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# TOPOLOGICAL AND METRIC RIGIDITY TEOREMS FOR HYPERSURFACES IN A HYPERBOLIC SPACE

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*Abstract.* In this paper we study the topological and metric rigidity of hypersurfaces in $\mathbb{H}^{n+1}$, the $(n+1)$-dimensional hyperbolic space of sectional curvature $-1$. We find conditions to ensure a complete connected oriented hypersurface in $\mathbb{H}^{n+1}$ to be diffeomorphic to a Euclidean sphere. We also give sufficient conditions for a complete connected oriented closed hypersurface with constant norm of the second fundamental form to be totally umbilic.

*Keywords*: rigidity, hypersurfaces, topology, hyperbolic space

*MSC 2000*: 53C20, 53C42

## 1. Introduction

A classical result in the theory of submanifolds states that an $n$-dimensional ($n \geqslant 2$) connected closed oriented hypersurface in a Euclidean space $\mathbb{R}^{n+1}$ with non-vanishing Gauss-Kronecker curvature is diffeomorphic to an $n$-sphere [5]. Recall that the Gauss-Kronecker curvature of a hypersurface $M$ in a Euclidean space is defined to be the product of all principal curvatures of $M$. Various differentiable sphere theorems for hypersurfaces in a Riemannian manifold were obtained in the past years. For example, Sacksteder [17] showed that an immersed closed orientable hypersurface with nonnegative curvature in $\mathbb{R}^{n+1}$ is the boundary of a convex body and thus is diffeomorphic to a sphere. Do Carmo and Warner [6] proved that an $n$-dimensional ($n \geqslant 2$) connected closed oriented hypersurface with sectional curvature not less than $-1$ in $\mathbb{H}^{n+1}$ is diffeomorphic to an $n$-sphere. Alexander [3] obtained a similar theorem for compact connected orientable hypersurfaces in a complete simply connected Riemannian manifold of nonpositive sectional curvature.

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In the first part of this paper, we prove some differential sphere theorems for hypersurfaces in $\mathbb{H}^{n+1}$. In order to state our results, we fix some notation. Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^{n+1} x_iy_i \tag{1.1}$$

for $x = (x_0, \ldots, x_{n+1})$, $y = (y_0, \ldots, y_{n+1}) \in \mathbb{R}^{n+2}$.

For any nonzero $a \in \mathbb{L}^{n+2}$, the hyperplane $a^\perp := \{x \in \mathbb{L}^{n+2}\colon \langle x, a\rangle = 0\}$ divides $\mathbb{L}^{n+2}$ into two open connected components which will be called the *open hemispaces determined by $a$*.

We use the Minkowski model for the simply connected hyperbolic space of constant sectional curvature $-1$ which is the submanifold

$$\mathbb{H}^{n+1} = \{x \in L^{n+2}\colon \langle x, x\rangle = -1, x_0 > 0\},$$

with the induced metric from $\langle\, ,\, \rangle$. Let $\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2}\colon \langle x, x\rangle = 1\}.$$

Let $M$ be an oriented hypersurface immersed in $\mathbb{H}^{n+1}$. Denote by $N$ a unit normal vector field globally defined on $M$ which can be regarded as a map from $M$ to $\mathbb{S}_1^{n+1}$. We will call $N$ the *Gauss map* of $M$ and $N(M)$ the *Gauss image* of $M$. Assume that $\tilde\nabla$, $\overline\nabla$ and $\nabla$ are the connections of $\mathbb{L}^{n+2}$, $\mathbb{H}^{n+1}$ and $M$, respectively. We have

$$\tilde\nabla_X Y = (Xy_1, \ldots, Xy_{n+2}) \tag{1.2}$$

for $X$, $Y = (y_1, \ldots, y_{n+2}) \in \mathcal{X}(\mathbb{L}^{n+2})$. Let $A$ be the shape operator of $M$ in $\mathbb{H}^{n+1}$ associated to $N$. Then

$$\tilde\nabla_X Y = \overline\nabla_X Y + \langle X, Y\rangle x = \nabla_X Y + \langle AX, Y\rangle N + \langle X, Y\rangle x \tag{1.3}$$

and

$$A(X) = -\tilde\nabla_X N = -\overline\nabla_X N \tag{1.4}$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$.

**Definition 1.1.** The eigenvalues of $A$ are called the principal curvatures of $M$ and the Gauss-Kronecker curvature of $M$ is the product of all the principal curvatures.

We can state our first two results as follows.

**Theorem 1.2.** *Let $M$ be an $n$-dimensional $(n \geqslant 2)$ complete connected oriented closed hypersurface immersed in $\mathbb{H}^{n+1}$ with non-vanishing Gauss-Kronecker curvature. Then $M$ is diffeomorphic to the Euclidean $n$-sphere.*

**Theorem 1.3.** *Let $M$ be an $n$-dimensional $(n \geqslant 2)$ complete connected oriented closed hypersurface immersed in $\mathbb{H}^{n+1}$. If there exists a point $a \in \mathbb{S}_1^{n+1}$ such that the Gauss image of $M$ is contained in an open half-space determined by $a$, then $M$ is diffeomorphic to the Euclidean $n$-sphere.*

Hypersurfaces with constant mean or scalar curvature in a space form have received much attention in the past years. Alexandrov [1] proved that an imbedded compact hypersurface in a Euclidean space having constant mean curvature is a geodesic sphere. This was extended by Ros [15], [16] and by Montiel and Ros [11] who proved that an imbedded compact hypersurface in $\mathbb{R}^{n+1}$ or in $\mathbb{H}^{n+1}$ is also a geodesic sphere if the scalar curvature is constant. For surfaces in the three dimensional Euclidean space $\mathbb{R}^3$ a classical result of Hartman-Nirenberg classifies complete surfaces with nonzero constant curvature as standard spheres and those with zero curvature as planes or cylinders ([8]). A well-known result of Cheng and Yau [7] classifies complete hypersurfaces $M^n$ $(n > 2)$ with constant scalar curvature and non-negative sectional curvature in $E^{n+1}$ as a round hypersphere, a hyperplane, or a generalized cylinder $S^k(c) \times E^{n-k}$, $1 \leqslant k \leqslant n-1$. Work of Li [9] showed that a compact hypersurface $M^n$ of constant scalar curvature $n(n-1)r$ with $r \geqslant 1$ in $S^{n+1}(1)$, whose squared norm of the second fundamental form is bounded above by a certain constant which depends only on $n$ and $r$, is isometric to the totally umbilical sphere $S^n(r)$ or the Riemannian product $S^1\bigl(\sqrt{1-c^2}\bigr) \times S^{n-1}(c)$ for a certain value of the constant $c$. Recently, Alencar, do Carmo and Santos [2] have given an interesting gap theorem for closed hypersurfaces with constant scalar curvature $n(n-1)$ in a unit sphere. Also, one finds some interesting results about minimal hypersurfaces in a sphere with constant scalar curvature in, e.g., [4], [13], [14], [18].

On the other hand, one can also consider hypersurfaces with constant norm of the second fundamental form. The second part of this paper is devoted to prove the following theorem for hypersurfaces with constant norm of the second fundamental form in $\mathbb{H}^{n+1}$.

**Theorem 1.4.** *Let $M$ be an $n$-dimensional $(n \geqslant 2)$ connected closed orientable hypersurface with Ricci curvature bounded below by $-(n-1)$ in $\mathbb{H}^{n+1}$ and denote by $h$ the second fundamental form of $M$. Assume that $|h|^2$ is a constant not less than $n$. If the mean curvature $H$ of $M$ satisfies*

$$|H|^2 > \frac{1}{2n^2}\bigl(n|h|^2 + 4(n-1) + (n-2)\sqrt{|h|^4 - 4(n-1)}\bigr), \tag{1.5}$$

*then $M$ is a geodesic sphere.*

## 2. Proofs of theorems

P r o o f of Theorem 1.2. Fix a point $a \in \mathbb{S}_1^{n+1}$ and consider the spacelike hyperplane

$$a^{\perp} = \{x \in \mathbb{L}^{n+2} \colon \langle a, x\rangle = 0\}.$$

Let

$$\mathbb{S}_a^n = \{x \in S_1^{n+1} \colon \langle a, x\rangle = 0\}$$

be the intersection of $\mathbb{S}_1^{n+1}$ and $a^{\perp}$. Observe that $\mathbb{S}_a^n$ defines a round $n$-sphere of radius one, which is a totally geodesic hypersurface in $\mathbb{S}_1^{n+1}$. We will show that $M$ is diffeomorphic to $\mathbb{S}_a^n$. In order to see this, we define a map $\psi$ from $M$ to $\mathbb{S}_a^n$ by

$$\psi(x) = \frac{N(x) + \langle N(x), a\rangle a}{\sqrt{1 + \langle N(x), a\rangle^2}}.$$

Let $d\psi$ be the tangent map of $\psi$. For any $x \in M$ and any $v \in T_xM$, it is easy to see from (1.2) and (1.4) that

$$d\psi_x(v) = \frac{-Av - \langle Av, a\rangle a}{\sqrt{1 + \langle N(x), a\rangle^2}} + \frac{\langle N(x), a\rangle\langle Av, a\rangle}{(1 + \langle N(x), a\rangle^2)^{3/2}}(N(x) + \langle N(x), a\rangle a).$$

Thus, noticing that $\langle Av, N(x)\rangle = 0$, we get

$$\begin{aligned}(2.1)\qquad \langle d\psi_x(v), d\psi_x(v)\rangle &= \frac{|Av|^2(1 + \langle N(x), a\rangle^2) + \langle Av, a\rangle^2}{(1 + \langle N(x), a\rangle^2)^2}\\ &\geqslant \frac{|Av|^2}{1 + \langle N(x), a\rangle^2}.\end{aligned}$$

Since $M$ has non-vanishing Gauss-Kronecker curvature, we know that if $v \neq 0$, then $Av \neq 0$. Thus, we conclude from (2.2) that if $v \neq 0$, then $d\psi_x(v) \neq 0$. Hence $\psi$ is a local diffeomorphism by the inverse function theorem. Since $M$ is compact, $\psi$ is a covering map (cf. [10], [5]). But $M$ is connected and $S_a^n$ is simply connected, hence we know that $\psi$ is a diffeomorphism (cf. [10], [5]). This completes the proof of Theorem 1.2. □

P r o o f of Theorem 1.3. Define a map $\varphi$ from $M$ to $a^{\perp} \cap \mathbb{H}^{n+1}$ by

$$\varphi(x) = \frac{1}{\sqrt{1 + \langle a, x\rangle^2}}(x - \langle a, x\rangle a).$$

For any $x \in M$ and any $v \in T_xM$ we have

$$d\varphi_x(v) = \frac{1}{\sqrt{1 + \langle a, x\rangle^2}}(v - \langle a, v\rangle a) - \frac{\langle a, v\rangle\langle a, x\rangle}{(1 + \langle a, x\rangle^2)^{3/2}}(x - \langle a, x\rangle a),$$

and so

$$\langle d\varphi_x(v), d\varphi_x(v)\rangle = \frac{1}{(1+\langle a,x\rangle^2)^2}(|v|^2(1+\langle a,x\rangle^2) - \langle a,v\rangle^2).$$

The Schwarz inequality gives

$$\langle a,v\rangle^2 = \langle a^T, v\rangle^2 \leqslant |a^T|^2|v|^2.$$

Also, we have

$$1 = |a^T|^2 + \langle a, N(x)\rangle^2 - \langle a,x\rangle^2.$$

Thus, we conclude that

$$\langle d\varphi_x(v), a\varphi_x(v)\rangle \geqslant \frac{|v|^2(1+\langle a,x\rangle^2 - |a^T|^2)}{(1+\langle a,x\rangle^2)^2} = \frac{|v|^2\langle a, N(x)\rangle^2}{(1+\langle a,x\rangle^2)^2}. \tag{2.2}$$

Since $\langle N(x), a\rangle \neq 0$, $\forall\, x \in M$, we know that if $v \neq 0$, then $d\varphi_x(v) \neq 0$.

Let $a = (a_0, \ldots, a_{n+1})$. For any $x = (x_0, x_1, \ldots, x_{n+1}) \in a^\perp \cap \mathbb{H}^{n+1}$, since

$$\langle x,x\rangle = -1, \quad \langle x,a\rangle = 0, \quad \langle a,a\rangle = 1,$$

we have

$$0 = \langle x+a, x+a\rangle = -(x_0+a_0)^2 + \sum_{i=1}^{n+1}(x_i+a_i)^2,$$

which implies that $x_0 + a_0 \neq 0$. Thus, we can define a map $\tilde{\varphi}$ from $a^\perp \cap \mathbb{H}^{n+1}$ to the unit Euclidean $n$-sphere $S^n(1)$ by

$$\tilde{\varphi}(x_0, x_1, \ldots, x_{n+1}) = \frac{1}{x_0+a_0}(x_1+a_1, \ldots, x_{n+1}+a_{n+1}).$$

Fix a point $z = (z_0, \ldots, z_{n+1}) \in a^\perp \cap \mathbb{H}^{n+1}$ and a vector

$$u = (u_0, \ldots, u_{n+1}) \in T_z(a^\perp \cap \mathbb{H}^{n+1}).$$

Take a smooth curve

$$\gamma(t) = (z_0(t), \ldots, z_{n+1}(t))\colon\ (-\varepsilon, \varepsilon) \to a^\perp \cap \mathbb{H}^{n+1}$$

satisfying $\gamma(0) = z$ and $\gamma'(0) = u$. From the definition of $\tilde{\varphi}$, one concludes that

$$\begin{aligned} d\tilde{\varphi}_z(u) &= \frac{\mathrm{d}(\tilde{\varphi}\circ\gamma)}{\mathrm{d}t}(0) \\ &= \frac{-u_0}{(z_0+a_0)^2}(z_1+a_1, \ldots, z_{n+1}+a_{n+1}) + \frac{1}{z_0+a_0}(u_1, \ldots, u_{n+1}), \end{aligned}$$

and so

(2.3)

$$\begin{aligned}&\langle d\tilde{\varphi}_z(u), d\tilde{\varphi}_z(u)\rangle\\&= \frac{1}{(z_0+a_0)^4}\left(u_0^2\sum_{i=1}^{n+1}(z_i+a_i)^2 + (z_0+a_0)^2\sum_{i=1}^{n+1}u_i^2 - 2u_0(z_0+a_0)\sum_{i=1}^{n+1}u_i(z_i+a_i)\right).\end{aligned}$$

Observe that the function $f\colon a^\perp \cap \mathbb{H}^{n+1} \to \mathbb{R}$ given by

$$f(y) = \langle y+a, y+a\rangle$$

is identically zero. Thus

$$0 = \frac{1}{2}uf = \langle z+a, u\rangle,$$

that is

$$\sum_{i=1}^{n+1} u_i(z_i+a_i) = u_0(z_0+a_0). \tag{2.4}$$

Substituting (2.4) and

$$\sum_{i=1}^{n+1}(z_i+a_i)^2 = (z_0+a_0)^2$$

into (2.3), we get

$$\langle d\tilde{\varphi}_z(u), d\tilde{\varphi}_z(u)\rangle = \frac{1}{(z_0+a_0)^2}\left(\sum_{i=1}^{n+1}u_i^2 - u_0^2\right) = \frac{|u|^2}{(z_0+a_0)^2}.$$

Hence, if $u \neq 0$, then $d\tilde{\varphi}_z(u) \neq 0$.

Consider now the map

$$\Phi = \tilde{\varphi}\circ\varphi\colon\ M \to S^n(1).$$

For any $x \in M$ and any nonzero $w \in T_xM$ we have $d\varphi_x(w) \neq 0$, which in turn implies that $d\Phi_x(w) = d\tilde{\varphi}_{\varphi(x)}(d\varphi_x(w)) \neq 0$. Consequently, for any $x \in M$, $d\Phi_x$ is injective and so it is an isomorphism. Thus $\Phi$ is a local diffeomorphism by the inverse function theorem. Since $M$ is compact and connected and $S^n(1)$ is simply connected, we know that $\Phi$ is a diffeomrphism. This completes the proof of Theorem 1.3. □

Before proving Theorem 1.4, we list an algebraic lemma.

**Lemma 2.1** ([12]). *Let $a_i$, $i=1,\dots,n$, be real numbers satisfying $\sum\limits_i a_i = 0$ and $\sum\limits_i a_i^2 = S$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}S^{3/2} \leqslant \sum_i a_i^3 \leqslant \frac{n-2}{\sqrt{n(n-1)}}S^{3/2} \tag{2.5}$$

*and one of the equalities holds if and only if at least $(n-1)$ of the $a_i$'s are equal.*

Proof of Theorem 1.4. Let $h = \sum\limits_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j$ be the second fundamental form of $M$; then $h$ is a Codazzi tensor. Assume that $h_{ij} = \lambda_i \delta_{ij}$, where $\lambda_i$, $i = 1,\dots,n$ are the principal curvatures of $M^n$. From [7] we have

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (\operatorname{tr} h)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2, \tag{2.6}$$

where $h_{ijk}$ and $R_{ijkl}$ are the covariant derivative of $h$ and the curvature tensor of $M$, respectively.

It follows from the Gauss equation that

$$R_{ijij} = -1 + \lambda_i \lambda_j \tag{2.7}$$

for any $i \neq j$, which gives

$$\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = -n|h|^2 + n^2 H^2 - |h|^4 + nH\sum_i \lambda_i^3. \tag{2.8}$$

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum\limits_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{2.9}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{2.10}$$

Putting (2.9), (2.10) into (2.6) and using Lemma 2.1, we get

$$\begin{aligned} \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 &= |Z|^2(-n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3 \\ &\geqslant (|h|^2 - nH^2)A(n,|H|,|h|^2) \end{aligned} \tag{2.11}$$

where

$$A(n,|H|,|h|^2) = -n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|h|^2 - nH^2}.$$

By (1.5),

$$(2n^2H^2 - (n|h|^2 + 4(n-1))^2 > (n-2)^2(|h|^4 - 4(n-1)),$$

which is equivalent to

$$\begin{aligned}(2.12)\qquad 0 &< \frac{n^4}{n-1}H^4 - \Big(4n^2 + \frac{n^3}{n-1}|h|^2\Big)H^2 + (n+|h|^2)^2 \\ &= (2n|H|^2 - (n+|h|^2))^2 - \frac{n(n-2)^2}{n-1}H^2(|h|^2 - nH^2).\end{aligned}$$

Since $|h|^2 \geqslant n$, it follows from (1.5) that

$$\begin{aligned}(2.13)\qquad &2nH^2 - |h|^2 - n \\ &> \frac{1}{n}\big(4(n-1) - n^2 + (n-2)\sqrt{|h|^4 - 4(n-1)}\big) \geqslant 0.\end{aligned}$$

Thus, we have from (2.12) and (2.13) that

$$(2.14)\qquad 2nH^2 - (n+|h|^2) > \sqrt{\frac{n}{n-1}}(n-2)|H|\sqrt{|h|^2 - nH^2}.$$

That is,

$$(2.15)\qquad A(n, |H|, |h|^2) > 0.$$

Since $|h|^2$ is constant, we have from (2.6) and (2.11) that

$$(2.16)\qquad 0 \geqslant \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i=1}^{n} \lambda_i (\operatorname{tr} h)_{ii} + (|h|^2 - nH^2)A(n, |H|, |h|^2).$$

Since the Ricci curvature of $M$ is bounded from below by $-(n-1)$, the Gauss equation gives

$$(2.17)\qquad \Big(\sum_{i=1}^{n} \lambda_i\Big)\lambda_j - \lambda_j^2 \geqslant 0, \quad j = 1, \dots, n, \text{ on } M^n.$$

Suppose without loss of generality that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$. We *claim* that for any $p \in M$, either

$$\lambda_i(p) \geqslant 0, \quad i = 1, \dots, n,$$

or

$$\lambda_i(p) \leqslant 0, \quad i = 1, \dots, n.$$

Let us prove this fact by contradiction. Thus suppose that there exists a $q \in M$ such that

$$\lambda_1(q) > 0 \quad \text{and} \quad \lambda_n(q) < 0.$$

If $\sum_{i=1}^{n} \lambda_i(q) \geqslant 0$, then we have

$$\sum_{i=1}^{n-1} \lambda_i(q) \geqslant -\lambda_n(q) > 0.$$

Hence

$$\left(\sum_{i=1}^{n} \lambda_i(q)\right)\lambda_n(q) - \lambda_n^2(q) = \left(\sum_{i=1}^{n-1} \lambda_i(q)\right)\lambda_n(q) < 0,$$

which contradicts (2.17). On the other hand, if $\sum_{i=1}^{n} \lambda_i(q) < 0$, then

$$\sum_{i=2}^{n} \lambda_i(q) < -\lambda_1(q) < 0,$$

which gives

$$\left(\sum_{i=1}^{n} \lambda_i(q)\right)\lambda_1(q) - \lambda_1^2(q) = \left(\sum_{i=2}^{n} \lambda_i(q)\right)\lambda_1(q) < 0,$$

contradicting (2.17) again. This proves our *claim*. The Gauss equation then implies that the sectional curvatures of $M$ are bounded from below by $-1$ and so $M$ is the boundary of a convex body in $\mathbb{H}^{n+1}$ ([6]). Thus, the second fundamental form of $M$ is always semi-positive definite if we choose the unit normal vector field of $M$ properly. Therefore, we can assume that $\lambda_i \geqslant 0$ on $M$, $\forall\, i = 1, \ldots, n$.

Take a point $p \in M$ such that

$$(\mathrm{tr}\, h)(p) = \min_{x \in M^n} (\mathrm{tr}\, h)(x). \tag{2.18}$$

Then we have from the maximal principle that

$$(\mathrm{tr}\, h)_{ii}(p) \geqslant 0, \quad i = 1, \ldots, n. \tag{2.19}$$

Thus, we have from (2.16), (2.19) and $A(n, |H|, |h|^2) > 0$ that $(|h|^2 - nH^2)(p) = 0$ and so

$$\lambda_1(p) = \ldots \lambda_n(p). \tag{2.20}$$

Now, for any $q \in M$ we have from (2.18), (2.20) and $|h|^2(q) = |h|^2(p)$ that

$$\sum_{i,j}(\lambda_i(q) - \lambda_j(q))^2 = 2n\sum_i^n(\lambda_i(q))^2 - 2\Big(\sum_i^n \lambda_i(q)\Big)^2 \leqslant 2n\sum_i^n(\lambda_i(p))^2 - 2\Big(\sum_i^n \lambda_i(p)\Big)^2 = 0. \tag{2.21}$$

This shows that $M$ is totally umbilic and so it is a geodesic sphere. The proof of Theorem 1.4 is complete.

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