophantine equation

\[
0 = \frac{1}{[y_1, y_2, y_3, y_4]} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{(y_1, y_2)} + \frac{1}{(y_1, y_3)} + \frac{1}{(y_2, y_3)}
\]

is solvable with the following constraints:

(a) \( y_i \nmid y_j \) for \( 1 \leq i \neq j \leq 4 \).

(b) \( (y_1, y_4) = (y_2, y_4) = (y_3, y_4) = (y_1, y_2, y_3) = (y_1, y_2, y_3, y_4) \).

(c) Let \( a_1 = (y_1, y_2), a_2 = (y_1, y_3), a_3 = (y_2, y_3) \), then \( a_i \nmid a_j \) for \( 1 \leq i \neq j \leq 3 \).

We call such a Diophantine equation with these constraints the LCM equation. If the LCM equation has one solution in which every element is the \( t \)-th power of some positive integer, we say it has a \( t \)-th power solution. In Section 5, we will explain the relation between the solvability of the LCM equation and Conjecture 1.1.

To summarize, we have proved the following:

**Lemma 4.3.** If \( \mathcal{M}^{(4)} = \{ G, a_1, a_2, a_3 \} \), then \( \alpha_k \neq 0 \) in any of the following cases:

(a) \( \mathcal{M}^{(4)} \) has 1 GTD.

(b) \( \mathcal{M}^{(4)} \) has 2 GTDs.

(c) \( \mathcal{M}^{(4)} \) has 3 GTDs and \( \left| \bigcup_{i=1}^{3} G(a_i) \right| = 4 \).

5. Conclusions

Now we give the main results of this paper.

**Theorem 5.1.** Let \( t \geq 2 \). If \( n \leq 8 \), then the power LCM matrix \( ([x_i, x_j]^t) \) defined on any gcd-closed set \( S = \{ x_1, \ldots, x_n \} \) of \( n \) distinct positive integers is nonsingular.

**Proof.** For the same reason as in the first paragraph of Section 2, we can just consider the gcd-closed set \( S^t = \{ x_1, \ldots, x_n \} \) in which every element is the \( t \)-th power of some positive integer. Without loss of generality, we may let \( 1 \leq x_1 < x_2 < \ldots < x_n \). For \( 1 \leq k \leq n \), let \( R_k \) and \( \mathcal{M}^{(\lfloor R_k \rfloor)} \) be defined as in Section 2. We have proved in Lemma 2.2 that \( R_k \cap \mathcal{M}^{(m)} = \emptyset \). Since \( S^t \) is gcd-closed, \( m + |\mathcal{M}^{(m)}| \leq k - 1 \). Together with (2), for \( m \geq 2 \) we have

\[
1 \leq |\mathcal{M}^{(m)}| \leq \min\{k - m - 1, 2^m - m - 1\}.
\]

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We claim that $\alpha_k \neq 0$ for $1 \leq k \leq 8$. For $k = 1$, $\alpha_1 = 1/x_1 \neq 0$. In what follows let $2 \leq k \leq 9$. By (7) we have $m \leq k - 2 \leq 6$, namely, $m = 6, 5, 4, 3, 2, 1$.

If $m = 6$, then $|M(6)| \leq 1$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 5$, then $|M(5)| \leq 2$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 4$, then $|M(4)| \leq 3$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 3$, then $|M(3)| \leq 4$ by (7). If $|M(3)| = 4$, by Corollary 2.14, we have $\alpha_k \neq 0$; if $|M(3)| \leq 3$, by Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 2$, then $|M(2)| = 1$ by (7). By Corollary 2.14, we have $\alpha_k \neq 0$.

If $m = 1$, then $\alpha_k = (1/x_k) - (1/y_1) < 0$.

Thus we have $\alpha_k \neq 0$ for $1 \leq k \leq 8$. So if $n \leq 8$, by (1) we have $\det[S']_n \neq 0$.

The proof is complete. \qed

Similarly, to prove $k(t) \geq 9$ we need only to prove that $\alpha_k \neq 0$ in the cases of $|M^{(4)}| = 4$ have been proved. Suppose $M^{(4)} = \{G, a_1, a_2, a_3\}$. Lemma 4.3 tells us that there remains only one case of $|M^{(4)}| = 4$ to discuss, i.e. $a_1, a_2, a_3$ are all GTDs and $\bigcup_{i=1}^{3} G(a_i) = 3$. If there exists a set of $\{y_1, y_2, y_3, y_4\}$ such that $\alpha_k = 0$, namely, the LCM equation is solvable then $k(t) = 8$; if such a set does not exist, namely, the LCM equation is unsolvable then $k(t) \geq 9$. In brief, we have

**Theorem 5.2.** $k(t) \geq 9$ iff the LCM equation has no $t$-th power solution.

**Remark 5.3.** As $|M^{(m)}|$ grows, the “free” elements in $R_k$, which have no relations with other elements in $R_k$, will be more and more numerous, and this makes it more possible that $\alpha_k = 0$ when $l(G) = 0$. We can see this clearly by letting $l(G) = 0$ in (5).

It is easy to show that if $t = t_1 t_2$ then $k(t_1), k(t_2) \leq k(t)$. So we have:

**Corollary 5.4.** If the LCM equation has one $t$-th power solution then $k(t') = 8$ for any $t' \mid t$ and $1 < t'$.

In fact, we conjecture that for every $t \geq 2$ the LCM equation has at least one $t$-th power solution. Assume that $S' = \{x_k = x', y_1 = y_1', y_2 = y_2', y_3 = y_3', y_4 = y_4', a_1 = a_1', a_2 = a_2', a_3 = a_3', (y_1, y_2, y_3, y_4) = G'\}$ is a set of some $t$-th power solution to the
LCM equation. As in [9, 11], for any integers $n \geq 9$ and $a > 1$, let
\[
\begin{align*}
    x_i &= G' a^{(i-1)t} \text{ for } 1 \leq i \leq n-8, \\
x_{n-7} &= a_1' a^{(n-9)t}, \quad x_{n-6} = a_2' a^{(n-9)t}, \quad x_{n-5} = a_3' a^{(n-9)t}, \\
x_{n-4} &= y_1' a^{(n-9)t}, \quad x_{n-3} = y_2' a^{(n-9)t}, \\
x_{n-2} &= y_3' a^{(n-9)t}, \quad x_{n-1} = y_4' a^{(n-9)t}, \quad x_n = x' a^{(n-9)t}.
\end{align*}
\]

It is easy to check that $S = \{x_1, \ldots, x_n\}$ is a gcd-closed set and the set of GTDs of $x_n$ is just $S'$. So by (1) $\det[S]_n = 0$ since $\alpha_n = 0$. Thus we have proved that if for some $t \geq 2$ the LCM equation has one $t$-th power solution, then for any $n \geq 9$ we can find a gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ on $S$ is singular. Therefore we raise the following conjecture.

**Conjecture 5.5.** $k(t) = 8$ for all $t \geq 2$. This is equivalent to the LCM equation having at least one $t$-th power solution.

This should not be surprising since the Riemann zeta function $\zeta(t)$ has the similar character, that is, $\zeta(t)$ diverges for $t = 1$ and converges for all $t \geq 2$. From Lemma 2.2 we can also sense some relationship between $k(t)$ and $\zeta(t)$. However, to prove that the LCM equation has $t$-th power solution for every $t \geq 2$ will not be as easy as to prove that $\zeta(t)$ converges for all $t \geq 2$.

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