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ON HONG'S CONJECTURE FOR POWER LCM MATRICES

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Abstract. A set $\mathcal{S} = \{x_1, \ldots, x_n\}$ of n distinct positive integers is said to be gcd-closed if $(x_i, x_j) \in \mathcal{S}$ for all $1 \leqslant i, j \leqslant n$. Shaofang Hong conjectured in 2002 that for a given positive integer t there is a positive integer $k(t)$ depending only on t, such that if $n \leqslant k(t)$, then the power LCM matrix $([x_i, x_j]^t)$ defined on any gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ is nonsingular, but for $n \geqslant k(t) + 1$, there exists a gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ on $\mathcal{S}$ is singular. In 1996, Hong proved $k(1) = 7$ and noted $k(t) \geqslant 7$ for all $t \geqslant 2$. This paper develops Hong's method and provides a new idea to calculate the determinant of the LCM matrix on a gcd-closed set and proves that $k(t) \geqslant 8$ for all $t \geqslant 2$. We further prove that $k(t) \geqslant 9$ iff a special Diophantine equation, which we call the LCM equation, has no t-th power solution and conjecture that $k(t) = 8$ for all $t \geqslant 2$, namely, the LCM equation has t-th power solution for all $t \geqslant 2$.

Keywords: gcd-closed set, greatest-type divisor(GTD), maximal gcd-fixed set(MGFS), least common multiple matrix, power LCM matrix, nonsingularity

MSC 2000: 11C20, 11A25

1. Introduction

Let $\mathcal{S} = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. For any $x_i, x_j \in \mathcal{S}$, we use (x_i, x_j) and $[x_i, x_j]$ to denote their greatest common divisor and least common multiple respectively. If $(x_i, x_j) \in \mathcal{S}$ for all $1 \leqslant i, j \leqslant n$, the set $\mathcal{S}$ is said to be *gcd-closed*. There is a special case for gcd-closed set $\mathcal{S}$ when it contains every divisor of x for any $x \in \mathcal{S}$, in which case we say it is *factor-closed*. The matrix $((x_i, x_j))$, whose i, j-entry is (x_i, x_j), is called the *greatest common divisor* (GCD) *matrix* and denoted by $(\mathcal{S})_n$. Similarly, the matrix $([x_i, x_j])$, whose i, j-entry is $[x_i, x_j]$, is called the *least common multiple* (LCM) *matrix* and denoted by $[\mathcal{S}]_n$.

Smith [17] obtained the formulae for the determinants of those two matrices on a factor-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$: $\det(\mathcal{S})_n = \prod_{i=1}^{n} \varphi(x_i)$ where φ is Euler's totient function and $\det[\mathcal{S}]_n = \prod_{i=1}^{n} \varphi(x_i)\pi(x_i)$ where π is the multiplicative function which is defined for the prime power p^r by $\pi(p^r) = -p$. Bourque and Ligh [4] generalized Smith's result to the LCM matrix $[\mathcal{S}]_n$ on a gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ by showing that

$$(1) \qquad \det[\mathcal{S}]_n = \prod_{k=1}^{n} x_k^2 \alpha_k \quad \text{where} \quad \alpha_k = \alpha_k(x_1, \ldots, x_k) = \sum_{\substack{d \mid x_k \\ d \nmid x_t, x_t < x_k}} g(d)$$

with the arithmetical function g defined by $g(m) = \frac{1}{m} \sum_{d \mid m} d \cdot \mu(d)$ and the function μ is the Möbius function.

What interests us is the nonsingularity of those matrices. From Beslin and Ligh's result [2], one knows that the GCD matrix $(\mathcal{S})_n$ on any set $\mathcal{S} = \{x_1, \ldots, x_n\}$ of n distinct integers is always nonsingular. However, this is not true for LCM matrices in general [1]. From Smith's result [17], one also knows that the LCM matrix on any factor-closed set is nonsingular. Further, it has been conjectured by Bourque and Ligh [4] that the LCM matrix $[\mathcal{S}]_n$ on any gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ is nonsingular. In [8]–[11], Hong systematically investigated the Bourque-Ligh conjecture. In fact, Hong [8] found a simple formula of the determinant of LCM matrix on a gcd-closed set. Using this reduced formula, Hong [8] confirmed the Bourque-Ligh conjecture when $n \leqslant 5$ while Hong [10] showed that the Bourque-Ligh conjecture holds for a certain class of gcd-closed sets. In [9], [11], Hong introduced the concept of greatest-type divisor to reduce greatly the formula of the determinant of LCM matrices on a gcd-closed set. Based on this new reduced formula, Hong [9], [11] showed that the Bourque-Ligh conjecture is true if $n \leqslant 7$, but not true if $n \geqslant 8$. Note that Haukkanen et al. [7] also found a counterexample to the Bourque-Ligh conjecture when $n = 9$. We also remark that according to the method found in [9], [11], Hong [16] confirmed Sun's conjecture which claims that the LCM matrix defined on any gcd-closed set such that each of this set has no more than two distinct prime factors is nonsingular. In [13]–[15], Hong further developed his method.

For any given integer $t \geqslant 2$ and any set $\mathcal{S} = \{x_1, \ldots, x_n\}$ of n distinct positive integers, it follows from Bourque and Ligh's result [3] that the power GCD matrix $((x_i, x_j)^t)$ on $\mathcal{S}$ is nonsingular. But it is not clear that the power LCM matrix $([x_i, x_j]^t)$ on $\mathcal{S}$ is also nonsingular. For the factor-closed case, one knows by [5] that the answer to this question is affirmative. For the gcd-closed case, Hong [12] raised the following conjecture which can be viewed as the generalization of Hong's solution [9], [11] to the Bourque-Ligh conjecture:

Conjecture 1.1 [(Hong, [12]). *Let t be a given positive integer and n any positive integer. Then there is a positive integer $k(t)$, depending only on t, such that if $n \leqslant k(t)$, then the power LCM matrix $([x_i, x_j]^t)$ defined on any gcd-closed set $\mathcal{S} = \{x_1, \dots, x_n\}$ is nonsingular. But for $n \geqslant k(t)+1$, there exists a gcd-closed set $\mathcal{S} = \{x_1, \dots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ is singular.*

By [9], [11], we know $k(1) = 7$. In [12], Hong noted that $k(t) \geqslant 7$ for all $t \geqslant 2$. We note that Chun [6] guessed that $k(t) = \infty$ for all $t \geqslant 1$. The current paper follows and develops Hong's method by providing a new idea to calculate the determinant of LCM matrix on a gcd-closed set and proves that $k(t) \geqslant 8, t \geqslant 2$. We further prove that $k(t) \geqslant 9$ iff a special Diophantine equation, which we call the LCM equation, has no t-th power solution and conjecture that $k(t) = 8$ for all $t \geqslant 2$, namely, the LCM equation has t-th power solution for all $t \geqslant 2$. The paper is organized as follows: Section 2 introduces the notations, conceptions and lemmas used in this paper and meanwhile discusses a few special cases. Some more complicated cases are discussed in Section 3 and Section 4. The last section gives the main results of this paper.

2. Preparations and some special cases

Let $\mathcal{S} = \{x_1, \dots, x_n\}$ be a gcd-closed set and $1 \leqslant x_1 < \dots < x_n$. Since $(x_i, x_j)^t = (x_i^t, x_j^t)$ and $[x_i, x_j]^t = [x_i^t, x_j^t]$, we can regard the t-th power LCM matrix $([x_i, x_j]^t)$ on $\mathcal{S} = \{x_1, \dots, x_n\}$ as the LCM matrix $([x_i^t, x_j^t])$ on a gcd-closed set $\mathcal{S}^t := \{x_1^t, \dots, x_n^t\}$. Since the case $t = 1$ of the nonsingularity problem of the power LCM matrices has been solved by Hong [8]–[11], throughout this paper we always suppose $t \geqslant 2$ and any $x \in \mathcal{S}^t$ is the t-th power of some positive integer. Let $|\mathcal{A}|$ denote the cardinality of a finite set $\mathcal{A}$.

Definition 2.1 (see [9], [11]). For $a, b \in \mathcal{S}$, we say that a is a *greatest-type divisor* (GTD) *of b in $\mathcal{S}$*, if $a|b, a < b$ and it can be deduced that $c = a$ from $a|c, c|b, c < b$ and $c \in \mathcal{S}$.

Note that the concept of greatest-type divisor played key roles in Hong's solution [9], [11] to the Bourque-Ligh conjecture [4] and in Hong's solution [16] to Sun's conjecture. As in [9], [11], let $\mathcal{R}_k = \{y_1, \dots, y_m\}$ be the set of GTDs of x_k $(1 \leqslant k \leqslant n)$ in $\mathcal{S}^t$. Clearly, $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_k \neq \emptyset$ for $k \geqslant 2$. Suppose $(y_1, \dots, y_m) = G$ and hence $y_i = G y_i'$ for $1 \leqslant i \leqslant m$ where $(y_1', \dots, y_m') = 1$. Define $\mathcal{M}^{(m)} := \bigcup\limits_{r=2}^{m} \mathcal{M}_r^{(m)}$ where $\mathcal{M}_r^{(m)} = \{(y_{i_1}, \dots, y_{i_r})\colon 1 \leqslant i_1 < \dots < i_r \leqslant m\}$ $(2 \leqslant r \leqslant m)$. Suppose $\mathcal{M}^{(m)} = \{a_0 = G, a_1, \dots, a_s\}$. It is easy to see that $G \mid a$ for any $a \in \mathcal{M}^{(m)}$ and

$s \leqslant 2^m - m - 2$ since

$$(2) \qquad |\mathcal{M}^{(m)}| \leqslant \binom{m}{2} + \binom{m}{3} + \ldots + \binom{m}{m} = 2^m - m - 1.$$

Lemma 2.2. *If $n = |\mathcal{S}^t| \geqslant 2$, we have*

$$\sum_{x \in \mathcal{S}^t \setminus \{1\}} \frac{1}{x} < 1.$$

In particular, for $m = |\mathcal{R}_k| \geqslant 2$, we have

$$(3) \qquad \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{G}.$$

Proof. Noting that any $x \in \mathcal{S}^t$ is the t-th power ($t \geqslant 2$) of some positive integer and that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645$, we have

$$\sum_{x \in S^t \setminus \{1\}} \frac{1}{x} < \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 \approx 0.645 < 1.$$

Multiplying both sides of (3) by G, we get

$$(3') \qquad \frac{1}{x_k/G} + \sum_{i=1}^{m} \frac{1}{y_i/G} + \sum_{j=1}^{s} \frac{1}{a_j/G} < 1.$$

It is easy to see that $x_k/G, y_1/G, \ldots, y_m/G, a_1/G, \ldots, a_s/G$ are all ($t \geqslant 2$) t-th powers of positive integers. So we only need to prove that they are distinct and none of them is equal to 1. It is equivalent to prove that $x_k, y_1, \ldots, y_m, a_1, \ldots, a_s$ are distinct and none of them is equal to G. Obviously, $x_k > y$ for any $y \in \mathcal{R}_k$, and hence $x_k > a \geqslant G$ for any $a \in \mathcal{M}^{(m)}$. We claim that $\mathcal{R}_k \cap \mathcal{M}^{(m)} = \emptyset$ for $m \geqslant 2$. If not, assuming $y \in \mathcal{R}_k \cap \mathcal{M}^{(m)}$, there exist $y_{i_1}, \ldots, y_{i_r} \in \mathcal{R}_k$ such that $(y_{i_1}, \ldots, y_{i_r}) = y$ which contradicts the fact that y is a GTD in $\mathcal{R}_k$. The proof is complete. □

Remark 2.3. It is well known that the Riemann zeta function $\zeta(t) = \sum\limits_{n=1}^{\infty} \frac{1}{n^t}$ converges rapidly as t grows: $\zeta(3) \approx 1.202$, $\zeta(4) \approx 1.082, \ldots$. Similarly, we can show that:

$$\frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{4G} \qquad \text{for} \quad t \geqslant 3 \quad \text{and}$$

$$\frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} < \frac{1}{12G} \qquad \text{for} \quad t \geqslant 4 \quad \ldots.$$

Lemma 2.4. *For any distinct $y_{i_1}, \ldots, y_{i_r} \in \mathcal{R}_k$ where $r \geqslant 2$, we have*

$$\frac{1}{y_{i_1}} + \ldots + \frac{1}{y_{i_r}} < \frac{1}{(y_{i_1}, \ldots, y_{i_r})}.$$

In particular, for $r = 2$ and $r = m$, we have

$$\frac{1}{y_i} + \frac{1}{y_j} < \frac{1}{(y_i, y_j)} \qquad \text{and} \qquad \sum_{i=1}^{m} \frac{1}{y_i} < \frac{1}{G}.$$

Proof. Let $(y_{i_1}, \ldots, y_{i_r}) = a$. Note that $y_{i_1}/a, \ldots, y_{i_r}/a$ are distinct t-th integer powers. For the same reason as in the above lemma, we have

$$\frac{1}{y_{i_1}/a} + \ldots + \frac{1}{y_{i_r}/a} < 1.$$

The desired result follows by letting a divide both sides of the inequality above. □

Definition 2.5. For any finite set $\mathcal{T}$ in $\mathbb{Z}$ and $r, a \in \mathbb{N}$, define

$$\mathcal{L}_{\mathcal{T},r}(a) := \{\{z_1, \ldots, z_r\} \colon\ z_1, \ldots, z_r \in \mathcal{T} \text{ are distinct, and } (z_1, \ldots, z_r) = a\},$$

$$\mathcal{G}_{\mathcal{T},r}(a) := \{z \colon\ \exists w \in \mathcal{L}_{\mathcal{T},r}(a) \text{ such that } z \in w\}, \quad \mathcal{G}_{\mathcal{T}}(a) := \bigcup_{r=2}^{|\mathcal{T}|} \mathcal{G}_{\mathcal{T},r}(a),$$

$$g_{\mathcal{T},r}(a) := |\mathcal{G}_{\mathcal{T},r}(a)|,\ l_{\mathcal{T},r}(a) := |\mathcal{L}_{\mathcal{T},r}(a)|, \quad l_{\mathcal{T}}(a) := \sum_{r=2}^{|\mathcal{T}|} (-1)^r l_{\mathcal{T},r}(a).$$

If $\mathcal{T} = \mathcal{R}_k$, we omit the subscript "$_{\mathcal{R}_k}$" and simply denote $\mathcal{L}_{\mathcal{R}_k,r}(a)$ by $\mathcal{L}_r(a)$, $l_{\mathcal{R}_k,r}(a)$ by $l_r(a)$ and $l_{\mathcal{R}_k}(a)$ by $l(a)$, etc.

Proposition 2.6. *For $\mathcal{M}^{(m)} = \{a_0 = G, a_1, \ldots, a_s\}$ and $G < a \in \mathcal{M}^{(m)}$, we have:*
(a) $\sum\limits_{j=0}^{s} l_r(a_j) = \binom{m}{r}$.
(b) *$a \mid y$ for any $y \in \mathcal{G}(a)$.*
(c) *$l_r(a) \leqslant \binom{g_r(a)}{r}$ and $g_r(a) \leqslant |\mathcal{G}(a)|$.*
(d) *$|\mathcal{G}(a)| \leqslant m - 1$.*

Proof. (a), (b) and (c) are trivial by definitions. To prove (d), assuming $|\mathcal{G}(a)| = m$, then by (b) we have $G < a \mid (y_1, \ldots, y_m)$ which contradicts the fact that $(y_1, \ldots, y_m) = G$. □

Now we need Hong's formula for α_k:

Lemma 2.7 ([14], Lemma 2.6). *For $1 \leqslant k \leqslant n$, we have*

$$\alpha_k = \frac{1}{x_k} + \sum_{r=1}^{m}(-1)^r \sum_{1\leqslant i_1<\ldots<i_r\leqslant m} \frac{1}{(y_{i_1},\ldots,y_{i_r})}.$$

Using $l(a)$, α_k can be rewritten as follows:

Lemma 2.8.

$$\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m}\frac{1}{y_i} + \sum_{j=0}^{s}\frac{l(a_j)}{a_j}, \qquad \text{where } \sum_{j=0}^{s} l(a_j) = m-1. \tag{4}$$

P r o o f. Using $l_r(a)$ and $l(a)$, α_k can be expressed as

$$\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m}\frac{1}{y_i} + \sum_{r=2}^{m}(-1)^r\sum_{j=0}^{s}\frac{l_r(a_j)}{a_j} = \frac{1}{x_k} - \sum_{i=1}^{m}\frac{1}{y_i} + \sum_{j=0}^{s}\frac{l(a_j)}{a_j}.$$

By Proposition 2.6 (a), we have

$$\begin{aligned}\sum_{j=0}^{s} l(a_j) &= \sum_{j=0}^{s}\sum_{r=2}^{m}(-1)^r l_r(a_j)\\ &= \sum_{r=2}^{m}(-1)^r\sum_{j=0}^{s} l_r(a_j)\\ &= \sum_{r=2}^{m}(-1)^r\binom{m}{r} = m-1.\end{aligned}$$

The result follows. □

Lemma 2.9. *If $l(G) \geqslant 1$ and $l(a_j) \geqslant 0$ for all $1 \leqslant j \leqslant s$ then $\alpha_k > 0$.*

P r o o f. This follows immediately from (4) and Lemma 2.4. □

Corollary 2.10. *If $|\mathcal{M}^{(m)}| = 1$, then $\alpha_k > 0$.*

P r o o f. $|\mathcal{M}^{(m)}| = 1$ means $\mathcal{M}^{(m)} = \{G\}$. By (4), $l(G) = m-1$. Since $m \geqslant 2$, we have $l(G) \geqslant 1$. The result follows by Lemma 2.9. □

Lemma 2.11. *If $l(a_j) \geqslant 0$ for all $0 \leqslant j \leqslant s$ and $\Big|\bigcup_{l(a_j)>0} \mathcal{G}(a_j)\Big| = m$, then $\alpha_k > 0$.*

Proof. $\Big|\bigcup_{l(a_j)>0} \mathcal{G}(a_j)\Big| = m$ implies that $\bigcup_{l(a_j)>0} \mathcal{G}(a_j) = \mathcal{R}_k$. Thus for any $y \in \mathcal{R}_k$ there must exist $1 \leqslant j \leqslant s$, $2 \leqslant r \leqslant m$ and $y_{i_1}, \ldots, y_{i_{r-1}} \in \mathcal{G}_r(a)$, such that $l(a_j) > 0$ and $(y_{i_1}, \ldots, y_{i_{r-1}}, y) = a_j$. By Lemma 2.4, we have

$$\frac{1}{y_{i_1}} + \ldots + \frac{1}{y_{i_{r-1}}} + \frac{1}{y} < \frac{1}{(y_{i_1}, \ldots, y_{i_{r-1}}, y)} = \frac{1}{a_j}.$$

Repeat the similar step for $y' \in \mathcal{R}_k \setminus \{y_{i_1}, \ldots, y_{i_{r-1}}, y\}$, ... Finally, we will get

$$\sum_{i=1}^{m} \frac{1}{y_i} < \sum_{l(a_j)>0} \frac{1}{a_j} = \alpha_k - \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} - l(G).$$

This implies that $\alpha_k > 0$. This completes the proof. □

Lemma 2.12. *If $l(G) \neq 0$ and $|l(a_j)| \leqslant G$ for all $1 \leqslant j \leqslant s$ then $\alpha_k \neq 0$.*

Proof. By Lemma 2.8, we have

$$\begin{aligned} |\alpha_k - l(G)| &= \left| \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{l(a_j)}{a_j} \right| \\ &\leqslant \frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{G}{a_j} \\ &\leqslant G\left(\frac{1}{x_k} + \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{1}{a_j} \right) < 1. \end{aligned}$$

The last inequality follows from Lemma 2.2. So we have

$$l(G) - 1 < \alpha_k < l(G) + 1,$$

which implies $\alpha_k > 0$ if $l(G) \geqslant 1$ and $\alpha_k < 0$ if $l(G) \leqslant -1$. □

Remark 2.13. By Remark 2.3, we can relax the condition on $|l(a_j)|$ $(1 \leqslant j \leqslant s)$ in the above lemma as t grows: $|l(a_j)| \leqslant 4G$ for $t \geqslant 3$, and $|l(a_j)| \leqslant 12G$ for $t \geqslant 4$, etc. This will be very useful in the proof of $\alpha_k \neq 0$ for $t \geqslant 3$, because we can just estimate the bound on $l(a)$ instead of calculating its exact value. This method is also effective for some special cases when $t = 2$ which we will see later on.

Corollary 2.14. *If* $|\mathcal{M}^{(m)}| = 2^m - m - 1$, *then* $\alpha_k \neq 0$.

Proof. By (2), $|\mathcal{M}^{(m)}| = 2^m - m - 1$ means $|l(a_j)| = 1$ for $1 \leqslant j \leqslant 2^m - m - 2$ and $l(G) = (-1)^m$. By the proof of Lemma 2.12, we have $\alpha_k > 0$ if $2 \mid m$ and $\alpha_k < 0$ if $2 \nmid m$. $\square$

3. MGFS and the case of $|\mathcal{M}^{(m)}| \leqslant 3$

In this section, we first introduce the concept of the so-called "MGFS", which will play an important role in the proof of our main lemmas.

Definition 3.1. Let $G < a \in \mathcal{M}^{(m)}$. Suppose that a set $\mathcal{F}$ in $\mathcal{G}(a)$ satisfies:

(a) For any $y_{i_1}, \ldots, y_{i_r} \in \mathcal{F}$ where $r \geqslant 2$, we have $(y_{i_1}, \ldots, y_{i_r}) = a$.

(b) For any $y \in \mathcal{G}(a) \setminus \mathcal{F}$, $\exists y' \in \mathcal{F}$ such that $(y, y') \neq a$.

We call $\mathcal{F}$ a *maximal gcd-fixed set* (MGFS) *of* a *in* $\mathcal{G}(a)$, and denote it by $\mathcal{F}(a)$.

Proposition 3.2. *For* $G < a, b \in \mathcal{M}^{(m)}$, *we have:*

(a) *If* $\mathcal{F}(a) \neq \emptyset$, *then* $2 \leqslant |\mathcal{F}(a)| \leqslant m - 1$.

(b) *If* $a \neq b$, *then* $|\mathcal{F}(a) \cup \mathcal{F}(b)| \leqslant m$ *and* $|\mathcal{F}(a) \cap \mathcal{F}(b)| \leqslant 1$.

(c) *If* $\mathcal{F}(a) = \mathcal{G}(a)$, *then* $l(a) = |\mathcal{F}(a)| - 1$.

Proof. (a) Suppose $\mathcal{F}(a) \neq \emptyset$. It is easy to see that $2 \leqslant |\mathcal{F}(a)|$. $|\mathcal{F}(a)| \leqslant m - 1$ follows from $\mathcal{F}(a) \subset \mathcal{G}(a)$ and $|\mathcal{G}(a)| \leqslant m - 1$ by Proposition 2.6 (d).

(b) Clearly, $(\mathcal{F}(a) \cup \mathcal{F}(b)) \subset (\mathcal{G}(a) \cup \mathcal{G}(b)) \subset \mathcal{R}_k$. It follows that $|\mathcal{F}(a) \cup \mathcal{F}(b)| \leqslant |\mathcal{R}_k| \leqslant m$. If $|\mathcal{F}(a) \cap \mathcal{F}(b)| \geqslant 2$, there exist at least two distinct $y, y' \in \mathcal{F}(a) \cap \mathcal{F}(b)$. So we get $a = (y, y') = b$. This is a contradiction.

(c) Let $r \geqslant 2$ and $|\mathcal{F}(a)| = n'$. By the definition of MGFS, it is clear that $\mathcal{F}(a) \subset \mathcal{G}_r(a)$. On the other hand, for any $y_{i_1}, \ldots, y_{i_r} \in \mathcal{G}_r(a)$, since $\mathcal{G}_r(a) \subset \mathcal{G}(a)$, it follows that $\{y_{i_1}, \ldots, y_{i_r}\} \subset \mathcal{G}(a) = \mathcal{F}(a)$. This means that $\mathcal{G}_r(a) \subset \mathcal{F}(a)$. So we get $\mathcal{G}_r(a) = \mathcal{F}(a)$. Thus $l_r(a) = \binom{g_r(a)}{r} = \binom{n'}{r}$, and hence

$$l(a) = \sum_{r=2}^{m} (-1)^r l_r(a) = \sum_{r=2}^{m} (-1)^r \binom{n'}{r} = n' - 1.$$

The proof is complete. $\square$

As seen from above, $l(a)$ is easy to calculate if $\mathcal{F}(a) = \mathcal{G}(a)$. Naturally, we want to know when this condition is satisfied? The following proposition gives us an equivalent statement.

Proposition 3.3. *Let $a \in \mathcal{M}^{(m)}$. $\mathcal{F}(a) = \mathcal{G}(a)$ iff a is a GTD of x_k in $\mathcal{M}^{(m)}$.*

Proof. "$\Rightarrow$" Assume a is not a GTD of x_k in $\mathcal{M}^{(m)}$. Then there exists $b \in \mathcal{M}^{(m)}$ such that $a < b$ and $a \mid b$. Since $a, b \in \mathcal{M}^{(m)}$, we must have $y_{i_1}, \ldots, y_{i_r} \in \mathcal{R}_k$ such that $(y_{i_1}, \ldots, y_{i_r}) = a$ and $y_{j_1}, \ldots, y_{j_{r'}} \in \mathcal{R}_k$ such that $(y_{j_1}, \ldots, y_{j_{r'}}) = b$. It follows that $(y_{i_1}, \ldots, y_{i_r}, y_{j_1}, \ldots, y_{j_{r'}}) = (a, b) = a$. So we get $y_{j_1}, \ldots, y_{j_{r'}} \in \mathcal{G}(a) = \mathcal{F}(a)$ which implies $(y_{j_1}, \ldots, y_{j_{r'}}) = a$. This is a contradiction.

"$\Leftarrow$" Assume $\mathcal{F}(a) \neq \mathcal{G}(a)$. Since $\mathcal{F}(a) \subset \mathcal{G}(a)$, there must exist $y_{i_1}, \ldots, y_{i_r} \in \mathcal{G}(a)$ such that $(y_{i_1}, \ldots, y_{i_r}) \neq a$. By Proposition 2.6 (b) we have $a \mid y_{i_1}, \ldots, a \mid y_{i_r}$. It follows that $a \mid (y_{i_1}, \ldots, y_{i_r})$ which contradicts that a is a GTD of x_k in $\mathcal{M}^{(m)}$. □

For convenience, if $a \in \mathcal{M}^{(m)}$ is a GTD of x_k in $\mathcal{M}^{(m)}$, we just say a is a GTD.

Corollary 3.4. *Let $\mathcal{M}^{(m)} = \{a_0 = G, a_1, \ldots, a_s\}$ with $G < a_1 < \ldots < a_s$.*

(a) *If $a_1, \ldots, a_s$ are all GTDs in $\mathcal{M}^{(m)}$, suppose $n_j = |\mathcal{F}(a_j)|$, then*

$$\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{j=1}^{s} \frac{n_j - 1}{a_j} + \frac{m + s - 1 - \sum_{j=1}^{s} n_j}{G}. \tag{5}$$

(b) $l(a_s) = |\mathcal{G}(a_s)| - 1$.

Proof. (a) This follows immediately from Proposition 3.3, 3.2 (c) and (4).

(b) Note that since a_s is the greatest in $\mathcal{M}^{(m)}$ it must be a GTD in $\mathcal{M}^{(m)}$.

The proof is complete. □

Remark 3.5. As seen from above, GTDs are "good" elements. Unfortunately as $|\mathcal{M}^{(m)}|$ grows, the number of non-GTDs in $\mathcal{M}^{(m)}$ may also increase. This makes the discussion of α_k more complicated. However, it is enough for this paper to consider the cases when s is very small.

Corollary 3.6. *If $|\mathcal{M}^{(m)}| = 2$, then $\alpha_k > 0$.*

Proof. Let $\mathcal{M}^{(m)} = \{G, a_1\}$. Obviously $\mathcal{M}^{(m)}$ has only one GTD, i.e. a_1. Suppose $|\mathcal{F}(a_1)| = n_1$, by Proposition 3.2 (a) and (c) we have that $2 \leqslant n_1 \leqslant m - 1$ and $l(a_1) = n_1 - 1$. So by (4) and Lemma 2.9 it follows that $\alpha_k > 0$. □

There is a special case of the so-called *divisor chain* (see [10]), in which $a_{i-1} \mid a_i$ for all $1 \leqslant i \leqslant s$. We can obtain the general formula for α_k in this case and hence show that $\alpha_k > 0$. To do this, we first need:

Lemma 3.7. *For $G < a' \in \mathcal{M}^{(m)}$, define*

$$\mathcal{M}' := \{a \in \mathcal{M}^{(m)} \colon\ a' \mid a\}, \quad \mathcal{G}' := \bigcup_{a\in\mathcal{M}'} \mathcal{G}(a), \quad m' := |\mathcal{G}'|,$$
$$\mathcal{L}'_r(a) := \mathcal{L}_{\mathcal{G}',r}(a), \quad l'_r(a) := l_{\mathcal{G}',r}(a), \quad l'(a) := l_{\mathcal{G}'}(a).$$

We have: (a) $m' < m$. (b) $l(a) = l'(a)$ *for any* $a \in \mathcal{M}'$.

P r o o f. (a) Obviously, $m' \leqslant m$. We claim that $m' = m$ is impossible. Otherwise $\mathcal{G}' = \mathcal{R}_k$. For any $a \in \mathcal{M}'$, by Proposition 2.6 (b), we have $a \mid y$ for all $y \in \mathcal{G}(a)$. Therefore $a' \mid y$ for all $y \in \mathcal{G}(a)$ and hence $a' \mid y$ for all $y \in \bigcup\limits_{a\in\mathcal{M}'} \mathcal{G}(a) = \mathcal{G}' = \mathcal{R}_k$. It follows that $G < a' \mid (y_1, \ldots, y_m)$ which contradicts the fact that $(y_1, \ldots, y_m) = G$.

(b) It is sufficient to show that $\mathcal{L}_r(a) = \mathcal{L}'_r(a)$ for $a \in \mathcal{M}'$. Obviously, $\mathcal{L}_r(a) \supset \mathcal{L}'_r(a)$. We show that $\mathcal{L}_r(a) \subset \mathcal{L}'_r(a)$ is also true. Otherwise there exist $y_{i_1}, \ldots, y_{i_r} \in \mathcal{R}_k$ where $y_{i_j} \notin \mathcal{G}'$ $(1 \leqslant j \leqslant r)$ such that $(y_{i_1}, \ldots, y_{i_r}) = a$. So we have $y_{i_j} \in \mathcal{G}(a) \subset \mathcal{G}'$. This is a contradiction. $\square$

Lemma 3.8. *Suppose that $\mathcal{M}^{(m)}$ is a divisor chain, that is, $a_{i-1} \mid a_i$ for all $1 \leqslant i \leqslant s$. If $m_i = \Bigl|\bigcup\limits_{j=i}^{s} \mathcal{G}(a_j)\Bigr|$, then we have*

$$\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \frac{m_s - 1}{a_s} + \sum_{j=0}^{s-1} \frac{m_j - m_{j+1}}{a_j} > 0.$$

P r o o f. For $a_i \in \mathcal{M}^{(m)}$ define $\mathcal{G}^{(i)} := \bigcup\limits_{j=i}^{s} \mathcal{G}(a_j)$ and for $a \in \mathcal{M}^{(m)}$ define $l^{(i)}(a) := l_{\mathcal{G}^{(i)}}(a)$.

If $\mathcal{G}^{(s)} = \mathcal{G}(a_s)$, we have $l(a_s) = l^{(s)}(a_s) = m_s - 1$ by Lemma 3.7.

If $\mathcal{G}^{(s-1)} = \mathcal{G}(a_s) \cup \mathcal{G}(a_{s-1})$, we have $l^{(s-1)}(a_s) + l^{(s-1)}(a_{s-1}) = m_{s-1} - 1$ by (4) and $l^{(s-1)}(a_s) = l(a_s) = m_s - 1$ by Lemma 3.7. Therefore $l(a_{s-1}) = l^{(s-1)}(a_{s-1}) = m_{s-1} - m_s$ and $m_s < m_{s-1}$ by Lemma 3.7 again.

Repeat the similar step in $\mathcal{G}^{(s-2)}, \ldots, \mathcal{G}^{(0)} = \mathcal{R}_k$. Finally we get $l(a_s) = m_s - 1$, and $l(a_j) = m_j - m_{j+1}$, $m_{j+1} < m_j$ for $s - 1 \geqslant j \geqslant 0$. The result follows by (4) and Lemma 2.9. $\square$

Remark 3.9. Corollary 3.6 can also obtained as be a corollary of Lemma 3.8, since if $|\mathcal{M}^{(m)}| = 2$ it is certainly a divisor chain. In fact, $\mathcal{M}^{(m)}$ is a divisor chain satisfying in addition that all a_j $(1 \leqslant j \leqslant s)$ are GTDs iff $s = 1$, i.e. $|\mathcal{M}^{(m)}| = 2$.

Corollary 3.10. *If* $|\mathcal{M}^{(m)}| = 3$, *then* $\alpha_k > 0$.

Proof. Let $\mathcal{M}^{(m)} = \{G, a_1, a_2\}$ with $G < a_1 < a_2$. According as a_1 divides a_2, there are two cases to deal with:

Case 1. $a_1 \nmid a_2$. It is clear that a_1, a_2 are both GTDs in $\mathcal{M}^{(m)}$. Suppose $|\mathcal{F}(a_i)| = n_i$ for $i = 1, 2$, then by Proposition 3.2 (a) and (c) we have $l(a_i) = n_i - 1$ $(i = 1, 2)$ and $l(G) = m + 1 - (n_1 + n_2)$. By Proposition 3.2 (b) we have

$$n_1 + n_2 = |\mathcal{F}(a_1)| + |\mathcal{F}(a_2)| = |\mathcal{F}(a_1) \cup \mathcal{F}(a_2)| + |\mathcal{F}(a_1) \cap \mathcal{F}(a_2)| \leqslant m + 1.$$

It follows that $l(G) \geqslant 0$ and $l(G) = 0$ iff $|\mathcal{F}(a_1) \cup \mathcal{F}(a_2)| = m$ and $|\mathcal{F}(a_1) \cap \mathcal{F}(a_2)| = 1$. If $l(G) \geqslant 1$ then $\alpha_k > 0$ by Lemma 2.9; if $l(G) = 0$ then $\alpha_k > 0$ by Lemma 2.11.

Case 2. $a_1 \mid a_2$. Clearly, $\mathcal{M}^{(m)}$ is a divisor chain, so by Lemma 3.8 we have $\alpha_k > 0$.

The proof is complete. □

To better understand the role of MGFS in $\mathcal{R}_k$, we can imagine them as a family of circles in a plane. In general, those circles may have different centers and meet each other. Corollary 3.4 and Lemma 3.8 just deal with two extreme cases: isolated circles and concentric circles.

We integrate Corollary 2.10, 3.6 and 3.10 into the following corollary:

Corollary 3.11. *If* $|\mathcal{M}^{(m)}| \leqslant 3$, *then* $\alpha_k > 0$.

4. The case of $|\mathcal{M}^{(4)}| = 4$ and the LCM equation

For the case of $|\mathcal{M}^{(4)}| = 4$, there are two methods to examine whether $\alpha_k = 0$: by estimating the bound on $l(a)$, or by discussing the distribution of GTDs in $\mathcal{M}^{(4)}$. Here we use the former method, which will yield the same result as the latter. In analysis, we naturally introduce a special Diophantine equation that we call the LCM equation. The solvability of the LCM equation is vital to deciding whether $k(t) \geqslant 9$.

Lemma 4.1. *Let* $G < a \in \mathcal{M}^{(4)}$. *We have* $l(a) \in \{-1, 0, 1, 2\}$, *and if* $l(a) = 2$ *there cannot exist* $G < b \in \mathcal{M}^{(4)}$ *such that* $b \neq a$ *and* $l(b) = 2$.

Proof. Since $l_4(G) = 1$ and $l_4(a) = 0$ for $G < a \in \mathcal{M}^{(4)}$, we have $l(a) = l_2(a) - l_3(a)$. First, it follows from Proposition 2.6 (c) that $l_2(a) \leqslant \binom{3}{2} = 3$ and $l_3(a) \leqslant \binom{3}{3} = 1$. Second, if $l_2(a) \geqslant 2$ there must be three (four is impossible, since $g_2(a) \leqslant 3$ by Proposition 2.6 (c)) distinct $y_a, y_b, y_c \in \mathcal{R}_k$ such that $(y_a, y_b) = (y_a, y_c) = a$ which implies $(y_a, y_b, y_c) = a$. Thus $l(a) \leqslant 3 - 1 = 2$. Moreover, if $l_2(a) = 3$ we must have $(y_b, y_c) = a$. And we claim that there cannot exist another $b \in \mathcal{M}^{(4)}$

such that $l_3(b) = 3$. Otherwise, we must have $(y_a, y_d) = (y_b, y_d) = (y_c, y_d) = b$. This contradicts the fact that $g_2(b) \leqslant 3$ by Proposition 2.6 (c). Hence we conclude that the possible values of $l(a)$ are -1, 0, 1 and 2, and there is at most one element $G < a \in \mathcal{M}^{(4)}$ such that $l(a) = 2$. This is just what is desired. □

Lemma 4.2. *For $\mathcal{M}^{(4)}$, if $l(G) \neq 0$ then $\alpha_k \neq 0$.*

Proof. By Lemmas 2.8 and 4.1 and the similar analysis as in Lemma 2.2, we have

$$\begin{aligned} G|\alpha_k - l(G)| &\leqslant \frac{1}{x_k/G} + \sum_{i=1}^{4} \frac{1}{y_i/G} + \sum_{j=1}^{s} \frac{|l(a_j)|}{a_j/G} \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4} + \frac{1}{2} \\ &= \frac{\pi^2}{6} - \frac{3}{4} \approx 0.895 < 1. \end{aligned}$$

By the similar discussion as in Lemma 2.12, the result follows. □

From Lemma 4.2 above, we know that to examine whether $\alpha_k = 0$ for the case of $|\mathcal{M}^{(4)}| = 4$, we only need to consider the case of $l(G) = 0$. Let $\mathcal{M}^{(4)} = \{G, a_1, a_2, a_3\}$. By (4) and Lemma 4.1, we need to solve a simple Diophantine equation: $l(a_1) + l(a_2) + l(a_3) = 3$ in $\{-1, 0, 1, 2\}$ with the constraint that there is at most one $l(a_j)$ $(1 \leqslant j \leqslant 3)$ equal to 2. Without loss of generality, let $l(a_1) \geqslant l(a_2) \geqslant l(a_3)$. Easily, we get two solutions: $(l(a_1), l(a_2), l(a_3))$ =(2,1,0) or (1,1,1).

For the case of $(l(a_1), l(a_2), l(a_3))$ =(2,1,0), we claim that $|\mathcal{G}(a_1) \cup \mathcal{G}(a_2)| = 4$. Since $l(a_1) = 2$, there must exist $y_a, y_b, y_c \in \mathcal{R}_k$ such that $(y_a, y_b) = (y_a, y_c) = (y_b, y_c) = a_1$ by Proposition 2.6 (c). Since $l(a_2) = 1$, we must have $(y_e, y_d) = a_2$ where $e \in \{a, b, c\}$. Thus the claim is true. By Lemma 2.11, we have $\alpha_k > 0$.

So there remains only one case to deal with, namely, $l(a_1) = l(a_2) = l(a_3) = 1$. Without loss of generality, let $(y_1, y_2) = a_1$. If $(y_3, y_4) = a_2$, then we again get $|\mathcal{G}(a_1) \cup \mathcal{G}(a_2)| = 4$ and hence $\alpha_k > 0$ by Lemma 2.11. Thus without loss of generality, suppose $(y_1, y_3) = a_2$. Consider $\mathcal{G}_2(a_3)$. If $y_4 \in \mathcal{G}_2(a_3)$, then again we get $|\mathcal{G}(a_1) \cup \mathcal{G}(a_2) \cup \mathcal{G}(a_3)| = 4$ and hence $\alpha_k > 0$ by Lemma 2.11. So there remains only one case deserving our consideration: $(y_1, y_2) = a_1$, $(y_1, y_3) = a_2$ and $(y_2, y_3) = a_3$. Note that since $\mathcal{F}(a_i) = \mathcal{G}(a_i)$ for $1 \leqslant i \leqslant 3$, by Proposition 3.3 they are all GTDs in $\mathcal{M}^{(4)}$, namely, they cannot be divided by each other. By (4) we have

$$\alpha_k = \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}. \tag{6}$$

From (6), we see that y_4 is a "free" element that has no relation with a_i. By Lemma 2.4 we have $\alpha_k < 0$ if $a_i \gg y_4$; and $\alpha_k > 0$ if $a_i \ll y_4$. Thus there may exist a set $\{x_k, y_1, y_2, y_3, y_4\}$ such that $\alpha_k = 0$. In fact, if such a set exists we must have $x_k = [y_1, y_2, y_3, y_4]$. Suppose $x_k = [y_1, y_2, y_3, y_4]g$ with $g \geqslant 1$ and let x_k multiply both sides of (6), then we get that $1/g$ is an integer implying that $g = 1$. In detail, we wonder whether the following Diophantine equation

$$0 = \frac{1}{[y_1, y_2, y_3, y_4]} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{(y_1, y_2)} + \frac{1}{(y_1, y_3)} + \frac{1}{(y_2, y_3)}$$

is solvable with the following constraints:

(a) $y_i \nmid y_j$ for $1 \leqslant i \neq j \leqslant 4$.

(b) $(y_1, y_4) = (y_2, y_4) = (y_3, y_4) = (y_1, y_2, y_3) = (y_1, y_2, y_3, y_4)$.

(c) Let $a_1 = (y_1, y_2)$, $a_2 = (y_1, y_3)$, $a_3 = (y_2, y_3)$, then $a_i \nmid a_j$ for $1 \leqslant i \neq j \leqslant 3$.

We call such a Diophantine equation with these constraints the *LCM equation*. If the LCM equation has one solution in which every element is the t-th power of some positive integer, we say it has a *t-th power solution*. In Section 5, we will explain the relation between the solvability of the LCM equation and Conjecture 1.1.

To summarize, we have proved the following:

Lemma 4.3. *If $\mathcal{M}^{(4)} = \{G, a_1, a_2, a_3\}$, then $\alpha_k \neq 0$ in any of the following cases:*

(a) *$\mathcal{M}^{(4)}$ has 1 GTD.*

(b) *$\mathcal{M}^{(4)}$ has 2 GTDs.*

(c) *$\mathcal{M}^{(4)}$ has 3 GTDs and* $\left|\bigcup_{i=1}^{3} \mathcal{G}(a_i)\right| = 4$.

5. Conclusions

Now we give the main results of this paper.

Theorem 5.1. *Let $t \geqslant 2$. If $n \leqslant 8$, then the power LCM matrix $([x_i, x_j]^t)$ defined on any gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ of n distinct positive integers is nonsingular.*

Proof. For the same reason as in the first paragraph of Section 2, we can just consider the gcd-closed set $\mathcal{S}^t = \{x_1, \ldots, x_n\}$ in which every element is the t-th power of some positive integer. Without loss of generality, we may let $1 \leqslant x_1 < x_2 < \ldots < x_n$. For $1 \leqslant k \leqslant n$, let $\mathcal{R}_k$ and $\mathcal{M}^{(|\mathcal{R}_k|)}$ be defined as in Section 2. We have proved in Lemma 2.2 that $\mathcal{R}_k \cap \mathcal{M}^{(m)} = \emptyset$. Since S^t is gcd-closed, $m + |\mathcal{M}^{(m)}| \leqslant k - 1$. Together with (2), for $m \geqslant 2$ we have

$$1 \leqslant |\mathcal{M}^{(m)}| \leqslant \min\{k - m - 1, 2^m - m - 1\}. \tag{7}$$

We claim that $\alpha_k \neq 0$ for $1 \leqslant k \leqslant 8$. For $k = 1$, $\alpha_1 = 1/x_1 \neq 0$. In what follows let $2 \leqslant k \leqslant 9$. By (7) we have $m \leqslant k - 2 \leqslant 6$, namely, $m = 6, 5, 4, 3, 2, 1$.

If $m = 6$, then $|\mathcal{M}^{(6)}| \leqslant 1$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 5$, then $|\mathcal{M}^{(5)}| \leqslant 2$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 4$, then $|\mathcal{M}^{(4)}| \leqslant 3$ by (7). By Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 3$, then $|\mathcal{M}^{(3)}| \leqslant 4$ by (7). If $|\mathcal{M}^{(3)}| = 4$, by Corollary 2.14, we have $\alpha_k \neq 0$; if $|\mathcal{M}^{(3)}| \leqslant 3$, by Corollary 3.11, we have $\alpha_k \neq 0$.

If $m = 2$, then $|\mathcal{M}^{(2)}| = 1$ by (7). By Corollary 2.14, we have $\alpha_k \neq 0$.

If $m = 1$, then $\alpha_k = (1/x_k) - (1/y_1) < 0$.

Thus we have $\alpha_k \neq 0$ for $1 \leqslant k \leqslant 8$. So if $n \leqslant 8$, by (1) we have $\det[\mathcal{S}^t]_n \neq 0$.

The proof is complete. □

Similarly, to prove $k(t) \geqslant 9$ we need only to prove that $\alpha_k \neq 0$ in the cases of $|\mathcal{M}^{(7)}| \leqslant 1$, $|\mathcal{M}^{(6)}| \leqslant 2$, $|\mathcal{M}^{(5)}| \leqslant 3$, $|\mathcal{M}^{(4)}| \leqslant 4$, $|\mathcal{M}^{(3)}| \leqslant 4$, $|\mathcal{M}^{(2)}| = 1$ and $m = 1$. From Section 2 and Section 3 we know that all these except the case of $|\mathcal{M}^{(4)}| = 4$ have been proved. Suppose $\mathcal{M}^{(4)} = \{G, a_1, a_2, a_3\}$. Lemma 4.3 tells us that there remains only one case of $|\mathcal{M}^{(4)}| = 4$ to discuss, i.e. a_1, a_2, a_3 are all GTDs and $\left|\bigcup_{i=1}^{3} \mathcal{G}(a_i)\right| = 3$. If there exists a set of $\{y_1, y_2, y_3, y_4\}$ such that $\alpha_k = 0$, namely, the LCM equation is solvable then $k(t) = 8$; if such a set does not exist, namely, the LCM equation is unsolvable then $k(t) \geqslant 9$. In brief, we have

Theorem 5.2. *$k(t) \geqslant 9$ iff the LCM equation has no t-th power solution.*

Remark 5.3. As $|\mathcal{M}^{(m)}|$ grows, the "free" elements in $\mathcal{R}_k$, which have no relations with other elements in $\mathcal{R}_k$, will be more and more numerous, and this makes it more possible that $\alpha_k = 0$ when $l(G) = 0$. We can see this clearly by letting $l(G) = 0$ in (5).

It is easy to show that if $t = t_1 t_2$ then $k(t_1), k(t_2) \leqslant k(t)$. So we have:

Corollary 5.4. *If the LCM equation has one t-th power solution then $k(t') = 8$ for any $t' \mid t$ and $1 < t'$.*

In fact, we conjecture that for every $t \geqslant 2$ the LCM equation has at least one t-th power solution. Assume that $\mathcal{S}' = \{x_k = x', y_1 = y'_1, y_2 = y'_2, y_3 = y'_3, y_4 = y'_4, a_1 = a'_1, a_2 = a'_2, a_3 = a'_3, (y_1, y_2, y_3, y_4) = G'\}$ is a set of some t-th power solution to the

LCM equation. As in [9, 11], for any integers $n \geqslant 9$ and $a > 1$, let

$$x_i = G'a^{(i-1)t} \text{ for } 1 \leqslant i \leqslant n-8,$$
$$x_{n-7} = a_1'a^{(n-9)t},\ x_{n-6} = a_2'a^{(n-9)t},\ x_{n-5} = a_3'a^{(n-9)t},$$
$$x_{n-4} = y_1'a^{(n-9)t},\ x_{n-3} = y_2'a^{(n-9)t},$$
$$x_{n-2} = y_3'a^{(n-9)t},\ x_{n-1} = y_4'a^{(n-9)t},\ x_n = x'a^{(n-9)t}.$$

It is easy to check that $\mathcal{S} = \{x_1, \ldots, x_n\}$ is a gcd-closed set and the set of GTDs of x_n is just $\mathcal{S}'$. So by (1) $\det[\mathcal{S}]_n = 0$ since $\alpha_n = 0$. Thus we have proved that if for some $t \geqslant 2$ the LCM equation has one t-th power solution, then for any $n \geqslant 9$ we can find a gcd-closed set $\mathcal{S} = \{x_1, \ldots, x_n\}$ such that the power LCM matrix $([x_i, x_j]^t)$ on $\mathcal{S}$ is singular. Therefore we raise the following conjecture.

Conjecture 5.5. *$k(t) = 8$ for all $t \geqslant 2$. This is equivalent to the LCM equation having at least one t-th power solution.*

This should not be surprising since the Riemann zeta function $\zeta(t)$ has the similar character, that is, $\zeta(t)$ diverges for $t = 1$ and converges for all $t \geqslant 2$. From Lemma 2.2 we can also sense some relationship between $k(t)$ and $\zeta(t)$. However, to prove that the LCM equation has t-th power solution for every $t \geqslant 2$ will not be as easy as to prove that $\zeta(t)$ converges for all $t \geqslant 2$.

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References

[1] *S. Beslin*: Reciprocal GCD matrices and LCM matrices. Fibonacci Quart. *29* (1991), 271–274. Zbl 0738.11026
[2] *S. Beslin and S. Ligh*: Greatest common divisor matrices. Linear Algebra Appl. *118* (1989), 69–76. Zbl 0672.15005
[3] *K. Bourque and S. Ligh*: Matrices associated with classes of arithmetical functions. J. Number Theory *45* (1993), 367–376. Zbl 0784.11002
[4] *K. Bourque and S. Ligh*: On GCD and LCM matrices. Linear Algebra Appl. *174* (1992), 65–74. Zbl 0761.15013
[5] *K. Bourque and S. Ligh*: Matrices associated with classes of multiplicative functions. Linear Algebra Appl. *216* (1995), 267–275. Zbl 0826.15013
[6] *S. Z. Chun*: GCD and LCM power matrices. Fibonacci Quart. *34* (1996), 290–297. Zbl 0861.11017
[7] *P. Haukkanen, J. Wang and J. Sillanpää*: On Smith's determinant. Linear Algebra Appl. *258* (1997), 251–269. Zbl 0883.15002

[8] *S. Hong*: LCM matrix on an r-fold gcd-closed set. J. Sichuan Univ. Nat. Sci. Ed. *33* (1996), 650–657. Zbl 0869.11021
[9] *S. Hong*: On Bourque-Ligh conjecture of LCM matrices. Adv. in Math. (China) *25* (1996), 566–568. Zbl 0869.11022
[10] *S. Hong*: On LCM matrices on GCD-closed sets. Southeast Asian Bull. Math. *22* (1998), 381–384. Zbl 0936.15011
[11] *S. Hong*: On the Bourque-Ligh conjecture of least common multiple matrices. J. Algebra *218* (1999), 216–228. Zbl 1015.11007
[12] *S. Hong*: Gcd-closed sets and determinants of matrices associated with arithmetical functions. Acta Arith. *101* (2002), 321–332. Zbl 0987.11014
[13] *S. Hong*: On the factorization of LCM matrices on gcd-closed sets. Linear Algebra Appl. *345* (2002), 225–233. Zbl 0995.15006
[14] *S. Hong*: Notes on power LCM matrices. Acta Arith. *111* (2004), 165–177. Zbl 1047.11022
[15] *S. Hong*: Nonsingularity of matrices associated with classes of arithmetical functions. J. Algebra *281* (2004), 1–14. Zbl pre02134414
[16] *S. Hong*: Nonsingularity of least common multiple matrices on gcd-closed sets. J. Number Theory *113* (2005), 1–9.
[17] *H. J. S. Smith*: On the value of a certain arithmetical determinant. Proc. London Math. Soc. *7* (1875–1876), 2080–212. Jfm 08.0074.03

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