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ON k-PAIRABLE GRAPHS FROM TREES

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Abstract. The concept of the k-pairable graphs was introduced by Zhibo Chen (On k-pairable graphs, Discrete Mathematics 287 (2004), 11–15) as an extension of hypercubes and graphs with an antipodal isomorphism. In the same paper, Chen also introduced a new graph parameter $p(G)$, called the pair length of a graph G, as the maximum k such that G is k-pairable and $p(G) = 0$ if G is not k-pairable for any positive integer k. In this paper, we answer the two open questions raised by Chen in the case that the graphs involved are restricted to be trees. That is, we characterize the trees G with $p(G) = 1$ and prove that $p(G \,\square\, H) = p(G) + p(H)$ when both G and H are trees.

Keywords: k-pairable graph, pair length, Cartesian product, G-layer, tree

MSC 2000: 05C75, 05C60, 05C05

1. Introduction

In [2], N. Graham, R. C. Entringer and L. A. Székely proved that for every spanning tree T of the hypercube Q_k, there is an edge of Q_k outside T whose addition to T forms a cycle of length at least $2k$. They also extended the result to graphs with an antipodal isomorphism. Recently, Chen [1] further extended their result to a greater class of graphs which he introduced as the k-pairable graphs. Chen pointed out that the k-pairable graphs have some special kind of symmetry that is different from the well-known type of symmetry such as vertex-transitivity, edge-transitivity, or distance transitivity. In the same paper, Chen also introduced a new graph parameter $p(G)$, the pair length of a graph G, and raised some open questions, which motivated our work here.

All graphs in this paper are connected and simple if not specified. We use the similar terminology here as in [1]. For example, the distance between two vertices x and y in a graph G is denoted as $d_G(x, y)$ or simply as $d(x, y)$ if it will cause no confusion. We write $x \operatorname{adj} y$ to mean that the two vertices x and y are adjacent.

The *eccentricity* of a vertex u in a graph G is $e(u) = \max_{v\in V(G)} d(u,v)$. The *diameter* of G is $d(G) = \max_{u\in V(G)} e(u)$. The *radius* of G is $r(G) = \min_{u\in V(G)} e(u)$. If $e(u) = r(G)$, then u is called a *central vertex* of G. The *center* of G, denoted as $C(G)$, is the set of all central vertices of G. It is well known that the center of a tree is either a vertex or a pair of adjacent vertices. The *degree* of a vertex u in G, denoted by $\deg(u)$, is the number of vertices that are adjacent to u in G. An *isomorphism* of a graph G is a one to one map $f\colon V(G) \to V(G)$ such that $u \operatorname{adj} v$ in G if and only if $f(u) \operatorname{adj} f(v)$ in G. A graph G has an *antipodal isomorphism* if for every vertex $v \in V(G)$, $e(v) = d(G)$ and there is a unique $\bar{v} \in V(G)$ such that $d(v,\bar{v}) = d(G)$ and the map $\varphi\colon V(G) \to V(G)$ defined by $\varphi(v) = \bar{v}$ is an isomorphism of G.

Definition 1.1 ([1]). Let k be a positive integer. A graph G is said to be k-pairable if

1. $V(G)$ can be partitioned into disjoint pairs, that is, $V(G) = P_1 \cup P_2 \cup \ldots \cup P_n$ with $|P_i| = 2$ for all i, and $P_i \cap P_j = \emptyset$ for all $i \neq j$. If two vertices x and y are in the same pair P_i, then we say x is the mate of y and y is the mate of x, which is denoted by $x = y'$ and $y = x'$.
2. $d(x,x') \geqslant k$, for every $x \in V(G)$, and
3. for any vertices x, y of G, $x \operatorname{adj} y$ implies $x' \operatorname{adj} y'$.

Any partition of $V(G)$ satisfying the above three conditions is called a k-pair partition of G. From the definition, we can see that for any k-pair partition Π of G, there is an induced isomorphism $f_\Pi\colon G \to G$ that maps each vertex x to its mate x', i.e., $f_\Pi(x) = x'$ and $f_\Pi(x') = x$ for each vertex x of G. This isomorphism does not fix any vertex of G since k is a positive integer.

Definition 1.2 ([1]). The pair length of a graph G, denoted as $p(G)$, is the maximum k such that G is k-pairable; $p(G) = 0$ if G is not k-pairable for any positive integer k.

It has been pointed out in [1] that the order of G has to be even to have the pair length $p(G) > 0$. For example, any complete graph K_{2n} has $p(K_{2n}) = 1$; any cycle C_{2n} has $p(C_{2n}) = n$; any path P_{2n} has $p(P_{2n}) = 1$. The pair length $p(G)$ measures the maximum distance between a subgraph G_1 of G induced by half the vertices of G and its isomorphic subgraph G_2 of G induced by the other half of $V(G)$ in the sense that $d(G_1,G_2) = \min_{g\in V(G_1)} d(g,g')$ where g' is the isomorphic image of g.

In [1], an upper bound for $p(G)$ was given, that is, $p(G) \leqslant \min\{r(G), \frac{1}{2}|V(G)|\}$. Properties of the k-pairable Cartesian product graphs were also studied. Recall that the Cartesian product of two graphs G and H is denoted by $G \,\square\, H$. It has the vertex set $V(G)\times V(H)$ and $(g_1,h_1)\operatorname{adj}(g_2,h_2)$ if either $g_1 = g_2$ and $h_1 \operatorname{adj} h_2$ in H or $h_1 = h_2$

and $g_1 \operatorname{adj} g_2$ in G. Chen showed that $p(G)+p(H) \leqslant p(G \square H) \leqslant r(G)+r(H)$ and he also gave a sufficient condition for $p(G \square H) = p(G) + p(H)$, that is, if $p(G) = r(G)$ and $p(H) = r(H)$, then $p(G \square H) = p(G) + p(H) = r(G) + r(H)$. But $p(G) = r(G)$ and $p(H) = r(H)$ is not a necessary condition for $p(G \square H) = p(G) + p(H)$. For example, let G be a path with $2n$ vertices, then $p(G \square K_2) = 2 = 1 + 1 = p(G) + p(K_2)$, but $1 = p(G) \neq r(G) = n$.

The following open questions were raised by Chen in [1]:

1. How to characterize the graphs for which $p(G) = k$?
2. Is it true that $p(G \square H) = p(G) + p(H)$ in general?

In this paper, we shall answer these questions when both G and H are trees.

2. Preliminaries

The following lemmas give some basic facts about the k-pairable graphs. (Note that we always assume $k > 0$ from now on.)

Lemma 2.1. *Let G be a k-pairable graph. Then for an arbitrary k-pair partition Π of G, the following hold:*

1. *$\deg(u) = \deg(u')$ for any vertex u of G where u' is the mate of u. In particular, if G is a tree and $e(u) = d(G)$, then $\deg(u) = \deg(u') = 1$.*
2. *$d(u, v) = d(u', v')$ for any vertices u and v of G where u', v' are the mates of u, v respectively.*
3. *$e(u) = e(u')$ for any vertex u of G where u' is the mate of u.*

Proof. 1. $\deg(u) = \deg(u')$ is trivial since $u \operatorname{adj} x$ if and only if their mates $u' \operatorname{adj} x'$ by the definition of a k-pairable graph G. If $e(u) = d(G)$, then $d(u, v) = d(G)$ for some $v \in V(G)$. Also $e(v) = d(u, v)$ since $d(G) \geqslant e(v) \geqslant d(u, v) = d(G)$. Assume that G is a tree and let P be the shortest path joining u and v. If $\deg(u) \neq 1$, then there is a vertex $x \in G - P$ and x is adjacent to u. Then $x \cup P$ is a path joining x and v and it is the unique path between x and v since G is a tree. It follows that $d(x, v) = d(u, v) + 1 > d(u, v)$. This is a contradiction since $e(v) = d(u, v) \geqslant d(x, v)$. Therefore, $\deg(u) = \deg(u') = 1$.

2. Suppose that $d(u, v) = n$ for some $n \geqslant 1$ and $uu_1 \ldots u_{n-1}v$ is a shortest path joining u and v in G, then $u'u'_1 \ldots u'_{n-1}v'$ is a path of length n joining their mates u' and v' in G where u'_i is the mate of u_i for $1 \leqslant i \leqslant n - 1$. It must be a shortest path joining u' and v'. Otherwise, there is a path $u's_1 \ldots s_{m-1}v'$ joining u' and v' in G with length m less than n. It implies that $us'_1 \ldots s'_{m-1}v$ is a path joining u and v where s'_i is the mate of s_i for $1 \leqslant i \leqslant m - 1$. This path has length m less than n, which contradicts the assumption that $d(u, v) = n$.

3. Suppose $e(u) = d(u, v)$ for some vertex v in G. If $e(u') = d(u', v')$ where u', v' are the mates of u, v respectively, then $e(u') = e(u)$ since $d(u, v) = d(u', v')$. If $e(u') \neq d(u', v')$, then there exists $w \in V(G)$ such that $e(u') = d(u', w) > d(u', v')$. Let w' be the mate of w. Then $e(u) \geqslant d(u, w') = d(u', w) > d(u', v') = d(u, v) = e(u)$. This is a contradiction. □

Lemma 2.2. *Let G be a k-pairable graph. Then we have the following:*

1. *If $|V(G)| > 2$ and $G_1 = G - \bigcup\{u \in V(G)\colon \deg(u) = 1\}$, then $p(G_1) \geqslant p(G) > 0$. The equality holds when G is a tree.*
2. *Let H be an induced subgraph of G. If there is a k-pair partition Π of G such that H does not have any two vertices in the same pair of Π, then H is isomorphic to some induced subgraph of $G - H$.*
3. *If $p(G) = r(G)$ and u is a central vertex of G, then for any $p(G)$-pair partition Π of G, $d(u, u') = r(G)$ where u' is the mate of u in Π.*

Proof. 1. It is easy to see that $G_1 = G - \bigcup\{u \in V(G)\colon \deg(u) = 1\}$ is an induced subgraph of G. For any $p(G)$-pair partition of G, there is an inherited $p(G)$-pair partition of G_1 since the mate of a vertex of G with degree 1 is still a vertex of G with degree 1. Therefore, $p(G_1) \geqslant p(G)$. If G is a tree, then we can delete the vertices with degree 1 repeatedly until a graph G_n is obtained that is either a vertex or an edge. It is easy to see that G_n is not a vertex if $p(G) \geqslant k > 0$ since $p(G_n) \geqslant p(G_{n-1}) \geqslant \ldots \geqslant p(G_1) \geqslant p(G) > 0$. Therefore, G_n is an edge and $1 = p(G_n) \geqslant p(G) > 0$. It follows that $p(G) = 1$ and the equality holds for each step from G to G_n by deleting the vertices of degree 1.

2. Let H' be the subgraph induced by the mates of vertices of H in the partition Π. Then H' is an induced subgraph of $G - H$ since H is an induced subgraph of G. If f_Π is the isomorphism of G induced by the partition Π, then it is clear that the restriction of f_Π to H is an isomorphism between H and H'.

3. For any $p(G)$-pair partition Π of G, $e(u) \geqslant d(u, u') \geqslant p(G) = r(G)$ where u' is the mate of u. Since u is a central vertex of G, then $e(u) = r(G)$. It follows that $d(u, u') = r(G)$. □

By part 1 of Lemma 2.1, we immediately have

Corollary 2.3. *If T is a star with more than 2 vertices, then $p(T) = 0$.*

From part 1 of Lemma 2.2, we can easily get the following result of Chen in [1].

Corollary 2.4. *If T is a tree, then $p(T) = 0$ or 1.*

This result tells that in order to answer Chen's first open question for trees, we only need to characterize the trees T with $p(T) = 1$.

3. Main results

Theorem 3.1. *A tree T has $p(T) = 1$ if and only if there is an edge $e = xy$ in T such that there exists an isomorphism f between the two connected components of $T - e$ satisfying $f(x) = y$.*

Proof. We first prove the sufficiency. Assume that there is an edge $e = xy$ in T such that there exists an isomorphism f between the two connected components $T - e$ satisfying $f(x) = y$. Let H_1 and H_2 denote these two components with x in H_1 and y in H_2. Then $\bigcup_{u \in V(H_1)} \{(u, f(u))\}$ gives a partition of $V(T)$ as disjoint pairs, since $f(u) \neq f(v)$ whenever $u \neq v$. For each vertex u in T, let the mate of u be $u' = f(u)$ if u is in H_1 and $u' = f^{-1}(u)$ if u is in H_2. Consider two adjacent vertices u_1 and u_2 in T. If both of them are in H_1, then their mates $f(u_1)$ and $f(u_2)$ are adjacent. If both of them are in H_2, then their mates $f^{-1}(u_1)$ and $f^{-1}(u_2)$ are adjacent. If they are in different components, then the two adjacent vertices u_1, u_2 must be x, y. It follows that u_1 is the mate of u_2 and u_2 is the mate of u_1. It is trivial that the mates of u_1 and u_2 are adjacent too. Furthermore, $\min_{u \in V(H_1)} d(u, f(u)) = d(x, y) = 1$. Therefore, $V(T) = \bigcup_{u \in V(H_1)} \{(u, f(u))\}$ is a 1-pair partition of T. This implies that $p(T) \geqslant 1$. On the other hand, $p(T) \leqslant 1$ for any tree T by Corollary 2.4. Therefore, $p(T) = 1$. This proves the sufficiency.

As to the necessity, we prove the following stronger statement: Let T be a tree with $p(T) = 1$. Then for any 1-pair partition Π of T, there is an edge $e = xy$ in T such that the two connected components of $T - e$ are isomorphic under f satisfying $f(x) = y$, where f is the isomorphism induced by Π.

The statement can be proved by the mathematical induction on $|V(T)|$. Obviously, $p(T) = 1$ implies that $|V(T)| \geqslant 2$. It is trivial if $|V(T)| = 2$ since $T = K_2$. Assume that it is true for tree T with less than $2n$ vertices where $n > 1$. Let T be a tree with $2n$ vertices, and let Π be a 1-pair partition of T. Take a vertex u of T with $e(u) = d(T)$. By Lemma 2.1, $\deg(u) = \deg(u') = 1$ where u' is the mate of u in the partition Π. Since $|V(T)| \geqslant 4$, it is clear that u' is not adjacent to u. Let v and w be the neighbors of u and u' in T respectively. Since $u \operatorname{adj} v$ implies that $u' \operatorname{adj} v'$ where v' is the mate of v and since $\deg(u') = 1$, we must have $w = v' \neq v$. Let $T' = T - u - u'$. Then T' is a tree with a 1-pair partition Π' inherited from the

1-pair partition Π of T, and so $p(T') = 1$. By the induction hypothesis, there is an edge $e = xy$ of T' such that the two connected components H_1' and H_2' of $T' - e$ are isomorphic under the isomorphism f' induced by Π' satisfying $f'(x) = y$. Without loss of generality, we may assume that v and x are in H_1' and v' and y are in H_2'. Since $T' = T - u - u'$, one of the two connected components of $T - e$ is obtained from H_1' by attaching the pendant edge joining the vertex u with v of H_1', and the other component of $T-e$ is obtained from H_2' by attaching the pendant edge joining the vertex u' with v' of H_2'. Extend f' to a map f on $V(T)$ such that $f|_{V(T')} \equiv f'$, $f(u) = u'$ and $f(u') = u$. Since $f'(v) = v'$ and $f'(v') = v$, it is easy to see that f is an isomorphism of T induced by Π, the two connected components of $T - e$ are isomorphic under f, and $f(x) = y$. □

Remarks.

1. It is not difficult to see that if a tree T has an edge $e = xy$ such that there exists an isomorphism f between the two connected components of $T - e$ satisfying $f(x) = y$, then the center of T is $\{x, y\}$.

2. If $p(T) = 1$, then the center of the tree T must be a pair of adjacent vertices. However, the converse is not true, which can be seen from Fig. 1.

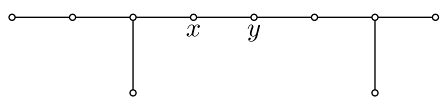

Figure 1. The center of the tree T is a pair of adjacent vertices but $p(T) = 0$

Theorem 3.1 solves Chen's first open question for trees. The next theorem is to solve Chen's second open question for trees.

Theorem 3.2. *If G and H are trees, then $p(G \,\square\, H) = p(G) + p(H)$.*

Before proving the theorem, we first prove some lemmas below.

For any graph G, we call its subgraph induced by the center $C(G)$ the *center subgraph* of G and denote it as $\langle C(G)\rangle$.

Lemma 3.3. *For any graphs G and H, $\langle C(G \,\square\, H)\rangle = \langle C(G)\rangle \,\square\, \langle C(H)\rangle$. In particular, if both G and H are trees, then $\langle C(G \,\square\, H)\rangle$ is either K_1 (if $\langle C(G)\rangle = \langle C(H)\rangle = K_1$), or K_2 (if $\{\langle C(G)\rangle, \langle C(H)\rangle\} = \{K_1, K_2\}$), or C_4 (if $\langle C(G)\rangle = \langle C(H)\rangle = K_2$).*

P r o o f. In the Cartesian product graph $G \,\square\, H$, $d_{G\square H}((u, v), (x, y)) = d_G(u, x) + d_H(v, y)$. It follows that $e_{G\square H}(u, v) = e_G(u) + e_H(v)$ and $r(G \,\square\, H) = r(G) + r(H)$.

Therefore, (u, v) is a central vertex of $G \square H$ if and only if u is a central vertex of G and v is a central vertex of H. That is, $\langle C(G \square H)\rangle = \langle C(G)\rangle \square \langle C(H)\rangle$.

The center of a tree is either a vertex or a pair of adjacent vertices. It follows that the center subgraph of a tree is either K_1 or K_2. If both G and H are trees, then either $\langle C(G \square H)\rangle = K_1 \square K_1 \cong K_1$ (when $\langle C(G)\rangle = \langle C(H)\rangle = K_1$); or $\langle C(G \square H)\rangle = K_1 \square K_2 \cong K_2$ (when $\{\langle C(G)\rangle, \langle C(H)\rangle\} = \{K_1, K_2\}$); or $\langle C(G \square H)\rangle = K_2 \square K_2 \cong C_4$ (when $\langle C(G)\rangle = \langle C(H)\rangle = K_2$). □

Given a Cartesian product graph $G \square H$, for a vertex h of H, we use $G \square \{h\}$ to denote the induced subgraph $\langle\{(g, h) \colon g \in V(\mathrm{G})\}\rangle$ and call it the G-layer at position h. Similarly, for a vertex g of G we call $\{g\} \square H \doteq \langle\{(g, h) \colon h \in V(\mathrm{H})\}\rangle$ the H-layer at position g. Note that a G-layer (H-layer) is an isomorphic copy of G (H), and that any two adjacent vertices in $G \square H$ must be either in the same G-layer or in the same H-layer.

Lemma 3.4. *Let f be an isomorphism of $G \square H$. If there is a G-layer that is mapped onto a G-layer by f, then each G-layer is mapped onto a G-layer by f, and each H-layer is mapped onto an H-layer by f.*

Proof. Let $G \square \{h\}$ be the G-layer such that $f(G \square \{h\}) = G \square \{h'\}$ for some $h' \in V(H)$. If h_1 is any vertex adjacent to h in H, then each vertex (g, h_1) in the G-layer $G \square \{h_1\}$ is adjacent to the corresponding vertex (g, h) in the G-layer $G \square \{h\}$. Thus, it is not difficult to see that any two adjacent vertices in $G \square \{h_1\}$ must be mapped into the same G-layer. Since G is connected, then $f(G \square \{h_1\}) = G \square \{h_1'\}$ for some h_1' adjacent to h' in H. It follows that each G-layer is mapped onto a G-layer since H is connected.

To prove that each H-layer is mapped onto an H-layer, we first prove the following: For any two adjacent vertices x and y in the same H-layer $\{g\} \square H$ of $G \square H$, $f(x)$ and $f(y)$ must be in the same H-layer.

From the proved fact that each G-layer is mapped onto a G-layer by f, it is clear that vertices in distinct G-layers must be mapped into distinct G-layers. For any two adjacent vertices x and y in the same H-layer of $G \square H$, x and y are in distinct G-layers, hence $f(x)$ and $f(y)$ must be in distinct G-layers. And x adj y implies $f(x)$ adj $f(y)$ in $G \square H$. Note that any two adjacent vertices in $G \square H$ must be either in the same G-layer or in the same H-layer. So $f(x)$ and $f(y)$ must be in the same H-layer.

Since H is connected, it is then easily seen that for any vertices x and y in the same H-layer, $f(x)$ and $f(y)$ must be in the same H-layer. That is, each H-layer is mapped onto an H-layer by f. □

Lemma 3.5. *Let G and H be trees. Let f_Π be the induced isomorphism of a k-pair partition Π of $G \Box H$. Then each G-layer is mapped onto a G-layer by f_Π.*

Proof. From the given condition, $G \Box H$ is k-pairable. Thus by Lemma 3.3, the center subgraph $\langle C(G \Box H)\rangle$ is either K_2 (if $\{\langle C(G)\rangle, \langle C(H)\rangle\} = \{K_1, K_2\}$), or C_4 (if $\langle C(G)\rangle = \langle C(H)\rangle = K_2$).

Case 1. $\langle C(G \Box H)\rangle = K_2$. Without loss of generality, we can denote the center of $G \Box H$ as $C(G \Box H) = \{(g_1, h), (g_2, h)\}$, where g_1 and g_2 are the pair of adjacent central vertices of G and h is the unique central vertex of H. Then $f_\Pi((g_1, h)) = (g_2, h)$ since the mate of a central vertex is a central vertex by Lemma 2.1. We will show that the G-layer $G \Box \{h\}$ is mapped onto itself by f_Π. This is trivial when $|V(G)| = 2$. So we may assume $|V(G)| > 2$. Since G is connected, we only need to show that if (g, h) is adjacent to (g_1, h) in $G \Box \{h\}$, then $f_\Pi(g, h) \in G \Box \{h\}$. If $f_\Pi(g, h) \notin G \Box \{h\}$, then $f_\Pi(g, h) = (g_2, h_1)$ for some vertex h_1 adjacent to h in H. Then f_Π maps the set $\{(g, h), (g_1, h), (g_2, h)\}$ into the set $\{(g_2, h_1), (g_2, h), (g_1, h), (g_1, h_1)\}$ that induces a four cycle in $G \Box H$. This is impossible since the set $\{(g, h), (g_1, h), (g_2, h)\}$ is not contained in any four cycle in $G \Box H$ since G is a tree. Hence, $f(G \Box \{h\}) = G \Box \{h\}$. Then by Lemma 3.4, each G-layer is mapped onto a G-layer by f_Π.

Case 2. $\langle C(G \Box H)\rangle = C_4$. Let $C(G \Box H) = \{(g_1, h_1), (g_2, h_1), (g_2, h_2), (g_1, h_2)\}$, where g_1 and g_2 are the pair of adjacent central vertices of G, and h_1 and h_2 are the pair of adjacent central vertices of H. Since the mate of a central vertex is a central vertex by Lemma 2.1, we distinguish three subcases:

Subcase 1. $f_\Pi(g_1, h_1) = (g_2, h_1)$. Then by the proof of Case 1, $f_\Pi(G \Box \{h_1\}) = G \Box \{h_1\}$.

Subcase 2. $f_\Pi(g_1, h_1) = (g_1, h_2)$. Similarly as above, we can see that $f_\Pi(\{g_1\} \Box H) = \{g_1\} \Box H$.

Subcase 3. $f_\Pi(g_1, h_1) = (g_2, h_2)$. Then $f_\Pi(g_2, h_1) = (g_1, h_2)$. We can show that $f_\Pi(G \Box \{h_1\}) = G \Box \{h_2\}$ similarly.

Therefore, by Lemma 3.4, we see that each G-layer is mapped onto a G-layer by f_Π. □

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We first show that $p(G \Box H) = p(G) + p(H) = 0$ when $p(G) = p(H) = 0$, by contradiction. Assume that $p(G \Box H) > 0$. Let Π be an arbitrary $p(G \Box H)$-pair partition of $G \Box H$ and f_Π be its induced isomorphism. For each vertex h in H, $f_\Pi(G \Box \{h\}) = G \Box \{h'\}$ where h' is some vertex in H by Lemma 3.5. We will show that there must be some G-layer that is mapped onto itself by f_Π. Otherwise, $h' \neq h$ for all $h \in V(H)$. Define $f_H\colon V(H) \to V(H)$ such that $f_H(h) = h'$ if $f_\Pi(G \Box \{h\}) = G \Box \{h'\}$. It is easy to see that f_H is well defined.

Note that $f_H(h) = h'$ if and only if $f_H(h') = h$ since $f_\Pi(G \,\square\, \{h\}) = G \,\square\, \{h'\}$ if and only if $f_\Pi(G \,\square\, \{h'\}) = G \,\square\, \{h\}$. If $h_1 \neq h_2$ in $V(H)$, then $f_H(h_1) \neq f_H(h_2)$ since $f_\Pi(G \,\square\, \{h_1\}) \neq f_\Pi(G \,\square\, \{h_2\})$. If h_1 is adjacent to h_2 in H, then each vertex in $f_\Pi(G \,\square\, \{h_1\}) = G \,\square\, \{h_1'\}$ is adjacent to the corresponding vertex in $f_\Pi(G \,\square\, \{h_2\}) = G \,\square\, \{h_2'\}$. It follows that $f_H(h_1) = h_1'$ is adjacent to $f_H(h_2) = h_2'$ in H. Similarly, we can show that if $f_H(h_1) = h_1'$ is adjacent to $f_H(h_2) = h_2'$ in H, then h_1 is adjacent to h_2 in H. Therefore, f_H is an isomorphism of H. Let $k_H = \min_{h \in V(H)} d(h, h')$. Then $k_H > 0$ since $h' \neq h$ for all $h \in V(H)$. This implies that f_H is an isomorphism induced by a k_H-pair partition of H, which is impossible since $p(H) = 0$. Hence, there must be some $h \in V(G)$ such that $f_\Pi(G \,\square\, \{h\}) = G \,\square\, \{h\}$. So there is a $p(G \,\square\, H)$-pair partition of $G \,\square\, \{h\}$ inherited from Π. Since $G \cong G \,\square\, \{h\}$, $p(G) = p(G \,\square\, \{h\}) \geqslant p(G \,\square\, H) > 0$. This contradicts the fact that $p(G) = 0$. Therefore, $p(G \,\square\, H) = 0$ when $p(G) = p(H) = 0$.

Now we prove the remaining case where: $p(G) > 0$ or $p(H) > 0$. Without loss of generality, we may assume that $p(G) = 1$ (note that any tree has its pair length 0 or 1). It has been proved in [1] that $p(G \,\square\, H) \geqslant p(G) + p(H)$. So $p(G \,\square\, H) = k > 0$ and we only need to prove that $p(G \,\square\, H) \leqslant p(G) + p(H)$. Let Π be an arbitrary k-pair partition of $G \,\square\, H$ and let f_Π be its induced isomorphism. We will show $p(G \,\square\, H) \leqslant p(G) + p(H)$ using mathematical induction on $|V(G)|$.

If $|V(G)| = 2$, then we can denote $V(G) = \{g_1, g_2\}$. If $p(H) = 1$, then $\langle C(G \,\square\, H)\rangle = C_4$ by Lemma 3.3. It follows that $p(G \,\square\, H) \leqslant 2$ since the mate of a central vertex is a central vertex. Thus $p(G \,\square\, H) \leqslant p(G) + p(H)$. If $p(H) = 0$, then there must be some G-layer $G \,\square\, \{h\}$ such that $f_\Pi(G \,\square\, \{h\}) = G \,\square\, \{h\}$ by the proof in the first paragraph. It follows that $f_\Pi(g_1, h) = (g_2, h)$, and so $p(G \,\square\, H) \leqslant 1$, i.e., $p(G \,\square\, H) \leqslant p(G) + p(H)$.

Assume that $p(G \,\square\, H) \leqslant p(G) + p(H)$ when $|V(G)| < 2n$ where $n > 1$. If $|V(G)| = 2n$, let $G' = G - \{u \in V(G)\colon \deg(u) = 1\}$. Then G' is a tree with $p(G') = 1$ by Lemma 2.2. By the induction hypothesis, we can have $p(G' \,\square\, H) = p(G') + p(H) = p(G) + p(H)$. By Lemma 3.5, it is not difficult to see that there is a $p(G \,\square\, H)$-pair partition of $G' \,\square\, H$ inherited from Π. This implies that $p(G \,\square\, H) \leqslant p(G' \,\square\, H) = p(G) + p(H)$. This completes the mathematical induction. Therefore, $p(G \,\square\, H) = p(G) + p(H)$. $\square$

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