Poom Kumam; Somyot Plubtieng

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THE CHARACTERISTIC OF NONCOMPACT CONVEXITY AND RANDOM FIXED POINT THEOREM FOR SET-VALUED OPERATORS

Poom Kumam, Bangkok, and Somyot Plubtieng, Phitsanulok

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Abstract. Let (Ω, Σ) be a measurable space, X a Banach space whose characteristic of noncompact convexity is less than 1, C a bounded closed convex subset of X, $KC(C)$ the family of all compact convex subsets of C. We prove that a set-valued nonexpansive mapping $T\colon C \to KC(C)$ has a fixed point. Furthermore, if X is separable then we also prove that a set-valued nonexpansive operator $T\colon \Omega \times C \to KC(C)$ has a random fixed point.

Keywords: random fixed point, set-valued random operator, measure of noncompactness

MSC 2000: 47H10, 47H09, 47H40

1. Introduction

The study of random fixed points has been a very active area of research in probabilistic operator theory in the last decade. In this direction, there have appeared various papers concerning random fixed point theorems for single-valued and set-valued random operators; see, for example, [6], [8], [10], [11], [12], [15], [21] and the references therein.

In 2002, P. Lorenzo Ramírez [10] proved the existence of a random fixed point theorems for a random nonexpansive operator in the framework of Banach spaces with the characteristic of noncompact convexity $\varepsilon_\alpha(X)$ less than 1. On the other hand, Domínguez Benavides and Ramírez [4] proved a fixed point theorem for a set-valued nonexpansive and 1-χ-contractive mapping in the framework of Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1.

The purpose of the present paper is to prove a fixed point theorem for set-valued random nonexpansive operators in the framework of Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Moreover, we also prove a fixed point theorem for set-valued nonexpansive mappings in Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ less than 1. Our results can also be viewed as an extension of Theorem 6 in [10] and Theorem 4.2 in [4], respectively.

2. Preliminaries

Through out this paper we will consider a measurable space (Ω, Σ) (where Σ is a σ-algebra of subsets of Ω) and (X, d) will be a metric space. We denote by $CL(X)$ (resp. $CB(X), KC(X)$) the family of all nonempty closed (resp. closed bounded, compact convex) subsets of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A,B) = \max\big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\big\}$$

for $A, B \in CB(X)$, where $d(x,E) = \inf\{d(x,y)\colon\ y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a set-valued mapping $T\colon\ C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C\colon\ Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C\colon\ Tx \cap V \neq \emptyset\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf. [2] and [3] for details). There is another different kind of continuity for multivalued operators: $T\colon\ C \to CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [3]) that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

A set-valued operator $T\colon\ \Omega \to 2^X$ is called (Σ)-measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega\colon\ T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x\colon\ \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T\colon\ \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T\colon\ \Omega \times C \to 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x)\colon\ \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C\colon\ x \in T(\omega, x)\}\,.$$

Note that if we do not assume the existence of a fixed point for the deterministic mapping $T(\omega, \cdot)\colon\ C \to 2^X$, $F(\omega)$ may be empty. A measurable operator $x\colon\ \Omega \to C$ is said to be a *random fixed point of an operator* $T\colon\ \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T\colon\ \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T(\omega)\colon\ C \to 2^X$ is continuous.

If C is a closed convex subset of a Banach space X, then a set-valued mapping $T\colon\ C \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leqslant k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leqslant \|x - y\|, \quad x, y \in C.$$

A random operator $T\colon\ \Omega \times C \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$, the map $T(\omega)\colon\ C \to C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1 (Wagner cf. [14]). *Let (X, d) be a complete separable metric space and $F\colon\ \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2 (Itoh 1977, cf. [8]). *Suppose $\{T_n\}$ is a sequence of measurable set-valued operator from Ω to $CB(X)$ and $T\colon\ \Omega \to CB(X)$ is an operators. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3 (Tan and Yuan cf. [13]). *Let X be a separable metric space and Y a metric space. If $f\colon\ \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x\colon\ \Omega \to X$ is measurable, then $f(\cdot, x(\cdot))\colon\ \Omega \to Y$ is measurable.*

As an easy application of Proposition 3 of Itoh[8] we have the following result.

Lemma 2.4. *Let C be a closed separable subset of a Banach space X, $T\colon\ \Omega \times C \to C$ a random continuous operator and $F\colon\ \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G\colon\ \Omega \to 2^C$ given by*

$$G(\omega) = \{x \in F(\omega)\colon\ \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measurable and so is the operator $\operatorname{cl}\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5 (Domínguez Benavidel, Lopez Acedo and Xu cf. [6]). *Suppose C is a weakly closed nonempty separable subset of a Banach space X, $F\colon \Omega \to 2^X$ a measurable map with weakly compact values and $f\colon \Omega\times C \to \mathbb{R}$ a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r\colon \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x\in F(x)} f(\omega, x)$$

and the marginal map $R\colon \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x)\colon\ f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf\{r > 0\colon\ B \text{ can be covered by finitely many sets of diameter} \leqslant r\},$$
$$\chi(B) = \inf\{r > 0\colon\ B \text{ can be covered by finitely many balls of radius} \leqslant r\}.$$

The separation measure of noncompacness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup\{\varepsilon\colon \text{ there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \operatorname{sep}(\{x_n\}) \geqslant \varepsilon\}.$$

Let X be a Banach space and $\varphi = \alpha, \beta$ or χ. The modulus of noncompact convexity associated to φ is defined in the following way:

$$\Delta_{X,\varphi}(\varepsilon) = \inf\{1 - d(0, A)\colon\ A \subset B_X \text{ is convex, } \varphi(A) \geqslant \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness φ is defined by

$$\varepsilon_\varphi(X) = \sup\{\varepsilon \geqslant 0\colon\ \Delta_{X,\varphi}(\varepsilon) = 0\}.$$

The following relationships among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\varepsilon) \leqslant \Delta_{X,\beta}(\varepsilon) \leqslant \Delta_{X,\chi}(\varepsilon), \tag{2.1}$$

and consequently

$$\varepsilon_\alpha(X) \geqslant \varepsilon_\beta(X) \geqslant \varepsilon_\chi(X). \tag{2.2}$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated to β and χ.

$$\Delta_{X,\beta}(\varepsilon) = \inf\big\{1 - \|x\|\colon \{x_n\} \subset B_X, x = w - \lim_n x_n,\ \operatorname{sep}(\{x_n\}) \geqslant \varepsilon\big\},$$
$$\Delta_{X,\chi}(\varepsilon) = \inf\big\{1 - \|x\|\colon \{x_n\} \subset B_X, x = w - \lim_n x_n, \chi(\{x_n\}) \geqslant \varepsilon\big\}.$$

Let C be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. We use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$r(C, \{x_n\}) = \inf\big\{\limsup_n \|x_n - x\|\colon\ x \in C\big\},$$
$$A(C, \{x_n\}) = \big\{x \in C\colon\ \limsup_n \|x_n - x\| = r(C, \{x_n\})\big\}.$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\left\{\sup\{\|x - y\|\colon\ y \in D\}\colon\ x \in C\right\}.$$

Let $\{x_n\}$ and C be nonempty bounded closed subsets of a Banach space X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Moreover, we also need the following Lemmas.

Lemma 2.6 (Domínguez Benavides and Lorenzo Ramírez Theorem 4.3 cf. [4])**.** *Let C be a closed convex subset of a reflexive Banach space X, and let x_n be a bounded sequence in C which is regular with respect to C. Then*

$$r_C(A(C, x_n)) \leqslant (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}). \tag{2.3}$$

Moreover, if X satisfies the nonstrict Opial condition then

$$r_C(A(C, x_n)) \leqslant (1 - \Delta_{X,\chi}(1^-))r(C, \{x_n\}). \tag{2.4}$$

The following result are now basic in the fixed point theorem for multivalued mappings.

Lemma 2.7 (Xu cf. Theorem 1.6 of [19])**.** *Let E be a nonempty bounded closed closed convex subset of a Banach space and $T\colon\ E \to KC(X)$ a contraction. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point. (Here $I_E(x)$ is call the inward set at x defined by $I_E := \{x + \lambda(y - x)\colon\ \lambda \geqslant 0, y \in E\}$)*

Proposition 2.8 (Kirk-Massa Theorem cf. [16]). *Let C be a nonempty weakly compact separable subset of a Banach space X, $T\colon C \to K(C)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n, Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \ \forall x \in A := A(C, \{z_n\})$$

3. The results

We begin this section by showing that in Benavides-Ramírez's result, the 1-χ-contractive condition on T can be removed.

Theorem 3.1. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T\colon C \to KC(C)$ a nonexpansive mapping. Then T has a fixed point.*

Proof. The condition $\varepsilon_\beta(X) < 1$ implies reflexivity [2], so C is weakly compact. Let $x_0 \in C$ be fixed and, for each $n \geqslant 1$, define $T_n\colon C \to KC(C)$ by

$$T_n x = \frac{1}{n} x_0 + \Big(1 - \frac{1}{n}\Big) Tx, \ \ \forall x \in C.$$

Then T_n is a set-valued contraction and hence has a fixed point x_n. It is easily seen that $\operatorname{dist}(x_n, Tx_n) \leqslant \frac{1}{n} \operatorname{diam} C \to 0$ as $n \to \infty$. By Goebel and Kirk [7], we may assume that $\{x_n\}$ is regular with respect to C and using Proposition 2.8 we can also assume that

$$Tx \cap A \neq \emptyset, \ \ \forall x \in A := A(C, \{x_n\}).$$

We apply Lemma 2.6 to obtain

$$r_C(A) \leqslant \lambda r(C, \{x_n\}), \tag{3.1}$$

where $\lambda := (1 - \Delta_{X,\beta}(1^-)) < 1$.

It is clear that A is a weakly compact convex subset of C. Now fix $x_1 \in A$ and for each $n \geqslant 1$, define the contraction $T_n^1\colon A \to KC(C)$ by

$$T_n^1(x) = \frac{1}{n} x_1 + \Big(1 - \frac{1}{n}\Big) Tx, \ \ \forall x \in A.$$

Since A is convex, each T_n^1 satisfies the same boundary condition as T does, that is, we have

$$T_n^1 x \cap \overline{I}_A(x) \neq \emptyset, \ \ \forall x \in A,$$

Hence by Lemma 2.7, T_n^1 has a fixed point $z_n \in A$. Consequently, we can get a sequence $\{x_n^1\}$ in A satisfying $d(x_n^1, T(x_n^1)) \to 0$ as $n \to \infty$. Again, applying Lemma 2.6, we obtain

$$r_C(A^1) \leqslant \lambda r(C, \{x_n^1\}), \tag{3.2}$$

where $A^1 := A(C, \{x_n^1\})$. Since $\{x_n^1(\omega)\} \subset A$, we have

$$r(C, \{x_n^1\}) \leqslant r_C(A), \tag{3.3}$$

and then

$$r_C(A^1) \leqslant \lambda^2 r_C(A). \tag{3.4}$$

By induction, for each $m \geqslant 1$, we construct A^m, and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\}), x_n^m \subset A^{m-1}$ such that $d(x_n^m, Tx_n^m) \to 0$ as $n \to \infty$ and

$$r_C(A^m) \leqslant \lambda r_C(A) \leqslant \lambda^m r(C, \{x_n\}). \tag{3.5}$$

By assumption $\varepsilon_\beta(X) < 1$ and $\operatorname{diam} A^m \leqslant 2r_C(A^m)$ leads to $\lim\limits_{m\to\infty} \operatorname{diam} A^m = 0$. Since $\{A^m\}$ is a descending sequence of weakly compact subsets of C, we have $\cap_m A^m = \{z\}$ for some $z \in C$. Finally, we will show that z is a fixed point of T. Indeed, for each $m \geqslant 1$, we have

$$\begin{aligned} d(z, Tz) &\leqslant \|z - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tz) \\ &\leqslant 2\|z - x_n^m\| + d(x_n^m, Tx_n^m) \\ &\leqslant 2 \operatorname{diam} A^m + d(x_n^m, Tx_n^m). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z, Tz) \leqslant 2 \operatorname{diam} A^m.$$

Now taking the limit in m on both sides we obtain $z \in Tz$. □

Corollary 3.2 (Domínguez Benavides and Lorenzo Ramírez, Theorem 4.2 in [4]). *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T\colon C \to KC(C)$ a nonexpansive and 1-χ-contractive mapping. Then T has a fixed point.*

Now we are ready to prove the main result of this paper.

Theorem 3.3. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T\colon \Omega \times C \to KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.*

Proof. For each $\omega \in \Omega$, and for every $n \geqslant 1$, we set

$$F(\omega) = \{x \in C\colon\ x \in T(\omega, x)\},$$

and

$$F_n(\omega) = \Big\{x \in C\colon\ d(x, T(\omega, x)) \leqslant \frac{1}{n} \operatorname{diam} C.\Big\}$$

It follows from Theorem 3.1 that $F(\omega)$ is nonempty. Clearly $F(\omega) \subseteq F_n(\omega)$, and $F_n(\omega)$ is closed and convex. Furthermore, by [8, Proposition 3], each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leqslant \frac{1}{n} \operatorname{diam} C \to 0 \text{ as } n \to \infty.$$

Define a function $f_1\colon\ \Omega \times C \to \mathbb{R}^+$ by

$$f_1(\omega, x) = \limsup_n \|x_n(\omega) - x\|,\ \ \forall \omega \in \Omega.$$

By Lemma 2.3, it is easily seen that for each $x \in C, f_1(\cdot, x)\colon\ \Omega \to \mathbb{R}^+$ is measurable and for each $\omega \in \Omega$, $f_1(\omega, \cdot)\colon\ C \to \mathbb{R}^+$ is continuous and convex (and hence weakly lower semicontinuous (w-l.s.c.)). Note that the condition $\varepsilon_\beta(X) < 1$ implies reflexivity (see [2]) and so C is weakly compact. Hence, by Lemma 2.5 the marginal functions

$$r_1(\omega) := \inf_{x \in C} f_1(\omega, x),$$

and

$$R_1(\omega) := \{x \in C\colon\ f_1(\omega, x) = r_1(\omega)\}$$

are measurable. By Goebel [7], for any $\omega \in \Omega$ we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect to C. Observe that $R_1(\omega) = A(C, \{x_n(\omega)\})$ and $r_1(\omega) = r(C, \{x_n(\omega)\})$, thus we can apply Lemma 2.6 to obtain

$$r_C(R_1(\omega)) \leqslant \lambda r_1(\omega), \tag{3.6}$$

where $\lambda := 1 - \Delta_{X,\beta}(1^-) < 1$, since $\varepsilon_\beta(X) < 1$. It is clear that $R_1(\omega)$ is a weakly compact and convex subset of C. By Lemma 2.1 we can take $x_1(\omega)$ as a measurable selector of $R_1(\omega)$. For each $\omega \in \Omega$ and $n \geqslant 1$, we define the contraction $T_n^1(\omega, \cdot)\colon$ $R_1(\omega) \to KC(C)$ by

$$T_n^1(\omega, x) = \frac{1}{n} x_1(\omega) + \Big(1 - \frac{1}{n}\Big) T(\omega, x), \quad \forall x \in R_1(\omega).$$

Since $R_1(\omega)$ is convex, each T_n satisfies the same boundary condition as T does, that is, we have

$$T_n^1(\omega, x) \cap \overline{I}_{R_1}(\omega)(x) \neq \emptyset, \;\; \forall x \in R_1(\omega).$$

Hence by Lemma 2.7, $T_n^1(\omega, \cdot)$ has a fixed point $z_n(\omega) \in R_1(\omega)$, i.e. $F(\omega) \cap R_1(\omega) \neq \emptyset$. Also it is easily seen that

$$\operatorname{dist}(z_n(\omega), T(\omega, z_n(\omega))) \leqslant \frac{1}{n} \operatorname{diam} C \to 0 \text{ as } n \to \infty.$$

Thus $F_n^1(\omega) = \{x \in R_1(\omega)\colon\ d(x, T(\omega, x)) \leqslant \frac{1}{n} \operatorname{diam} C\} \neq \emptyset$ for each $n \geqslant 1$, is closed and, by Lemma 2.4, measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1, and from its definition we have $x_n^1(\omega) \in R_1(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. Consider the function $f_2\colon \Omega \times C \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \;\; \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontinuous function. Thus the marginal functions

$$r_2(\omega) := \inf_{x \in R_1(\omega)} f_2(\omega, x)$$

and

$$R_2(\omega) := \{x \in R_1(\omega)\colon\ f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $R_2(\omega) = A(R_1(\omega), \{x_n^1(\omega)\})$, it follows that $R_2(\omega)$ is weakly compact and convex. Also $r_2(\omega) = r(R_1(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $R_1(\omega)$. Again, applying Lemma 2.6, we obtain

$$r_C(R_2(\omega)) \leqslant \lambda r_2(\omega). \tag{3.7}$$

Furthermore, $\{x_n^1(\omega)\} \subset R_1(\omega)$. Hence

$$r_2(\omega) \leqslant r_C(R_1(\omega)), \tag{3.8}$$

and thus

$$r_C(R_2(\omega)) \leqslant \lambda^2 r_1(\omega). \tag{3.9}$$

By induction, for each $m \geqslant 1$, we construct $R_m(\omega), r_m(\omega)$ and $\{x_n^m(\omega)\}_n$ where $x_n^m(\omega) \in R_m(\omega)$ such that $d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \to 0$ as $n \to \infty$ and

$$r_C(R_m(\omega)) \leqslant \lambda r_m(\omega) \leqslant \lambda^m r_1(\omega). \tag{3.10}$$

Since $\operatorname{diam} R_m(\omega) \leqslant 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m\to\infty} \operatorname{diam} R_m(\omega) = 0$. Since $\{R_m(\omega)\}$ is a descending sequence of weakly compact subsets of C for each $\omega \in \Omega$, we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leqslant \operatorname{diam} R_m(\omega) \to 0 \quad \text{as } n \to +\infty.$$

Therefore, by Lemma 2.2, $z(\omega)$ is measurable. Finally, we will show that $z(\omega)$ is a fixed point of T. Indeed, for each $m \geqslant 1$, we have

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega)) &\leqslant \|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\quad + H(T(\omega, x_n^m(\omega)), T(\omega, z(\omega))) \\ &\leqslant 2\|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\leqslant 2 \operatorname{diam} R_m(\omega) + d(x_n^m(\omega), T(\omega, x_n^m(\omega))). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z(\omega), T(\omega, z(\omega)) \leqslant 2 \operatorname{diam} R_m(\omega).$$

Finally, taking limit in m in both sides we obtain $z(\omega) \in T(\omega, z(\omega))$. □

Corollary 3.4. *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\beta(X) < 1$, and $T\colon\ \Omega \times C \to C$ a random nonexpansive operator. Then T has a random fixed point.*

Corollary 3.5 (Lorenzo Ramírez, Theorem 6 in [10])**.** *Let C be a nonempty closed bounded convex separable subset of a Banach space X such that $\varepsilon_\alpha(X) < 1$, and $T\colon\ \Omega \times C \to C$ a random nonexpansive operator. Then T has a random fixed point.*

Proof. By (2.2) we see that $\varepsilon_\alpha(X) < 1$ implies $\varepsilon_\beta(X) < 1$. □

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Authors' addresses: Poom Kumam, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand, e-mail: `pooom.kum@kmutt.ac.th`; Somyot Plubtieng (Corresponding Author), Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand, e-mail: `somyotp@nu.ac.th`.