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SOME KURZWEIL-HENSTOCK-TYPE INTEGRALS AND THE WIDE DENJOY INTEGRAL

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Dedicated to the memory of Ralph Henstock (1923–2007)

Abstract. Kurzweil-Henstock integrals related to local systems and the wide Denjoy integral are discussed in the frame of their comparability and compatibility.

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1. Introduction

In the theory of integration on real line, various nonabsolute integrals have been considered. The most important seems to be the Denjoy-Perron integral (known also as the restricted Denjoy integral). This integral, discovered in early years of the 20th century, was intended to encompass both the Lebesgue (absolute) integral and the Newton integral (i.e., the integral that recovers a primitive from an everywhere differentiable function). Moving from less to more general integrals, we can observe that the Denjoy-Perron integral is the stage at which generalizations split into two wings. The first is the wing of descriptive definitions, based upon relaxing the notion of absolute continuity, in the direction traced Lusin with his definition of wide Denjoy integral via ACG-functions (Denjoy-Khintchine integral). While Denjoy-Khintchine integral is of the same age as the Denjoy-Perron integral, the second wing has its origin in the late fifties of the 20th century. At that time, Jaroslav Kurzweil and, independently, Ralph Henstock introduced a Riemann-type definition of integral that is equivalent to Denjoy-Perron integral. This definition opened the way for many generalizations that use Riemann integral sums, thus forming our second wing. Such an observation calls for a connection between the two wings described above.

The theory of Kurzweil-Henstock-type integrals on the real line can be presented in a (quite abstract) fashion of integration with respect to a base [26]. However, for a more subtle investigation, it is convenient to restrict the concern to a less abstract attitude. For this, there are at least two ways useful: free-point bases (called also *Busemann-Feller bases*) and bases related to *local systems*. Some Busemann-Feller bases, like the dyadic base or, more generally, the $\mathscr{P}$-adic base [2], play an important role in Harmonic Analysis. However, indefinite integrals or primitives considered in the theory of Kurzweil-Henstock integrals with respect to these bases are interval functions rather than ordinary point functions. In consequence, continuity, differentiation and variations related to these bases, considered for a point function, neglect the value at some points, which is not the case for bases related to local systems. And this is essential. For example, an indefinite continuous Kurzweil-Henstock integral related to dyadic base, extended continuously to all the subintervals of $\langle 0,1\rangle$, and considered as a point function $F\colon \langle 0,1\rangle \to \mathbb{R}$, can fail both to be VBG and to satisfy Lusin's condition $\mathscr{N}$ [22], while just for the path dyadic base this is not possible. Such features would not suit our purpose (a link to wide Denjoy integral). Thus, we confine our considerations to bases related to local systems.

One says that two integrals are *comparable*, if one of these integrals is a generalization of the other. One says that two integrals are *compatible* (or *consistent*), if for each function integrable in both senses the integrals agree. In the present paper we discuss Kurzweil-Henstock integrals related to local systems and the wide Denjoy integral in the frame of their comparability and compatibility.

2. Local systems

By *a local system* [25] we mean a family $\Delta = \{\Delta(x)\}_{x\in\mathbb{R}}$ such that each $\Delta(x)$ is a nonvoid collection of subsets of $\mathbb{R}$ with the following properties:

(i) $\{x\} \notin \Delta(x)$;

(ii) if $S \in \Delta(x)$, then $x \in S$;

(iii) if $S \in \Delta(x)$ and $R \supset S$, then $R \in \Delta(x)$;

(iv) if $S \in \Delta(x)$ and $\delta > 0$, then $(x-\delta, x+\delta) \cap S \in \Delta(x)$.

Every S belonging to $\Delta(x)$ we call a *path* leading to x. A function $\mathscr{C}$ on $\mathbb{R}$ such that $\mathscr{C}(x) \in \Delta(x)$ for each x, is called *a choice*. Given a choice $\mathscr{C}$, we write $(I,x) \in \beta_{\mathscr{C}}$ iff $x \in I$ and both extremities of I are in $\mathscr{C}(x)$.

We say that a local system Δ is *filtering down*, if $S_1 \cap S_2 \in \Delta(x)$ for each $x \in \mathbb{R}$ and each two paths $S_1, S_2 \in \Delta(x)$. In the sequel we will consider such local systems only.

We say that a local system Δ is *bilateral* if, $(x-\delta, x) \cap S \neq \emptyset$ and $(x, x+\delta) \cap S \neq \emptyset$ for each $x \in \mathbb{R}$, $S \in \Delta(x)$, $\delta > 0$.

We say that a local system Δ satisfies the *internal intersection condition* (abbr. iIC), the *intersection condition* (abbr. IC), the *weak intersection condition* (abbr. wIC), if for every choice $\mathscr{C}$, there exists a *gauge* δ; i.e., a $\delta \colon \mathbb{R} \to (0, \infty)$, such that if

$$0 < y - x < \min\{\delta(x), \delta(y)\},$$

then respectively

$$\mathscr{C}(x) \cap \mathscr{C}(y) \cap (x, y) \neq \emptyset, \quad \mathscr{C}(x) \cap \mathscr{C}(y) \cap \langle x, y\rangle \neq \emptyset, \quad \mathscr{C}(x) \cap \mathscr{C}(y) \neq \emptyset.$$

Given Δ, we say that a function $f \colon \mathbb{R} \to \mathbb{R}$ is Δ-continuous at $x \in \mathbb{R}$, if for each $\varepsilon > 0$ there exists an $S \in \Delta(x)$ such that

$$f(x) - \varepsilon < \liminf_{t \to x,\, t \in S} f(t) \leqslant \limsup_{t \to x,\, t \in S} f(t) < f(x) + \varepsilon.$$

We say that f is Δ-differentiable at x to a number g, if for each $\varepsilon > 0$ there exists an $S \in \Delta(x)$ such that

$$g - \varepsilon < \liminf_{t \to x,\, t \in S \setminus \{x\}} \frac{f(t) - f(x)}{t - x} \leqslant \limsup_{t \to x,\, t \in S \setminus \{x\}} \frac{f(t) - f(x)}{t - x} < g + \varepsilon.$$

With the aid of the filtering down property, one shows that the number g, if it exists, is unique.

As the most significant examples of local systems let us mention the neighbourhood local system [28, *Example* 1], the density local system Δ_{ap} [28, *Example* 2], the dyadic local system Δ_{d} [28, *Example* 3], the proximal density local system [8], [20], and the $\mathscr{I}$-density local system [8], [16]. All of them (except the proximal density local system) satisfy iIC.

3. Kurzweil-Henstock integrals related to local systems

Let $\langle a,b\rangle$ be a nondegenerate compact interval. By a *division* in $\langle a,b\rangle$ we understand any finite collection $\mathscr{P}$ of pairs (I,x) (the so-called *tagged intervals*), where I is a compact subinterval of $\langle a,b\rangle$ and its *tag* $x \in I$, such that for all $(I,x),(J,y) \in \mathscr{P}$, if $(I,x) \neq (J,y)$, then the intervals I and J are nonoverlapping. If δ is a gauge, then we say that $\mathscr{P}$ is δ*-fine*, if $I \subset \big(x-\delta(x), x+\delta(x)\big)$ for every $(I,x) \in \mathscr{P}$. Given local system Δ, if $\mathscr{C}$ is a choice, then we say that $\mathscr{P}$ is $\mathscr{C}$*-fine*, if $\mathscr{P} \subset \beta_{\mathscr{C}}$. We say that $\mathscr{P}$ is anchored in a set E if $x \in E$ for every $(I,x) \in \mathscr{P}$. If $\bigcup_{(I,x)\in\mathscr{P}} I = \langle a,b\rangle$, then the division $\mathscr{P}$ is called a *partition* of $\langle a,b\rangle$.

For a local system Δ, we will say that it has the *partitioning property*, if for every subinterval of real line and every choice $\mathscr{C}$ there exists a $\mathscr{C}$-fine partition of this subinterval. In the sequel, when speaking about an H_Δ-integral or about a G_Δ-integral, we will tacitly assume Δ has the partitioning property. Recall the following theorem due to Thomson [25]. *If Δ is bilateral and satisfies* IC*, then it has the partitioning property.*

Remark 3.1. It is not known if the proximal density local system has the partitioning property. It satisfies wIC, but not IC [8].

Definitions 3.2. We call a function $f\colon \langle a,b\rangle \to \mathbb{R}$ Δ*-integrable in the Kurzweil-Henstock sense* (abbr. H_Δ*-integrable*), if there exists a real number $\mathbf{I}$ (its integral) such that for any $\varepsilon > 0$ there is a choice $\mathscr{C}$ with the property that for every $\mathscr{C}$-fine partition $\pi = \{(I_i,x_i)\}_{i=1}^n$ of $\langle a,b\rangle$,

$$\left|\sum_{i=1}^n f(x_i)|I_i| - \mathbf{I}\right| < \varepsilon. \tag{1}$$

We call a function $f\colon \langle a,b\rangle \to \mathbb{R}$ Δ*-integrable in the sense of Gordon* (abbr. G_Δ*-integrable*), if there exists a real number $\mathbf{I}$ and a choice $\mathscr{C}$ such that for any $\varepsilon > 0$ there is a gauge δ such that for every $\mathscr{C}$-fine and δ-fine partition $\pi = \{(I_i,x_i)\}_{i=1}^n$ of $\langle a,b\rangle$, we have the inequality (1).

The H_Δ-integral was introduced explicitly by Wang and Ding [28], but its idea is in fact due to Thomson [25], [26]. The G_Δ-integral was considered first by Gordon [9] (for a particular case of the density local system). Thanks to the filtering down property of Δ, both integrals are uniquely defined. As a consequence of (iv) from the definition of a local system, each G_Δ-integrable function is H_Δ-integrable and the integrals coincide. In general, the two integrals are not equivalent (for example for the $\mathscr{I}$-density local system; see [16], [23]), however for the density local system the equivalence holds [15].

Let an $f\colon \langle a,b\rangle \to \mathbb{R}$ be H_Δ-integrable. Using the partitioning property one shows that f is H_Δ-integrable on each subinterval of $\langle a,b\rangle$. The indefinite H_Δ-integral of f, $F\colon \langle a,b\rangle \to \mathbb{R}$, is defined as

$$F(x) = \int_a^x f.$$

One shows that F is a Δ-continuous function. Moreover,

Theorem 3.3. *F is Δ-differentiable to $f(x)$ at almost every point x of $\langle a,b\rangle$.*

Theorem 3.4. *If a set $E \subset \langle a,b\rangle$ has $|E| = 0$, then for each $\varepsilon > 0$ there exists a choice $\mathscr{C}$ such that for every $\mathscr{C}$-fine division $\{(\langle a_i,b_i\rangle, x_i)\}_{i=1}^n$, anchored in E,*

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

For proofs see [28, *Theorem* 5].

4. Wide Denjoy integral

Let $F\colon E \to \mathbb{R}$. If a subset $A \subset E$ is nonvoid, then we set $\omega_F(A) = \sup F(A) - \inf F(A)$. By $\mathscr{D}_F$ we denote the set of points of E at which the function F is discontinuous. We will say that F satisfies the condition $\mathscr{N}$, if $|F(N)| = 0$ for each $N \subset E$ of null measure. F is said to be an AC-function, if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for any pairwise nonoverlapping intervals $\langle a_1,b_1\rangle, \dots, \langle a_n,b_n\rangle$, with both endpoints in E,

$$\sum_{i=1}^{n} (b_i - a_i) < \eta \quad \Rightarrow \quad \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

F is said to be a VB-function, if there is a number $M > 0$ such that for any pairwise nonoverlapping intervals $\langle a_1,b_1\rangle, \dots, \langle a_n,b_n\rangle$ with both endpoints in E,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < M.$$

The lower bound for all such M's we call the *variation* of F. F is said to be an ACG- and a VBG-function, if there exists a sequence $(E_n)_{n=1}^\infty$ with $E = \bigcup\limits_{n=1}^{\infty} E_n$ such that for each n, $F \restriction E_n$ is an AC- and a VB-function respectively. The F is said to be respectively an $[ACG]$- and a $[VBG]$-function if, moreover, the sets E_n above are assumed to be closed.

Definition 4.1. We call a function $f\colon \langle a,b\rangle \to \mathbb{R}$ *Denjoy integrable in the wide sense* (abbr. D-*integrable*), if there exists a continuous ACG-function $F\colon \langle a,b\rangle \to \mathbb{R}$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in \langle a,b\rangle$. The integral of f is defined as $(D)\int_a^b f = F(b) - F(a)$.

$F'_{\mathrm{ap}}(x)$ denotes the approximate derivative (i.e., the Δ_{ap}-derivative) of F at x.

5. Noncomparability of the H_Δ-integral and the wide Denjoy integral

Let $\mathbb{C}$ stand for Cantor ternary set, let

$$\{I_i^{(n)} = \big(a_i^{(n)}, b_i^{(n)}\big)\colon\ i = 1,\dots,2^n\}$$

be the family of open intervals contiguous to $\mathbb{C}$ of rank $n = 0,1,\dots$. By $c_i^{(n)}$ we denote the centre of $I_i^{(n)}$. Tolstov [27] proved that the (defined almost everywhere) derivative of the function

$$F(x) = \begin{cases} 0 & \text{for } x \in \mathbb{C},\\ 1/n & \text{for } x = c_i^{(n)},\\ \text{linear} & \text{on intervals } \big\langle a_i^{(n)}, c_i^{(n)}\big\rangle \text{ and } \big\langle c_i^{(n)}, b_i^{(n)}\big\rangle, \end{cases} \tag{2}$$

is not $H_{\Delta_{\mathrm{ap}}}$-integrable (in fact, Tolstov did not consider the approximate Kurzweil-Henstock integral, not known at that time, but Burkill's approximately continuous integral), while it is D-integrable. We will show that Tolstov's function is not H_Δ-integrable for a wide class of local systems Δ. To this purpose we need the notion of *strong porosity* [29]. Let $x \in \mathbb{R}$, $S \subset \mathbb{R}$. By $\lambda((a,b),S)$ we mean the length of the longest open interval contained in (a,b) that shares no point with the set S. We say that at x the set S is right strongly porous, if

$$\lambda_S^+(x) = \limsup_{h\to 0^+} \frac{\lambda((x,x+h),S)}{h} = 1.$$

Similarly we define $\lambda_S^-(x)$ and left strong porosity of S at x. We say that a local system $\Delta = \{\Delta(x)\}_{x\in\mathbb{R}}$ is right (left) strongly porous at $x \in \mathbb{R}$, if some path $S \in \Delta(x)$ is right (resp. left) strongly porous at x.

Example 5.1. If a local system Δ is either not left strongly porous or not right strongly porous at each point $x \in \mathbb{C}$, then the derivative F' of F from (2) is not H_Δ-integrable.

P r o o f. Suppose F' is H_Δ-integrable. Let G be the indefinite H_Δ-integral of F'. Notice that for each $I_i^{(n)}$ there is a constant C such that $G = F + C$ on $I_i^{(n)}$; C depends on $I_i^{(n)}$.

Consider an arbitrary choice $\mathscr{C} = \{\mathscr{C}(x)\}_{x\in\mathbb{R}}$. Put

$$E_n^+ = \Big\{x \in \mathbb{C}\colon\ \lambda^+_{\mathscr{C}(x)}(x) < 1 - \frac{1}{n}\Big\},\quad E_n^- = \Big\{x \in \mathbb{C}\colon\ \lambda^-_{\mathscr{C}(x)}(x) < 1 - \frac{1}{n}\Big\}.$$

Δ is either not left strongly porous or not right strongly porous at every point x of $\mathbb{C}$, so $\mathbb{C} = \bigcup\limits_{n=2}^{\infty} (E_n^+ \cup E_n^-)$. Fix an n and take $x \in E_n^+$. We write E_{nk}^+, $k \in \mathbb{N}$, if for each $h < 1/k$ one has $\lambda((x, x+h), \mathscr{C}(x)) < (1 - 1/n)h$. The set E_{nk}^- is defined in an analogous manner. We have obtained $\mathbb{C} = \bigcup\limits_{k=1}^{\infty}\bigcup\limits_{n=1}^{\infty} (E_{nk}^- \cup E_{nk}^+)$. The Baire Category Theorem implies there are n and k and a portion $I \cap \mathbb{C} \neq \emptyset$ of $\mathbb{C}$, such that one of the sets E_{nk}^- or E_{nk}^+, is dense in $I \cap \mathbb{C}$. We may assume that E_{nk}^+ is dense in $I \cap \mathbb{C}$ and $|I| < 1/k$. There is an integer N such that the interval I contains an interval contiguous to $\mathbb{C}$ of rank N, say $I_1^{(N)}$, an interval of rank $N+1$, say $I_1^{(N+1)}$, an interval of rank $N+2$, say $I_1^{(N+2)}$, etc.

Pick an $x_i \in E_{nk}^+ \cap I$, $i = N, N+1, \ldots$, with

$$0 < \frac{a_1^{(i)} - x_i}{c_1^{(i)} - a_1^{(i)}} < \frac{2n-1}{4n^3}. \tag{3}$$

Let us define the point $\tilde{x}_i \in \langle a_1^{(i)}, c_1^{(i)}\rangle$ as follows:

- if $G(a_1^{(i)}) \leqslant G(x_i) \leqslant G(c_1^{(i)})$, choose a unique point $\tilde{x}_i \in \langle a_1^{(i)}, c_1^{(i)}\rangle$ such that $G(x_i) = G(\tilde{x}_i)$;
- if $G(x_i) < G(a_1^{(i)})$, put $\tilde{x}_i = a_1^{(i)}$;
- if $G(x_i) > G(c_1^{(i)})$, put $\tilde{x}_i = c_1^{(i)}$.

Now, we must split the proof into two cases.

C a s e I. $\tilde{x}_i \geqslant a_1^{(i)} + (2n)^{-1}(c_1^{(i)} - a_1^{(i)})$.

Put $\tilde{z}_i = \tilde{x}_i - (2n)^{-1}(\tilde{x}_i - a_1^{(i)})$. We have

$$\tilde{z}_i - a_1^{(i)} = \Big(1 - \frac{1}{2n}\Big)(\tilde{x}_i - a_1^{(i)}) \geqslant \Big(1 - \frac{1}{2n}\Big)\frac{1}{2n}(c_1^{(i)} - a_1^{(i)}),$$

whence by (3)

$$\frac{\tilde{z}_i - a_1^{(i)}}{\tilde{z}_i - x_i} = 1 - \frac{a_1^{(i)} - x_i}{\tilde{z}_i - x_i} \geqslant 1 - \frac{a_1^{(i)} - x_i}{\tilde{z}_i - a_1^{(i)}} \geqslant 1 - \frac{4n^2}{2n-1}\cdot\frac{a_1^{(i)} - x_i}{c_1^{(i)} - a_1^{(i)}} > 1 - \frac{1}{n}.$$

Since $\tilde{z}_i - x_i < 1/k$ and $x_i \in E^+_{nk}$, $\lambda((x_i, \tilde{z}_i), \mathscr{C}(x_i)) < (1-1/n)(\tilde{z}_i - x_i)$. Thus, there is a $z_i \in \mathscr{C}(x_i) \cap (a_1^{(i)}, \tilde{z}_i)$. We have

$$\tilde{x}_i - z_i > \tilde{x}_i - \tilde{z}_i = \frac{1}{2n}(\tilde{x}_i - a_1^{(i)}) \geqslant \frac{1}{4n^2}(c_1^{(i)} - a_1^{(i)}).$$

So, since G is linear on $\langle a_1^{(i)}, c_1^{(i)} \rangle$,

$$G(x_i) - G(z_i) \geqslant G(\tilde{x}_i) - G(z_i) > \frac{1}{4n^2}(G(c_1^{(i)}) - G(a_1^{(i)})) = \frac{1}{4n^2 i}.$$

Case II. $\tilde{x}_i < a_1^{(i)} + (2n)^{-1}(c_1^{(i)} - a_1^{(i)})$.
Put $\tilde{z}_i = a_1^{(i)} + 2(c_1^{(i)} - a_1^{(i)})(3n)^{-1}$. Again by (3) we have

$$\frac{c_1^{(i)} - \tilde{z}_i}{c_1^{(i)} - x_i} = 1 - \frac{\tilde{z}_i - a_1^{(i)}}{c_1^{(i)} - x_i} - \frac{a_1^{(i)} - x_i}{c_1^{(i)} - x_i} > 1 - \frac{2}{3n} - \frac{2n-1}{4n^3} \geqslant 1 - \frac{11n^2}{12n^3} > 1 - \frac{1}{n}.$$

Since $c_1^{(i)} - x_i < 1/k$ and $x_i \in E^+_{nk}$, $\lambda\big((x_i, c_1^{(i)}), \mathscr{C}(x_i)\big) < (1-1/n)(c_1^{(i)} - x_i)$. Thus, there is a $z_i \in \mathscr{C}(x_i) \cap \big(\tilde{z}_i, c_1^{(i)}\big)$. We have

$$z_i - \tilde{x}_i > \tilde{z}_i - a_1^{(i)} - \frac{1}{2n}(c_1^{(i)} - a_1^{(i)}) = \frac{1}{n}\left(\frac{2}{3} - \frac{1}{2}\right)(c_1^{(i)} - a_1^{(i)}) = \frac{1}{6n}(c_1^{(i)} - a_1^{(i)})$$

and, consequently,

$$G(z_i) - G(x_i) \geqslant G(z_i) - G(\tilde{x}_i) > \frac{1}{6ni} > \frac{1}{4n^2 i}.$$

Take an index N_0 such that $\sum\limits_{i=N}^{N_0} 1/i > 4n^2$. The division $\mathscr{P} = \{(\langle x_i, z_i \rangle, x_i) \colon i = N, \ldots, N_0\}$ is $\mathscr{C}$-fine and it is anchored in $\mathbb{C}$ (if necessary, the x_i's may be shifted to be closer to $a_1^{(i)}$'s so that the intervals from $\mathscr{P}$ would not overlap; it does not inflict the argument). We have

$$\sum_{i=N}^{N_0} |G(z_i) - G(x_i)| > 1.$$

Since the choice $\mathscr{C}$ was arbitrary and $|\mathbb{C}| = 0$, we have arrived at a contradiction with Theorem 3.4. □

Remark 5.2. Notice that for the foregoing example we do not use the compatibility of H_Δ- and D-integrals. Moreover, instead of Cantor's ternary set $\mathbb{C}$, another nonempty perfect nullset would work (with the definition of F analogous to (2)).

For a converse example, assume that for the local system Δ and some point $x \in \mathbb{R}$, at least one of the following two conditions holds:

$\dagger_{\mathrm{r}}$ there is an $S \in \Delta(x)$ such that for every $h > 0$, $\lambda((x, x+h), S) > 0$;

$\dagger_{\mathrm{l}}$ there is an $S \in \Delta(x)$ such that for every $h > 0$, $\lambda((x-h, x), S) > 0$.

Then, it is an easy matter to construct an everywhere Δ-differentiable function which is differentiable (in the ordinary sense) everywhere except x and which fails to be continuous at the x. The Δ-derivative of the so-defined function is H_Δ-integrable (even G_Δ-integrable), while not D-integrable, since it is D-integrable on each interval not containing x and its primitive has no limit at x.

Recalling Remark 5.2, we can state

Corollary 5.3. *Let for a local system Δ*

- *the set of points at which Δ is both left and right strongly porous, contain no nonempty perfect set;*
- *at some $x \in \mathbb{R}$ one of the conditions $\dagger_{\mathrm{r}}$, $\dagger_{\mathrm{l}}$ be satisfied.*

Then the H_Δ-integral and the D*-integral are noncomparable.*

Notice that the two arguments used to establish Corollary 5.3, have completely different nature. The first is of variational kind, while the other uses just a continuity argument. And it must be so. Under a supplementary assumption, it can be shown that each function with continuous indefinite H_Δ-integral is D-integrable; see Theorem 7.1 below.

Remarks 5.4. All the examples of Δ mentioned on page 3, except the neighbourhood local system, satisfy both $\dagger_{\mathrm{r}}$ and $\dagger_{\mathrm{l}}$ at each $x \in \mathbb{R}$. Also, all of them, now except the dyadic local system, are neither left nor right strongly porous at each $x \in \mathbb{R}$. For Δ_{d} however (as for each $\mathscr{P}$-adic local system [3] with the sequence $\mathscr{P}$ being bounded), one can construct a nonempty perfect set C with Δ_{d} being neither left nor right strongly porous at each $x \in C$. Recently, we have learnt (from Professor Valentin Skvortsov) a construction of a continuous ACG-function which is not an indefinite H_Δ-integral for Δ being $\mathscr{P}$-adic local system associated with any given sequence $\mathscr{P}$. This construction will possibly appear elsewhere.

Now, we turn to the question under what assumption the D-integral and the H_Δ-integral are compatible. One may answer this question by defining an integral more general than both these integrals. If such an integral exists, it implies they are compatible. In this section we introduce four integrals that are straight generalizations to D-integral and investigate the relations between them (in terms of local systems).

The next section will provide the reader with assumptions under which these generalizations cover also H_Δ-integral.

Consider the following four classes of measurable functions defined on an $\langle a, b\rangle$.

- $\mathscr{L}_1$: $[ACG]$-functions,
- $\mathscr{L}_2$: $[VBG]$-functions satisfying $\mathscr{N}$,
- $\mathscr{L}_3$: ACG-functions,
- $\mathscr{L}_4$: VBG-functions satisfying $\mathscr{N}$.

For each i, the class $\mathscr{L}_i$ is a linear space. For $i = 1, 3$ it is evident, while for $i = 2$ it was verified by Sarkhel and Kar [21, *Corollary* 3.1.1 and *Theorem* 3.6], for $i = 4$ by Ene [5, *Corollary* 2]. It is well-known [18] that each member of $\mathscr{L}_i$ is approximately differentiable almost everywhere.

Let $\mathscr{F}$ be a linear space of Baire one Darboux functions defined on $\langle a, b\rangle$.

Definition 6.1. We call a function $f\colon \langle a, b\rangle \to \mathbb{R}$ $\mathscr{F}_i$-integrable, $i = 1, 2, 3, 4$, if there exists a function $F \in \mathscr{F}_i = \mathscr{L}_i \cap \mathscr{F}$ on $\langle a, b\rangle$ such that $F'_{\rm ap}(x) = f(x)$ for almost all $x \in \langle a, b\rangle$. The $\mathscr{F}_i$-integral of f is defined as $F(b) - F(a)$.

The $\mathscr{F}_i$-integral is uniquely defined since $\mathscr{L}_i$ is a linear space and since the following lemma holds.

Lemma 6.2 [14, Theorem 1]**.** *Assume that an $F\colon \langle a, b\rangle \to \mathbb{R}$ satisfies $\mathscr{N}$ and is Darboux. If $F'(x) \geqslant 0$ at almost every point $x \in \langle a, b\rangle$ at which the function F is differentiable* (in the usual sense), *then F is nondecreasing.*

Let us shortly recall the background of Definition 6.1. It was originally investigated in the particular case of $\mathscr{F}$ being the class of approximately continuous functions. In this case, the $\mathscr{F}_1$-integral is well-known under the name Kubota integral [12], [17]; the $\mathscr{F}_3$-integral is sometimes called the $\mathrm{AK}_{\mathscr{N}}$-integral of Gordon [10]; the $\mathscr{F}_3$-integral has been mentioned by Kubota in [11].

Definition 6.1 in its more general version appeared first in [13] for the $\mathscr{F}_1$-integral, in [6] for the $\mathscr{F}_2$- and $\mathscr{F}_4$-integrals. Sarkhel's TD-integral [19] is equivalent to the $\mathscr{F}_2$-integral, for $\mathscr{F}$ being the class of T-continuous $[VBG]$-functions, since T-continuity implies the Darboux property and each $[VBG]$-function is Baire one. For various equivalences for $\mathscr{F}_i$-integrals see [6], [7].

Lemma 6.3. *Let for a local system Δ, at each point $x \in E$, $\operatorname{cl} E \supset \langle a, b\rangle$, both the conditions $\dagger_{\mathrm{r}}$ and $\dagger_{\mathrm{l}}$ hold. Then, there is a perfect set C, nowhere dense in $\langle a, b\rangle$, and a countable set $X = \{x_n\}_{n=1}^{\infty} \subset C$, dense in C, with the displayed property:*

For each n there exist two sequences of intervals contiguous to C: $(R_k(x_n))_{k=1}^{\infty} \searrow x_n$, $(L_k(x_n))_{k=1}^{\infty} \nearrow x_n$, such that for some closed subintervals $\widetilde{L}_k \subset L_k$, $\widetilde{R}_k \subset R_k$, one has $\{x_n\} \cup \bigcup_{k=1}^{\infty} \big(\widetilde{L}_k \cup \tilde{R}_k\big) \in \Delta(x_n)$. Moreover, the intervals R_k and L_k are different for different x_n's and united give the entire set $\langle a, b\rangle \setminus C$.

Construction. Put $\mathscr{S}_0 = \{\langle a, b\rangle\}$. Pick an arbitrary $x_1 \in E \cap (a, b)$. From the assumption there is a path $S \in \Delta(x_1)$ such that for every positive h, $\lambda((x_1, x_1 + h), S) > 0$ and $\lambda((x_1 - h, x_1), S) > 0$. Choose points

$$b > d_1 > b_1 > a_1 > c_1 > d_2 > b_2 > a_2 > c_2 > d_3 > b_3 > \dots,$$

tending to x_1 and such that $(b_{k+1}, a_k) \cap S = \emptyset$ for each $k = 1, 2, \dots$. Set $R_k = (c_k, d_k)$, $\widetilde{R}_k = \langle a_k, b_k\rangle$. Symmetrically we construct interval sequences $L_k = (e_k, f_k)$ and $\widetilde{L}_k$. By (iii) we have $P_1 = \{x_1\} \cup \bigcup_{k=1}^{\infty} \big(\widetilde{L}_k \cup \widetilde{R}_k\big) \in \Delta(x_1)$. Denote $O_1 = \{x_1\} \cup \bigcup_{k=1}^{\infty} (L_k \cup R_k)$ and let $\mathscr{S}_1$ be the family of closed compound intervals of the set $\langle a, b\rangle \setminus O_1$. Enumerate these intervals according to the following recipe. The first is $K_1^{(1)} = \langle a, e_1\rangle$, the second $K_2^{(1)} = \langle d_1, b\rangle$, the third is $K_3^{(1)} = \langle f_1, e_2\rangle$, and so on. Next, we pick an $x_2 \in E \cap \operatorname{int} K_1^{(1)}$ and like for x_1 we choose within $K_1^{(1)}$, two couples of sequences $(\widetilde{R}_k(x_2) \subset R_k(x_2))_k \searrow x_2$ and $(\widetilde{L}_k(x_2) \subset L_k(x_2))_k \nearrow x_2$, of closed and open intervals, such that $P_2 = \{x_2\} \cup \bigcup_{k=1}^{\infty} \big(\widetilde{L}_k(x_2) \cup \widetilde{R}_k(x_2)\big) \in \Delta(x_2)$; we set $O_2 = \{x_2\} \cup \bigcup_{k=1}^{\infty} \big(L_k(x_2) \cup R_k(x_2)\big)$. The collection $\mathscr{S}_2 = \{K_i^{(2)}\}_{i=1}^{\infty}$ is formed by closed intervals that are components of the set $K_1^{(1)} \setminus O_2$. In an analogous way the interval $K_2^{(1)} \in \mathscr{S}_1$ is treated; (we pick an $x_3 \in E \cap \operatorname{int} K_2^{(1)}$, $P_3 \subset O_3 \subset \operatorname{int} K_2^{(1)}$, form an $\mathscr{S}_3 = \{K_i^{(3)}\}_{i=1}^{\infty}$). By induction, step by step we can do so for intervals

$$K_1^{(2)}, K_3^{(1)}, K_2^{(2)}, K_1^{(3)}, K_4^{(1)}, K_3^{(2)}, K_2^{(3)}, K_1^{(4)}, K_5^{(1)},$$

etc. It is seen that the set $C = \langle a, b\rangle \setminus \bigcup_{n=1}^{\infty} \big(O_n \setminus \{x_n\}\big)$ is perfect and nowhere dense and that it contains all x_n's. Moreover, each interval contiguous to C in $\langle a, b\rangle$ is of the kind (c_k, d_k) or (e_k, f_k). From the construction, we notice that C has the desired property.

Theorem 6.4. *Let a local system Δ satisfy the condition from Lemma* 6.3. *There exist Δ-continuous functions F_1, F_2, F_3 such that*

- $F_1 \in \mathscr{L}_2 \setminus \mathscr{L}_3$;
- $F_2 \in \mathscr{L}_3 \setminus \mathscr{L}_2$;
- $F_3 \in \mathscr{L}_2 \cap \mathscr{L}_3 \setminus \mathscr{L}_1$.

Proof. Take the sets C and X from Lemma 6.3. First, we define functions F_i, $i = 1, 2, 3$, on C and next we extend them to the entire segment $\langle a, b\rangle$.

In order to define F_2 and F_3 we have to enumerate the collection $\mathscr{A}$ of open intervals contiguous to C in $\langle a, b\rangle$. Fix any member of $\mathscr{A}$ and designate it as $I_1^{(1)}$. As $I_1^{(2)}$ and $I_2^{(2)}$ we designate one of the longest members of $\mathscr{A}$ that are between a and $I_1^{(1)}$, $I_1^{(1)}$ and b, respectively. Next, as $I_1^{(3)}$, $I_2^{(3)}$, $I_3^{(3)}$, $I_4^{(3)}$ we designate one of the longest members of $\mathscr{A}$ that are between a and $I_1^{(2)}$, $I_1^{(2)}$ and $I_1^{(1)}$, $I_1^{(1)}$ and $I_2^{(2)}$, $I_2^{(2)}$ and b, respectively. And so on. Each member of $\mathscr{A}$ is designated in this process. Denote by $J_1^{(n)}, \ldots, J_{2^n}^{(n)}$ the component intervals of the set $\langle a, b\rangle \setminus \bigcup_{k=1}^{n} \bigcup_{i=1}^{2^{k-1}} I_i^{(k)}$. Inductively, we choose a subset $\{\xi_i^{(n)}\}_{i,n} \subset X$ as follows:

- we pick a $\xi_i^{(1)} \in X \cap J_i^{(1)}$, $i = 1, 2$;
- for an $n \geqslant 1$, we set $R = \{\xi_i^{(k)} \colon k = 1, \ldots, n,\ i = 1, \ldots, 2^k\}$ and choose $\xi_i^{(n+1)} \in X \cap J_i^{(n+1)} \setminus R$, $i = 1, \ldots, 2^{n+1}$.

We put

$$F_2(x) = \begin{cases} \dfrac{1}{n} & \text{for } x = \xi_i^{(n)}, \\ 0 & \text{for } x \in C \setminus \{\xi_i^{(n)}\}_{i,n}, \end{cases} \qquad F_3(x) = \begin{cases} \dfrac{1}{2^{2n}} & \text{for } x = \xi_i^{(n)}, \\ 0 & \text{for } x \in C \setminus \{\xi_i^{(n)}\}_{i,n}. \end{cases}$$

Let an F_1 be defined via

$$F_1(x) = \sum_{n\colon x_n < x} \frac{1}{2^n},$$

where $\{x_n\}_{n=1}^{\infty}$ is a re-enumeration of $\{\xi_i^{(n)}\}_{i,n}$.

Now, let F_i, $i = 1, 2, 3$, be extended by $F_i(x) = F_i(x_n)$ for $x \in \bigcup_{k=1}^{\infty} \big(\widetilde{L}_k(x_n) \cup \widetilde{R}_k(x_n)\big)$, F_i linear on the closures of differences $L_k(x_n) \setminus \widetilde{L}_k(x_n)$, $R_k(x_n) \setminus \widetilde{R}_k(x_n)$, $n, k = 1, 2, \ldots$. Since $\{x_n\} \cup \bigcup_{k=1}^{\infty} \big(\widetilde{L}_k \cup \widetilde{R}_k\big) \in \Delta(x_n)$ for each n, F_i is Δ-continuous at x_n. At each $x \in C \setminus X$, F_i is continuous, since $\omega_{F_i}(\operatorname{cl} R_k(x_n))$ and $\omega_{F_i}(\operatorname{cl} L_k(x_n))$ tend to 0 as $n, k \to \infty$. Moreover, F_i is continuous at each point of $\langle a, b\rangle \setminus C$. So, F_i is a Δ-continuous function.

Consider F_1. The restriction $F_1 \restriction C$ is increasing, $|F_1(C)| = F(b) - F(a) - \sum_{n=1}^{\infty} 1/2^n = 0$, and F_1 is absolutely continuous on each $\operatorname{cl} R_k$ and $\operatorname{cl} L_k$. So, F_1 is a

$[VBG]$-function and it satisfies $\mathscr{N}$. Suppose F_1 is ACG. Then, by the Baire Category Theorem, there is a portion $I \cap C \neq \emptyset$ of C and a subset $E \subset I \cap C$, dense in $I \cap C$, such that $F_1 \restriction E$ is AC. However, there is an x_n within I. Hence for each $(\alpha, \beta) \ni x_n$ with both endpoints in C we have $F_1(\beta) - F_1(\alpha) > 1/2^n$. Since x_n is a bilateral accumulation point of $E \subset C$, we obtained a contradiction.

Consider F_2. The restriction $F_2 \restriction (C \setminus X)$ is constant, X is countable, and F_2 is absolutely continuous on each interval $\operatorname{cl} R_k$ and $\operatorname{cl} L_k$. So, F_2 is an ACG-function. Suppose F_2 is $[VBG]$. So, there is a portion $I \cap C \neq \emptyset$ of C such that $F_2 \restriction (I \cap C)$ is VB. There are k and n such that $J_k^{(n)} \subset I$ and so I contains points $\xi_{k_m}^{(m)} \in X$, $m = n, n+1, \ldots$. Since the preimage $F_2^{-1}(0)$ is dense in C, we see that the variation of $F_2 \restriction (I \cap C)$ is at least

$$\sum_{m=n}^{\infty} |F(\xi_{k_m}^{(m)})| = \sum_{m=n}^{\infty} \frac{1}{m} = \infty,$$

a contradiction.

Consider F_3. In the same way as for F_2 we verify F_3 is an ACG-function. Since the variation of $F_3 \restriction C$ equals

$$2 \sum_{n=1}^{\infty} \sum_{i=1}^{2^n} \left| F\left(\xi_i^{(n)}\right) \right| = 2 \sum_{n=1}^{\infty} \sum_{i=1}^{2^n} \frac{1}{2^{2n}} = 2,$$

F_3 is also a $[VBG]$-function. However, it is not $[ACG]$ since the set $\mathscr{D}_{F_3 \restriction C}$ is dense in C and thus the restriction of F_3 to no portion of C is AC (even continuous). $\square$

The construction of F_1 follows [21, *Example* 3.1]. For the particular case of the density local system, another example in known.

Example 6.5 [24, Example 3.5]**.** Let $\mathscr{F}$ be the class of approximately continuous functions. There exists an $F \in \mathscr{F}_4 \setminus (\mathscr{F}_2 + \mathscr{F}_3)$.

Corollary 6.6. *For a local system Δ, let each Δ-continuous function be Darboux Baire one. Assume, moreover, that the set of points at which Δ fulfils both the conditions $\dagger_{\mathrm{r}}$ and $\dagger_{\mathrm{l}}$, is not nowhere dense in $\mathbb{R}$. Then, denoting by $\mathscr{F}$ the class of Δ-continuous functions, we obtain the following inclusion relations between $\mathscr{F}_i$-integrals: $\mathscr{F}_1 \subsetneq \mathscr{F}_2 \cap \mathscr{F}_3$, $\mathscr{F}_2 \not\subset \mathscr{F}_3$, $\mathscr{F}_3 \not\subset \mathscr{F}_2$, $\mathscr{F}_2 \subsetneq \mathscr{F}_4$, $\mathscr{F}_3 \subsetneq \mathscr{F}_4$.*

The assumption to the foregoing corollary can be provided in terms of intersection conditions [25]:

Theorem 6.7. *If a bilateral local system* Δ *satisfies* IC, *then each* Δ*-continuous function* $f\colon \mathbb{R} \to \mathbb{R}$ *is Darboux.*

Theorem 6.8. *If a local system* Δ *satisfies* wIC, *then each* Δ*-continuous function* $f\colon \mathbb{R} \to \mathbb{R}$ *is Baire one.*

Remark 6.9. Let Δ be the proximal density local system. Despite Δ fails to have IC, each Δ-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is Darboux [20, *Theorem* 4.1].

7. $\mathscr{F}_i$-integrals versus H_Δ-integral

Theorem 7.1 [1, Theorem 3.1]**.** *Assume that* Δ *is bilateral and fulfils* IC. *Then each indefinite* H_Δ*-integral is a* $[VBG]$*-function and satisfies* $\mathscr{N}$.

Theorem 7.2 [1, Theorem 3.2]**.** *Assume* Δ *fulfils* iIC. *Then each indefinite* G_Δ*-integral is an* $[ACG]$*-function.*

Theorem 7.3 [25]**.** *Let two local systems,* Δ_1 *and* Δ_2*, both satisfy* IC. *Then, if a function* $F\colon \mathbb{R} \to \mathbb{R}$ *is* Δ_1*- and* Δ_2*-differentiable at each point of a set* $A \subset \mathbb{R}$*, the subset*

$$\{x \in A\colon\ F'_{\Delta_1}(x) \neq F'_{\Delta_2}(x)\}$$

is at most countable.

Consider a bilateral local system Δ with IC. Denote by $\mathscr{F}$ the class of Δ-continuous functions. In virtue of Theorems 6.7 and 6.8, each member of $\mathscr{F}$ is Darboux Baire one. From Theorems 7.1 and 7.2 we obtain

- each indefinite H_Δ-integral is a primitive for the $\mathscr{F}_2$-integral;
- if Δ satisfies iIC, then each indefinite G_Δ-integral is a primitive for the $\mathscr{F}_1$-integral.

In both cases, an indefinite integral is the indefinite integral of its (defined almost everywhere) Δ-derivative (Theorem 3.3), and is a primitive (and the indefinite integral) for the $\mathscr{F}_i$-integral, $i = 2, 1$, of its (also defined almost everywhere) approximate derivative. Since both local systems, Δ and the approximate one, satisfy IC, from Theorem 7.3 we get that the derivatives for these two systems (where they exist) coincide on a co-countable set. It implies the following corollary.

Corollary 7.4. *Let Δ be a bilateral local system satisfying* IC. *The $\mathscr{F}_1$-integral is more general than the G_Δ-integral, while the $\mathscr{F}_2$-integral is more general than the H_Δ-integral. Consequently, the H_Δ-integral and the wide Denjoy integral are compatible.*

As we have mentioned after Definition 3.2, there are local systems Δ for which $G_\Delta \subsetneq H_\Delta$. In fact, we have yet no example of indefinite H_Δ-integral which is not an $[ACG]$-function [1, *Problem* 3.3]. It would be interesting to know if the $\mathscr{F}_1$-integral covers not only the G_Δ-, but also the H_Δ-integral.

8. Incompatibility

In the end, let us remark that H_Δ-integrals for a wide class of local systems Δ are compatible with the wide Denjoy integral, as we have shown, nevertheless they are frequently incompatible among themselves, just because of different continuity conditions; for a simple example see [1, *Example* 4.1]. In what follows, also $\mathscr{F}_i$-integrals for different classes $\mathscr{F}$ of Δ-continuous functions, are frequently incompatible.

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