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A KOROVKIN TYPE APPROXIMATION THEOREMS VIA $\mathcal{I}$-CONVERGENCE

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Abstract. Using the concept of $\mathcal{I}$-convergence we provide a Korovkin type approximation theorem by means of positive linear operators defined on an appropriate weighted space given with any interval of the real line. We also study rates of convergence by means of the modulus of continuity and the elements of the Lipschitz class.

Keywords: $\mathcal{I}$-convergence, positive linear operator, the classical Korovkin theorem

MSC 2000: 41A10, 41A25

1. Introduction

Chlodovsky [3] was the first to notice that the Bernstein polynomials converge to the average of the left and right limits at the point of simple discontinuity of a function. However, this phenomenon does not always take place for general positive linear approximation operators. One such example was given by Bojanic and Cheng [1] who showed that the Hermit-Fejer interpolation operator does not converge at a point of simple discontinuity. On the other hand, Bojanic and Khan [2] showed that the Cesáro averages of the Hermit-Fejer operator converge to the midpoint of the jump discontinuity. In recent years another form of regular summability transformation has shown to be quite effective in summing non-convergent sequences which may have unbounded subsequences (see [8], [9]). Furthermore, some Korovkin type approximation theorems have been studied via statistical convergence in [5] and [6].

In the present paper we investigate the approximation properties of positive linear operators defined on an appropriate subspace of all real-valued continuous functions on an arbitrary interval of the real numbers by means of $\mathcal{I}$-convergence, which is a more general method than A-statistical convergence. We will also give an application concerning the $\mathcal{I}$-convergence of these operators.

We now recall some definitions and notation used in this paper.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The natural density of K is the nonnegative real number given by $\delta(K) := \lim_j \frac{1}{j}|\{n \leqslant j\colon\ n \in K\}|$ provided the limit exists, where $|B|$ denotes the cardinality of the set B (see [18] for details). Then, a sequence $x := (x_n)$ is called statistically convergent to a number L if for every $\varepsilon > 0$, $\delta\{n \in \mathbb{N}\colon\ |x_n - L| \geqslant \varepsilon\} = 0$. This is denoted by $\text{st-}\lim_n x_n = L$ (see [7], [9]). It is easy to see that every convergent sequence is statistically convergent but not conversely.

Let $A := (a_{jn})$ $(j, n = 1, 2, \ldots)$ be an infinite summability matrix. For a given sequence $x = (x_n)$, the A-transform of x, denoted by $Ax := \{(Ax)_j\}$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each $j \in \mathbb{N}$. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$ [11]. Freedman and Sember [8] introduced the following extension of statistical convergence: Assume that $A = (a_{jn})$ is a nonnegative regular summability matrix. The A-density of a subset K of $\mathbb{N}$ is given by $\delta_A(K) := \lim_j \sum_{n \in K} a_{jn}$ whenever the limit exists. Then, a sequence $x = (x_n)$ is called A-statistically convergent to L if $\delta_A\{n \in \mathbb{N}\colon\ |x_n - L| \geqslant \varepsilon\} = 0$, which is denoted by $\text{st}_A\text{-}\lim_n x_n = L$. It is known that if A is the identity matrix, then A-statistical convergence reduces to the classical convergence, and also if $A = C_1$, the Cesáro matrix of order one, then it coincides with the statistical convergence (see also [10], [13] and [17]).

Let X be a non-empty set. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided that

(i) $\varphi \in \mathcal{I}$;
(ii) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
(iii) $A \in \mathcal{I}$ and if $B \subseteq A$, then $B \in \mathcal{I}$.

An ideal is called nontrivial if $X \notin \mathcal{I}$. Also, a nontrivial ideal in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$ (see [16] for details). In [15], a unifying approach to the concept of statistical convergence has been introduced: Let $\mathcal{I}$ be a nontrivial ideal in $\mathbb{N}$. A sequence $x = (x_n)$ of real numbers is $\mathcal{I}$-convergent to a real number L if for every $\varepsilon > 0$, $\{n\colon\ |x_n - L| \geqslant \varepsilon\} \in \mathcal{I}$, which is denoted by $\mathcal{I}\text{-}\lim_n x_n = L$. We also know from [15] that if $\mathcal{I}$ is the class of all finite subsets of $\mathbb{N}$, then $\mathcal{I}$-convergence reduces to the classical convergence. Furthermore, $\mathcal{I}$-convergence coincides with the A-statistical convergence by taking $\mathcal{I} = \{K \subset \mathbb{N}\colon\ \delta_A(K) = 0\}$, where A is a nonnegative regular summability matrix; of course, choosing $A = C_1$ we have the statistical convergence.

In this section, using $\mathcal{I}$-convergence we prove a Korovkin type approximation theorem for positive linear operators defined on the linear space $C_g(U)$ given by

$$C_g(U) := \left\{ f \in C(U) \colon \lim_{\substack{|x|\to\infty \\ (x\in U)}} \frac{|f(x)|}{(g(|x|))^c} = 0, \text{ for any } c > 0 \right\},$$

where U is an arbitrary interval of $\mathbb{R}$, the set of all real numbers, and g is a nonnegative increasing function on $[0,\infty)$ with $g(0) = 1$, and also $C(U)$ denotes the linear space of all real-valued continuous functions on U (see, for instance, [12]; also [6]). If $U = [a,b]$, then $C(U)$ is a Banach space with the norm $\|f\|_{C[a,b]} := \sup\limits_{x\in[a,b]} |f(x)|$ for $f \in C[a,b]$. Note that when $U = [a,b]$, the notation $C_g(U)$ will stand for $C[a,b]$ with $g(x) \equiv 1$. Let U be an arbitrary interval of $\mathbb{R}$ and let $x \in U$ be fixed. Assume that $\{\mu_{n,x} \colon n \geqslant 1\}$ is a collection of measures defined on $(U,\mathcal{B})$, where $\mathcal{B}$ is the sigma field of Borel measurable subsets of U. Throughout the paper we assume, for $\delta > 0$, that

$$\sup_{n\in\mathbb{N}} \int_{U\setminus U_\delta} g(|y|)\,\mathrm{d}\mu_{n,x}(y) < \infty, \tag{2.1}$$

where $U_\delta := [x-\delta, x+\delta] \cap U$. We now consider operators L_n defined on $C_g(U)$ as follows:

$$L_n(f;x) = \int_U f(y)\,\mathrm{d}\mu_{n,x}(y), \quad n \in \mathbb{N} \text{ and } f \in C_g(U). \tag{2.2}$$

These operators were introduced in [6]. Note that condition (2.1) guarantees that the integral in (2.2) is well-defined. Now we have the following main result.

Theorem 2.1 Main Theorem. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and let U be an arbitrary interval of $\mathbb{R}$. Assume that g is a function such that $f_2(y) = y^2$ is in $C_g(U)$ and for any $\delta > 0$, (2.1) holds. Then, for the operators L_n given by (2.2), the following two statements are equivalent:*

(i) $\mathcal{I}\text{-}\lim\limits_n |L_n(f;x) - f(x)| = 0$ *for all* $f \in C_g(U)$.

(ii) $\mathcal{I}\text{-}\lim\limits_n |L_n(f_i;x) - f_i(x)| = 0$, *where* $f_i(y) = y^i$ *for* $i = 0,1,2$.

Proof. Under the hypotheses, since $f_i \in C_g(U)$ for each $i = 0,1,2$, (i) implies (ii) immediately. Assume now that (ii) holds. Let $f \in C_g(U)$ and fix $x \in U$.

As in the proof of Theorem 1 of [6], it follows from the continuity of f and the Cauchy-Bunyakowsky-Schwarz inequality that

$$\begin{aligned}\int_{U\setminus U_\delta} &|f(y)-f(x)|\,\mathrm{d}\mu_{n,x}(y)\\ &= \int_{U\setminus U_\delta} \chi_{U\setminus U_\delta}|f(y)-f(x)|\,\mathrm{d}\mu_{n,x}(y)\\ &\leqslant \Big[\int_U \chi_{U\setminus U_\delta}\,\mathrm{d}\mu_{n,x}(y)\Big]^{1/p}\Big[\int_{U\setminus U_\delta} |f(y)-f(x)|^q\,\mathrm{d}\mu_{n,x}(y)\Big]^{1/q},\end{aligned}$$

where $p>1$, $1/p+1/q=1$, $U_\delta := [x-\delta, x+\delta]\cap U$ for some $\delta>0$; here χ_A denotes the characteristic function of A. By the hypothesis and the definition of the function g, we conclude that $f\in C_g(U)$ implies that $f^q\in C_g(U)$ and also that there exists a number of K such that

$$\Big[\int_{U\setminus U_\delta} |f(y)-f(x)|^q\,\mathrm{d}\mu_{n,x}(y)\Big]^{1/q} < K.$$

Using this and following the proof of Theorem 1 in [6] we conclude that

$$\begin{aligned}|L_n(f;x)-f(x)| \leqslant \varepsilon + B(x)\big\{&|L_n(f_0;x)-f_0(x)|\\ &+|L_n(f_0;x)-f_0(x)|^{1/p}\\ &+|L_n(f_1;x)-f_1(x)|^{1/p}\\ &+|L_n(f_2;x)-f_2(x)|^{1/p}\big\}\end{aligned} \tag{2.3}$$

holds for every $n\in\mathbb{N}$ and $\varepsilon>0$, where

$$B(x) := \max\Big\{\varepsilon+|f(x)|, \frac{K}{\delta^{2/p}}, K\Big(\frac{|x|}{\delta}\Big)^{2/p}, K\Big(\frac{2|x|}{\delta^2}\Big)^{1/p}\Big\}.$$

Given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$. Consider the following sets:

$$\begin{aligned}H &= \Big\{n\colon\ |L_n(f;x)-f(x)| \geqslant r\Big\},\\ H_1 &= \Big\{n\colon\ |L_n(f_0;x)-f_0(x)| \geqslant \frac{r-\varepsilon}{4B(x)}\Big\},\\ H_2 &= \Big\{n\colon\ |L_n(f_0;x)-f_0(x)|^{1/p} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\},\\ H_3 &= \Big\{n\colon\ |L_n(f_1;x)-f_1(x)|^{1/p} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\},\\ H_4 &= \Big\{n\colon\ |L_n(f_2;x)-f_2(x)|^{1/p} \geqslant \frac{r-\varepsilon}{4B(x)}\Big\}.\end{aligned}$$

Then, it follows from (2.3) that $H \subseteq \bigcup_{j=1}^{4} H_j$. By (ii), $H_j \in \mathcal{I}$ for $j = 1, 2, 3, 4$. So, by the definition of an ideal, $\bigcup_{j=1}^{4} H_j \in \mathcal{I}$, which yields $H \in \mathcal{I}$. So we have

$$\{n\colon\ |L_n(f,x) - f(x)| \geqslant r\} \in \mathcal{I},$$

whence the result. □

We remark that Theorem 2.1 reduces to Theorem 1 in [6] if we take $\mathcal{I} = \{K \subseteq \mathbb{N}\colon\ \delta_A(K) = 0\}$, where A is a nonnegative regular summability matrix. Furthermore, in Theorem 2.1 the choice of $\mathcal{I}$ being the class of all finite subsets of $\mathbb{N}$ gives the next result immediately.

Corollary 2.2. *Under the hypotheses of Theorem* 2.1, *the following two statements are equivalent:*

(i) *For all $f \in C_g(U)$, the sequence $\{L_n(f)\}$ converges pointwise to f.*

(ii) *For each $i = 0, 1, 2$, the sequence $\{L_n(f_i)\}$ converges pointwise to f_i, where $f_i(y) = y^i$.*

On the other hand, the choice of $U = [a, b]$ in Theorem 2.1 leads to

Corollary 2.3. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and let $U = [a, b]$ be a closed and bounded interval of $\mathbb{R}$. Assume that the measure $\mu_{n,x}$ and the operators L_n are given by* (2.1) *and* (2.2). *Then the following two statements are equivalent:*

(i) $\mathcal{I}\text{-}\lim_n \|L_n(f;x) - f(x)\|_{C[a,b]} = 0$ *for all* $\in C[a, b]$.

(ii) $\mathcal{I}\text{-}\lim_n \|L_n(f_i;x) - f_i(x)\|_{C[a,b]} = 0$, *where* $f_i(x) = x^i$ *for* $i = 0, 1, 2$.

Of course, if I is the class of all finite subsets of $\mathbb{N}$, then the classical Korovkin theorem (see [14, p. 20]) follows from Corollary 2.3 at once.

3. An application to Theorem 2.1

In this section, as a special case, we deal with an application to positive linear operators satisfying Theorem 2.1.

When $U = \mathbb{R}$, the Gauss-Weierstrass operators are defined by

$$W_n(f;x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-\frac{1}{2}n(y-x)^2}\,dy. \tag{3.1}$$

Now by using Theorem 2.1 we obtain the following result.

Corollary 3.1. *Consider the function g defined by $g(x) = \mathrm{e}^x$ for $x \geqslant 0$. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then, for all $f \in C_g(\mathbb{R})$,*

$$\mathcal{I}\text{-}\lim_n |W_n(f;x) - f(x)| = 0$$

holds, where the operators W_n are given by (3.1).

P r o o f. Let $x \in \mathbb{R}$ be fixed and $f \in C_g(\mathbb{R})$. It is well-known that $W_n(1;x) = 1$, $W_n(y;x) = x$ and $W_n(y^2;x) = x^2 + 1/n$ for every $n \in \mathbb{N}$. So, it is clear that $\lim_n |W_n(f_i;x) - f_i(x)| = 0$ for $i = 0, 1, 2$ which implies $\mathcal{I}\text{-}\lim_n |W_n(f_i;x) - f_i(x)| = 0$ where $f_i(y) = y^i$ $(i = 0, 1, 2)$. We also know that each Borel measure $\mu_{n,x}$ represents a non-decreasing right continuous function $F_{n,x}$ (see, for instance, [19]). Now, for every $n \in \mathbb{N}$, define a function $F_{n,x}$ on $\mathbb{R}$ by

$$F_{n,x}(y) = \int_{-\infty}^{y} h_{n,x}(t)\,\mathrm{d}t,$$

where $h_{n,x}(t) := \sqrt{\frac{1}{2}n/\pi}\,\mathrm{e}^{-\frac{1}{2}n(t-x)^2}$. So we have $\mathrm{d}F_{n,x}(y)/\mathrm{d}y = h_{n,x}(y)$. For each $n \in \mathbb{N}$, $\mu_{n,x}$ is the Borel measure corresponding to the function $F_{n,x}$. Since $f \in C_g(\mathbb{R})$ is a measurable function, we conclude that

$$\int_{\mathbb{R}} f(y)\,\mathrm{d}\mu_{n,x}(y) = \int_{\mathbb{R}} f(y)\,\mathrm{d}F_{n,x}(y) = \int_{\mathbb{R}} f(y)h_{n,x}(y)\,\mathrm{d}(y)$$

(see [19] for details). Hence, we get

$$W_n(f;x) = \int_{\mathbb{R}} f(y)\,\mathrm{d}\mu_{n,x}(y), \quad n \in \mathbb{N}, \ \ f \in C_g(\mathbb{R}).$$

Thus, the operators W_n have the form given by (2.2).

On the other hand, for any $\delta > 0$ and $n \in \mathbb{N}$ we obtain that

$$\begin{aligned}\sqrt{\frac{n}{2\pi}}\int_{|y-x|\geqslant\delta} \mathrm{e}^{|y|}\mathrm{e}^{-\frac{1}{2}n(y-x)^2}\,\mathrm{d}y &\leqslant \mathrm{e}^{|x|}\sqrt{\frac{n}{2\pi}}\int_{|t|\geqslant\delta} \mathrm{e}^{|t|}\mathrm{e}^{-\frac{1}{2}nt^2}\,\mathrm{d}t\\ &\leqslant 2\mathrm{e}^{|x|}\sqrt{\frac{n}{2\pi}}\int_{0}^{\infty} \mathrm{e}^{t}\mathrm{e}^{-\frac{1}{2}nt^2}\,\mathrm{d}t\\ &\leqslant 2\mathrm{e}^{|x|}\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{t}\mathrm{e}^{-\frac{1}{2}nt^2}\,\mathrm{d}t\\ &= 2\mathrm{e}^{|x|+1/2n}.\end{aligned}$$

This gives

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus\mathbb{R}_\delta} \mathrm{e}^{|y|}\,\mathrm{d}\mu_{n,x}(y) < \infty.$$

Therefore, the operators W_n satisfy all hypotheses of Theorem 2.1, which completes the proof. □

4. Rates of convergence

In this section we compute the rates of $\mathcal{I}$-convergence in Theorem 2.1 by means of the modulus of continuity and elements of the Lipschitz class.

Since most of the approximating operators in approximation theory preserve the function $f_0(y) = 1$, throughout this section we assume that $L_n(f_0; x) = 1$.

Now, let $f \in C_g(U)$. The modulus of continuity of f, denoted by $w(f, \delta)$, is defined to be

$$w(f,\delta) = \sup_{|y-x|<\delta,\, x,y\in U} |f(y) - f(x)|.$$

Then it is clear that for any $\delta > 0$ and each $x, y \in U$

$$|f(y) - f(x)| \leqslant w(f,\delta)\Big(\frac{|y-x|}{\delta} + 1\Big), \quad (f \in C_g(U)). \tag{4.1}$$

Now we have

Theorem 4.1. *For the operators L_n given by* (2.2) *we have, for any $f \in C_g(U)$, $\delta > 0$, $n \in \mathbb{N}$ and for each $x \in U$,*

$$|L_n(f;x) - f(x)| \leqslant 2w(f, \delta_n),$$

where

$$\delta_n := \delta_n(x) = \sqrt{|L_n(f_2;x) - f_2(x)| + 2|x||L_n(f_1;x) - f_1(x)|}. \tag{4.2}$$

Proof. Let $f \in C_g(U)$ and $x \in U$. Since $L_n(f_0;x) = 1$, using (4.1) and the linearity and monotonicity of the operators L_n, we get, for any $\delta > 0$ and $n \in \mathbb{N}$, that

$$\begin{aligned} |L_n(f;x) - f(x)| &\leqslant L_n(|f(y) - f(x)|;x) \\ &\leqslant w(f,\delta)\Big\{\frac{1}{\delta}L_n(|y-x|;x) + 1\Big\}. \end{aligned}$$

Now applying the Cauchy-Schwarz inequality for positive linear operators, we obtain

$$|L_n(f;x) - f(x)| \leqslant w(f,\delta)\Big\{\frac{1}{\delta}\sqrt{L_n(\varphi_x;x)} + 1\Big\}, \tag{4.3}$$

where $\varphi_x(y) = (y-x)^2$. Since

$$\begin{aligned} L_n(\varphi_x;x) &= L_n(f_2;x) - 2xL_n(f_1;x) + x^2 \\ &\leqslant |L_n(f_2;x) - f_2(x)| + 2|x||L_n(f_1;x) - f_1(x)|, \end{aligned} \tag{4.4}$$

we conclude from (4.3) and (4.4) that

$$|L_n(f;x) - f(x)| \leqslant w(f,\delta) \times \left\{\frac{1}{\delta}\sqrt{|L_n(f_2;x) - f_2(x)| + 2|x||L_n(f_1;x) - f_1(x)|} + 1\right\}. \tag{4.5}$$

Choosing $\delta := \delta_n(x)$ given by (4.2) the proof follows from (4.5). □

We will now study the rate of convergence of the positive linear operators L_n by means of the elements of the Lipschitz class $\mathrm{Lip}_M(\alpha)$ for $0 < \alpha \leqslant 1$. We recall that a function $f \in C_g(U)$ belongs to $\mathrm{Lip}_M(\alpha)$ if the inequality

$$|f(y) - f(x)| \leqslant M|y - x|^\alpha \quad (y, x \in U) \tag{4.6}$$

holds. So we get

Theorem 4.2. *For the operators L_n given by* (2.2) *we have, for any $f \in \mathrm{Lip}_M(\alpha)$, $n \in \mathbb{N}$ and for each $x \in U$,*

$$|L_n(f;x) - f(x)| \leqslant M\delta_n^{\alpha/2},$$

where $\delta_n := \delta_n(x)$ is given by (4.2).

Proof. Let $f \in \mathrm{Lip}_M(\alpha)$ and $x \in U$. Then (4.6) implies, for all $n \in \mathbb{N}$, that

$$\begin{aligned}|L_n(f;x) - f(x)| &\leqslant L_n(|f(y) - f(x)|;x)\\ &\leqslant ML_n(|y-x|^\alpha;x).\end{aligned}$$

Now applying the Hölder inequality with $p = 2/\alpha$, $q = 2/(2-\alpha)$ we get

$$|L_n(f;x) - f(x)| \leqslant M\{L_n(\varphi_x;x)\}^{\alpha/2},$$

and also by (4.4),

$$|L_n(f;x) - f(x)| \leqslant M\{|L_n(f_2;x) - f_2(x)| + 2|x||L_n(f_1;x) - f_1(x)|\}^{\alpha/2}.$$

So the proof follows from the choice $\delta_n := \delta_n(x)$ given by (4.2). □

Concluding Remarks. Under condition (ii) of Theorem 2.1, observe that $\mathcal{I}\text{-}\lim_n \delta_n(x) = 0$ for each $x \in U$, which also guarantees that $\mathcal{I}\text{-}\lim_n w(f;\delta_n) = 0$ for all $f \in C_g(U)$. Therefore, Theorems 4.1 and 4.2 give us the rates of $\mathcal{I}$-convergence in the approximation $L_n(f;x)$ of $f(x)$.

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