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COMPLEMENTED COPIES OF ℓ_p SPACES IN TENSOR PRODUCTS

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Abstract. We give sufficient conditions on Banach spaces X and Y so that their projective tensor product $X \otimes_\pi Y$, their injective tensor product $X \otimes_\varepsilon Y$, or the dual $(X \otimes_\pi Y)^*$ contain complemented copies of ℓ_p.

Keywords: ℓ_p space, injective and projective tensor product

MSC 2000: 46B28, 46B20

It is proved in [3] that $C(K_1) \otimes_\pi C(K_2)$ contains a complemented copy of ℓ_2 whenever at least one of the spaces $C(K_i)$ contains an isomorphic copy of ℓ_1, and that $L_1(\mu_1) \otimes_\varepsilon L_1(\mu_2)$ contains a complemented copy of ℓ_2 whenever at least one of the spaces $L_1(\mu_i)$ does not have the Schur property. Moreover, it is also proved that, if X contains a copy of c_0, Y^* has the Orlicz property and there exists a surjective operator from Y onto ℓ_2, then $X \otimes_\pi Y$ contains a complemented copy of ℓ_2. In the present paper we extend these results, giving new conditions on X and Y so that $X \otimes_\pi Y$, $X \otimes_\varepsilon Y$, or the dual $(X \otimes_\pi Y)^*$ contain complemented copies of ℓ_p spaces.

Throughout, X and Y denote Banach spaces, X^* is the dual of X, and B_X stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers. The notation $X \equiv Y$ (respectively, $X \cong Y$) means that X and Y are isometrically isomorphic (respectively, isomorphic). By an *operator* from X into Y we always mean a bounded linear mapping. We use $\mathcal{L}(X, Y)$ for the space of all operators from X into Y, endowed with the supremum norm, and $\mathcal{K}(X, Y)$ for the subspace of compact operators.

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Given $1 \leqslant p \leqslant \infty$, we denote by p^* the conjugate index of p ($1/p + 1/p^* = 1$). Given $1 \leqslant r < \infty$, if a sequence $(x_n) \subset X$ is *weakly r-summable,* then there is a positive constant C such that

$$\|(x_n)_n\|_{w,r} := \sup_{x^* \in B_{X^*}} \left(\sum_{n=1}^{\infty} |x^*(x_n)|^r \right)^{1/r} \leqslant C$$

(see [8, page 32]). We denote by e_n the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with 1 in the n-th position. The sequence $(e_n)_{n=1}^{\infty}$ is weakly r-summable in ℓ_p ($1 < p < \infty$), for $r \geqslant p^*$, with $\|(e_n)_n\|_{w,r} = 1$.

The following result will be used without explicit mention.

Proposition 1. *Let $1 < p < \infty$ and let X be a Banach space. The following assertions are equivalent:*

(a) *$\mathcal{L}(\ell_p, X) \neq \mathcal{K}(\ell_p, X)$;*

(b) *there is a weakly p^*-summable sequence in X which is not norm null;*

(c) *there is a normalized weakly p^*-summable sequence in X.*

The equivalence (a) $\Leftrightarrow$ (b) is proved in [4, Corollary 5]. The equivalence (b) $\Leftrightarrow$ (c) is obvious. Note that in [4, Corollary 5] there is a misprint: instead of $C_p(X, Y)$, one should read $C_{p^*}(X, Y)$.

By $X \otimes_\pi Y$ (respectively, $X \otimes_\varepsilon Y$) we denote the projective (respectively, injective) tensor product of X and Y. Recall that $(X \otimes_\pi Y)^* \equiv \mathcal{L}(X, Y^*)$. We refer to [5] and [9] for the theory of injective and projective tensor products of Banach spaces.

For any undefined notion from Banach Space Theory, we refer to [7] or [8].

In what follows, $\Pi_r(X, Y)$ denotes the space of all absolutely r-summing operators from X into Y.

Theorem 2. *Let X and Y be Banach spaces such that $\mathcal{L}(X, Y^*) = \Pi_r(X, Y^*)$, for $1 < r < \infty$. Suppose that $\mathcal{L}(\ell_{r^*}, X) \neq \mathcal{K}(\ell_{r^*}, X)$ and $\mathcal{L}(\ell_r, Y^*) \neq \mathcal{K}(\ell_r, Y^*)$. Then $X \otimes_\pi Y$ contains a complemented copy of ℓ_{r^*}.*

Proof. Let $(x_n) \subset X$ (respectively, $(y_n^*) \subset Y^*$) be normalized weakly r-summable (respectively, weakly r^*-summable) sequences. We can assume that they are basic. There is a sequence $(x_n^*) \subset X^*$ such that $\|x_n^*\| \leqslant M$ ($n \in \mathbb{N}$) and $x_m^*(x_n) = \delta_{mn}$. The argument used in the proof of [11, Theorem 12] yields a sequence $(y_n) \subset Y$ such that $\|y_n\| \leqslant K$ and $y_m^*(y_n) = \delta_{mn}$.

Let $I\colon \ell_{r^*} \to X \otimes_\pi Y$ be the linear mapping given by $I(e_n) = x_n \otimes y_n$. We show that I is well-defined and continuous. Indeed, given $a = (a_n) \in \ell_{r^*}$, we have for

$k, m \in \mathbb{N}$,

$$\Big\| \sum_{n=k}^{m} a_n x_n \otimes y_n \Big\|_\pi = \sup_{T \in B_{\mathcal{L}(X,Y^*)}} \Big| \sum_{n=k}^{m} a_n \langle T(x_n), y_n \rangle \Big|$$
$$\leqslant K \Big(\sum_{n=k}^{m} |a_n|^{r^*} \Big)^{1/r^*} \sup_{T \in B_{\mathcal{L}(X,Y^*)}} \Big(\sum_{n=k}^{m} \|T(x_n)\|^r \Big)^{1/r},$$

where we have used Hölder's inequality. Since every $T \in \mathcal{L}(X, Y^*)$ is absolutely r-summing, we have

$$I(a) = \sum_{n=1}^{\infty} a_n x_n \otimes y_n \in X \otimes_\pi Y.$$

Thanks to the Open Mapping Theorem, there is a positive constant C independent of T such that the absolutely r-summing norm $\pi_r(T)$ of T satisfies

$$\pi_r(T) \leqslant C \|T\|_{\mathcal{L}(X,Y^*)},$$

so we have

$$\|I(a)\|_\pi = \Big\| \sum_{n=1}^{\infty} a_n x_n \otimes y_n \Big\|_\pi \leqslant KC \|a\|_{r^*} \|(x_n)_n\|_{w,r},$$

and I is continuous.

Now let $R\colon X \otimes_\pi Y \to \ell_{r^*}$ be the linear mapping given by

$$R(x \otimes y) = (x_n^*(x) y_n^*(y))_{n=1}^{\infty} \qquad (x \in X,\ y \in Y).$$

Note that R is well-defined since

$$\Big(\sum_{n=1}^{\infty} |x_n^*(x) y_n^*(y)|^{r^*} \Big)^{1/r^*} \leqslant M \|x\| \Big(\sum_{n=1}^{\infty} |y_n^*(y)|^{r^*} \Big)^{1/r^*} \leqslant M \|x\| \|y\| \|(y_n^*)_n\|_{w,r^*}.$$

Let $u \in X \otimes Y$ and let $\sum_{i=1}^{m} x_i \otimes y_i$ be one of its representations. Then

$$(1) \qquad \|R(u)\| = \Big\| \Big(\sum_{i=1}^{m} x_n^*(x_i) y_n^*(y_i) \Big)_{n=1}^{\infty} \Big\| = \Big(\sum_{n=1}^{\infty} \Big| \sum_{i=1}^{m} x_n^*(x_i) y_n^*(y_i) \Big|^{r^*} \Big)^{1/r^*}.$$

Consider now the operator $T \in \mathcal{L}(Y^*, X)$ defined by

$$T(y^*) = \sum_{i=1}^{m} y^*(y_i) x_i \qquad (y^* \in Y^*).$$

Clearly, T is nuclear and its nuclear norm satisfies

$$\|T\|_{\mathrm{N}} \leqslant \sum_{i=1}^{m} \|x_i\| \|y_i\|.$$

For every index n, we have

$$\left| \sum_{i=1}^{m} x_n^*(x_i) y_n^*(y_i) \right| = |\langle T(y_n^*), x_n^* \rangle| \leqslant M \|T(y_n^*)\|.$$

Then, from (1), using the fact that T is also absolutely r^*-summing, it follows that

$$\begin{aligned}
\|R(u)\| &\leqslant M \left(\sum_{n=1}^{\infty} \|T(y_n^*)\|^{r^*} \right)^{1/r^*} \\
&\leqslant M \pi_{r^*}(T) \|(y_n^*)_n\|_{w,r^*} \\
&\leqslant M \|T\|_{\mathrm{N}} \|(y_n^*)_n\|_{w,r^*} \\
&\leqslant M \|(y_n^*)_n\|_{w,r^*} \sum_{i=1}^{m} \|x_i\| \|y_i\|.
\end{aligned}$$

Since this holds for every representation of u as an element of $X \otimes Y$, we have $R(u) \leqslant M' \|u\|_\pi$. Therefore, R is continuous. Easily, $R \circ I$ is the identity map on ℓ_{r^*}, and so $I \circ R$ is a projection. □

Remark 3. The equality $\mathcal{L}(X, Y^*) = \Pi_2(X, Y^*)$ holds, for example, when X is an $\mathscr{L}_\infty$-space and Y^* has cotype 2 [8, Theorem 11.14(a)], while the equality $\mathcal{L}(X, Y^*) = \Pi_r(X, Y^*)$ for $r > 2$ holds, for example, when X is an $\mathscr{L}_\infty$-space and Y^* has cotype q $(2 < q < r)$ [8, Theorem 11.14(b)]. The disk algebra A is not an $\mathscr{L}_\infty$-space [2, page 4], nevertheless, whenever Y^* has cotype 2, we have $\mathcal{L}(A, Y^*) = \Pi_2(A, Y^*)$ [2, Corollary 2.8].

A Banach space X has the *Orlicz property* if the identity operator on X is absolutely $(2,1)$-summing. Every Banach space with cotype 2 has the Orlicz property (see [10, Definition 5.1] and [8, Corollary 11.17]). The converse is not true [18].

Theorem 4. *Suppose that X has the Orlicz property and contains a normalized weakly r-summable sequence, for $1 < r \leqslant 2$, and Y contains a complemented copy of ℓ_1. Then $X \otimes_\varepsilon Y$ contains a complemented copy of ℓ_{r^*}.*

Proof. Since $X \otimes_\varepsilon \ell_1$ is complemented in $X \otimes_\varepsilon Y$, it is enough to prove the result for $X \otimes_\varepsilon \ell_1$.

Let $(x_n) \subset X$ be a normalized weakly r-summable sequence, that can be assumed to be basic. Then there is a sequence $(x_n^*) \subset X^*$ with $\|x_n^*\| \leqslant M$ $(n \in \mathbb{N})$, such that $x_m^*(x_n) = \delta_{mn}$.

We give a linear mapping $R\colon X \otimes \ell_1 \to \ell_{r^*}$ by

$$R(x \otimes y) = (x_n^*(x) e_n(y))_{n=1}^{\infty} .$$

Clearly, R is well-defined.

Given $\sum_{i=1}^{m} x_i \otimes y_i \in X \otimes \ell_1$, we define the operator $T \in \mathcal{L}(\ell_\infty, X)$ by

$$T(y^*) = \sum_{i=1}^{m} y^*(y_i) x_i \qquad (y^* \in \ell_\infty).$$

Then

$$\|T\| = \left\| \sum_{i=1}^{m} x_i \otimes y_i \right\|_{\varepsilon}$$

[5, Examples 4.2]. Moreover, as in the proof of Theorem 2, since $r^* \geqslant 2$, we obtain

$$\left\| R\left(\sum_{i=1}^{m} x_i \otimes y_i \right) \right\|_{r^*} = \left(\sum_{n=1}^{\infty} |\langle T(e_n), x_n^* \rangle|^{r^*} \right)^{1/r^*} \leqslant M \left(\sum_{n=1}^{\infty} \|T(e_n)\|^2 \right)^{1/2}.$$

Since X has the Orlicz property, the identity map on X is absolutely $(2,1)$-summing. So there is a positive constant C such that

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \|T(e_n)\|^2 \right)^{1/2} &\leqslant C \sup_{x^* \in B_{X^*}} \left(\sum_{n=1}^{\infty} |\langle x^*, T(e_n) \rangle| \right) \\ &\leqslant K \sup_{x^* \in B_{X^*}} \|T^*(x^*)\| \\ &= K \|T\| \\ &= K \left\| \sum_{i=1}^{m} x_i \otimes y_i \right\|_{\varepsilon} . \end{aligned}$$

where we have used the Closed Graph Theorem as in [7, page 44]. Therefore,

$$\left\| R\left(\sum_{i=1}^{m} x_i \otimes y_i \right) \right\|_{r^*} \leqslant MK \left\| \sum_{i=1}^{m} x_i \otimes y_i \right\|_{\varepsilon}$$

and then R is continuous with respect to the injective norm.

Define the linear mapping $I\colon\ \ell_{r^*} \to X \otimes_\varepsilon \ell_1$ by $I(e_n) = x_n \otimes e_n$ $(n \in \mathbb{N})$. We show that I is well-defined and continuous. Indeed, given $a = (a_n) \in \ell_{r^*}$, by Hölder's inequality, we have for $k, m \in \mathbb{N}$,

$$\Big\| \sum_{n=k}^{m} a_n x_n \otimes e_n \Big\|_\varepsilon = \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{\ell_\infty}}} \Big| \sum_{n=k}^{m} a_n x^*(x_n) y^*(e_n) \Big|$$
$$\leqslant \Big(\sum_{n=k}^{m} |a_n|^{r^*} \Big)^{1/r^*} \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{\ell_\infty}}} \Big(\sum_{n=k}^{m} |x^*(x_n) y^*(e_n)|^r \Big)^{1/r},$$

and, since (x_n) is weakly r-summable, this implies that

$$I(a) = \sum_{n=1}^{\infty} a_n x_n \otimes e_n \in X \otimes_\varepsilon \ell_1.$$

Using again the fact that (x_n) is weakly r-summable, we have:

$$\|I(a)\|_\varepsilon \leqslant \|a\|_{r^*} \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{\ell_\infty}}} \Big(\sum_{n=1}^{\infty} |x^*(x_n) y^*(e_n)|^r \Big)^{1/r} = \|a\|_{r^*} \|(x_n)_n\|_{w,r},$$

so I is continuous. Easily, $R(I(e_n)) = e_n$ $(n \in \mathbb{N})$, and the proof is complete. □

Theorem 5. *Let X be a Banach space with finite cotype $q \geqslant 2$ containing a normalized weakly q^*-summable sequence. Let Y be a Banach space containing a complemented copy of ℓ_1. Then $X \otimes_\varepsilon Y$ contains a complemented copy of ℓ_q.*

Proof. Since $X \otimes_\varepsilon \ell_1$ is complemented in $X \otimes_\varepsilon Y$, it is enough to consider $Y = \ell_1$. If $q = 2$, the result is true by Theorem 4, since X has the Orlicz property. Suppose $q > 2$. Let $(x_n) \subset X$ be a normalized weakly q^*-summable sequence, which can be assumed to be basic. Then there is a bounded sequence $(x_n^*) \subset X^*$ such that $x_m^*(x_n) = \delta_{mn}$. Now let $R\colon\ X \otimes_\varepsilon \ell_1 \to \ell_q$ be the linear mapping given by

$$R(x \otimes y) = (x_n^*(x) e_n(y))_{n=1}^{\infty}.$$

Clearly, R is well-defined. Given $\sum_{i=1}^{m} x_i \otimes y_i \in X \otimes_\varepsilon \ell_1$, we define $T \in \mathcal{L}(\ell_\infty, X)$ as in the proof of Theorem 4. Since X has cotype $q > 2$, T is absolutely $(q,1)$-summing and there is a positive constant C independent of T such that the absolutely $(q,1)$-summing norm of T satisfies $\pi_{(q,1)}(T) \leqslant C\|T\|$ (see [8, Theorem 11.14(b) and its proof]). So, as in the proof of Theorem 4, R is continuous.

Let $I\colon\ \ell_q \to X \otimes_\varepsilon \ell_1$ be the linear mapping given by $I(e_n) = x_n \otimes e_n$ $(n \in \mathbb{N})$. As in the proof of Theorem 4, I is well-defined and continuous, and $R(I(e_n)) = e_n$ $(n \in \mathbb{N})$, so we are done. □

Theorem 6. *Suppose that Y^* contains a complemented copy of ℓ_1 and X^* has finite cotype $q \geqslant 2$. Let $r > q$ if $q > 2$ and let $r \geqslant 2$ if $q = 2$. If $\mathcal{L}(\ell_r, X^*) \neq \mathcal{K}(\ell_r, X^*)$, then $(X \otimes_\pi Y)^*$ contains a complemented copy of ℓ_r.*

Proof. Since $\mathcal{L}(X, \ell_1)$ is complemented in $\mathcal{L}(X, Y^*) \equiv (X \otimes_\pi Y)^*$ (see, for instance, the proof of [11, Theorem 15]), it is enough to prove the statement for $\mathcal{L}(X, \ell_1)$. Let $(x_n^*) \subset X^*$ be a normalized weakly r^*-summable sequence. We can assume that it is basic. As in the proof of [11, Theorem 12], we can find a sequence $(x_n) \subset X$ such that $x_m^*(x_n) = \delta_{mn}$ and $\|x_n\| \leqslant M$ $(n \in \mathbb{N})$. Let $j\colon \ell_1 \to \ell_r$ be the natural inclusion and let $R\colon \mathcal{L}(X, \ell_1) \to \ell_r$ be given by

$$R(T) = (\langle jT(x_n), e_n \rangle)_{n=1}^\infty .$$

We show that R is a well-defined operator. Indeed, given $T \in \mathcal{L}(X, \ell_1)$, its adjoint $T^* \in \mathcal{L}(\ell_\infty, X^*)$ is absolutely r-summing [8, Theorem 11.14]. Moreover, by the Open Mapping Theorem, there is a positive constant C independent of T such that $\pi_r(T^*) \leqslant C\|T^*\|$, so

$$\left(\sum_{n=1}^\infty |\langle jT(x_n), e_n \rangle|^r \right)^{1/r} \leqslant M \left(\sum_{n=1}^\infty \|T^* j^*(e_n)\|^r \right)^{1/r} \leqslant CM\|T\|.$$

Therefore, R is well-defined and continuous. Now let $I\colon \ell_r \to \mathcal{L}(X, \ell_1)$ be the linear mapping given by

$$I(a)(x) = (x_n^*(x) a_n)_{n=1}^\infty \qquad \text{for each } a = (a_n)_n \in \ell_r.$$

Since (x_n^*) is weakly r^*-summable, we have

$$\sum_{n=1}^\infty |x_n^*(x) a_n| \leqslant \|x\| \|a\|_r \|(x_n^*)_n\|_{w, r^*}.$$

It follows that I is a well-defined operator. Moreover,

$$I(e_m)(x_n) = (x_k^*(x_n)\delta_{mk})_{k=1}^\infty = x_m^*(x_n) e_m = \delta_{mn} e_m,$$

so

$$R(I(e_m)) = (\langle j(I(e_m)(x_n)), e_n \rangle)_{n=1}^\infty = (\langle \delta_{mn} e_m, e_n \rangle)_{n=1}^\infty = e_m,$$

and $I \circ R$ is a projection. □

Remark 7. Under the hypotheses of Theorem 6, the space $\mathcal{K}(X, Y^*)$ contains a complemented copy of ℓ_r.

Indeed, since $\mathcal{K}(X, \ell_1)$ is complemented in $\mathcal{K}(X, Y^*)$, it is enough to show that the range of I is contained in $\mathcal{K}(X, \ell_1)$. Given $a = (a_n) \in \ell_r$ and $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\left(\sum_{n=n_0}^{\infty} |a_n|^r\right)^{1/r} < \frac{\varepsilon}{\|(x_n^*)\|_{w,r^*}}.$$

Hence, by Hölder's inequality,

$$\begin{aligned}\sup_{x\in B_X} \sum_{n=n_0}^{\infty} |x_n^*(x)a_n| &\leqslant \left(\sum_{n=n_0}^{\infty} |a_n|^r\right)^{1/r} \sup_{x\in B_X} \left(\sum_{n=n_0}^{\infty} |x_n^*(x)|^{r^*}\right)^{1/r^*} \\ &< \frac{\varepsilon}{\|(x_n^*)\|_{w,r^*}} \cdot \|(x_n^*)\|_{w,r^*} = \varepsilon,\end{aligned}$$

so $I(a)(B_X)$ is relatively compact in ℓ_1.

The following result improves [11, Corollary 16].

Corollary 8. *Let X and Y be infinite-dimensional $\mathscr{L}_\infty$-spaces such that at least one of them contains a copy of ℓ_1. Then $(X \otimes_\pi Y)^*$ contains a complemented copy of ℓ_2.*

Proof. Suppose that X contains a copy of ℓ_1. Then there is a surjective operator $q\colon X \to \ell_2$ [8, Corollary 4.16]. The operator $q^*\colon \ell_2 \to X^*$ is not compact. Since X is an $\mathscr{L}_\infty$-space, X^* is an $\mathscr{L}_1$-space [14, Theorem III(a)] and then has cotype 2 [8, Corollary 11.7(a)]. Since Y is an infinite-dimensional $\mathscr{L}_\infty$-space, Y^* contains a complemented copy of ℓ_1 [13, Proposition 7.3]. Then it is enough to apply Theorem 6. □

Corollary 9. *Let X and Y be infinite-dimensional $\mathscr{L}_\infty$-spaces. Assume that Y is separable and $Y^* \ncong \ell_1$. Then $(X \otimes_\pi Y)^*$ contains a complemented copy of ℓ_2.*

Proof. Since Y is an infinite-dimensional separable $\mathscr{L}_\infty$-space and $Y^* \ncong \ell_1$, then $Y^* \cong C[0,1]^*$ [1, Theorem 3.1]. Therefore,

$$(X \otimes_\pi Y)^* \cong \mathcal{L}(X, C[0,1]^*) \equiv (X \otimes_\pi C[0,1])^*,$$

and it is enough to apply Corollary 8. □

Corollary 10. *Let X and Y be infinite-dimensional separable $\mathscr{L}_\infty$-spaces. Then the following assertions are equivalent:*
(a) $X^* \cong Y^* \cong \ell_1$;
(b) $(X \otimes_\pi Y)^*$ *has the Dunford-Pettis property;*
(c) $(X \otimes_\pi Y)^*$ *contains no complemented copy of* ℓ_2.

Proof. (a) $\Leftrightarrow$ (b) is proved in [11, Corollary 7];
(b) $\Rightarrow$ (c) is clear;
(c) $\Rightarrow$ (a) follows from Corollary 9. □

Corollary 11. *Let X and Y be infinite-dimensional $\mathscr{L}_\infty$-spaces. Then the following assertions are equivalent:*
(a) *X and Y contain no copy of* ℓ_1;
(b) $(X \otimes_\pi Y)^*$ *has the Schur property;*
(c) $(X \otimes_\pi Y)^*$ *has the Dunford-Pettis property;*
(d) $(X \otimes_\pi Y)^*$ *contains no complemented copy of* ℓ_2;
(e) $X^* \otimes_\varepsilon Y^*$ *has the Schur property;*
(f) $X^* \otimes_\varepsilon Y^*$ *has the Dunford-Pettis property;*
(g) $X^* \otimes_\varepsilon Y^*$ *contains no complemented copy of* ℓ_2.

Proof. (a) $\Rightarrow$ (b). Since X and Y have the Dunford-Pettis property and contain no copy of ℓ_1, their duals X^* and Y^* have the Schur property [6, Theorem 3]. By [17, Corollary 3.4], the space $(X \otimes_\pi Y)^*$ has the Schur property.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a) follows from Corollary 8.

(a) $\Rightarrow$ (e). Since X^* and Y^* have the Schur property, $X^* \otimes_\varepsilon Y^*$ has the Schur property [15].

(e) $\Rightarrow$ (f) $\Rightarrow$ (g) are obvious.

(g) $\Rightarrow$ (a). Suppose that Y contains a copy of ℓ_1. Then there exists a surjection $q\colon Y \to \ell_2$ [8, Corollary 4.16]. The sequence $(q^*(e_n))$ is weakly 2-summable in Y^* and is not norm null. Since Y^* is an $\mathscr{L}_1$-space, it has the Orlicz property. Since X is an infinite-dimensional $\mathscr{L}_\infty$-space, X^* contains a complemented copy of ℓ_1. By Theorem 4, $X^* \otimes_\varepsilon Y^*$ contains a complemented copy of ℓ_2. □

Remark 12.

(a) In the proof of Corollary 11, only the following assumptions on X and Y are used: X and Y are infinite-dimensional and have the Dunford-Pettis property, Y^* has the Orlicz property, and X^* contains a complemented copy of ℓ_1.

(b) If X and Y are $\mathscr{L}_\infty$-spaces and X contains no copy of ℓ_1, then

$$(X \otimes_\pi Y)^* \equiv X^* \otimes_\varepsilon Y^*.$$

Indeed, $(X \otimes_\pi Y)^* \equiv \mathcal{L}(X, Y^*)$. Every operator in $\mathcal{L}(X, Y^*)$ is completely continuous [8, Theorems 3.7 and 2.17] and, since X contains no copy of ℓ_1, also compact [16, page 377]. By the approximation property of X^* (or Y^*) [5, page 306], we have $\mathcal{K}(X, Y^*) \equiv X^* \otimes_\varepsilon Y^*$ [5, Proposition 5.3].

The following result is proved in [11, Corollary 14]:

Theorem 13. *Let X and Y be infinite-dimensional $\mathscr{L}_1$-spaces. The following assertions are equivalent:*

(a) *X and Y have the Schur property;*

(b) *$X \otimes_\varepsilon Y$ has the Schur property;*

(c) *$X \otimes_\varepsilon Y$ has the Dunford-Pettis property.*

We do not know if these assertions are equivalent to:

(d) $X \otimes_\varepsilon Y$ contains no complemented copy of ℓ_2.

As for the dual, it is shown in [12] that, if X and Y are infinite-dimensional $\mathscr{L}_1$-spaces, then $(X \otimes_\varepsilon Y)^*$ contains a complemented copy of ℓ_2. This was proved independently and by different techniques in [3]. Moreover, its isometric subspace $X^* \otimes_\pi Y^*$ [9, Theorem VIII.3.10] also contains a complemented copy of ℓ_2, by a result of [3] (see the introduction to the present paper).

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