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Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 2, 523–551

Persistent URL: http://dml.cz/dmlcz/128188

Czechoslovak Mathematical Journal, 57 (132) (2007), 523–551

DECOMPOSING COMPLETE TRIPARTITE GRAPHS INTO CLOSED TRAILS OF ARBITRARY LENGTHS

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(Received March 31, 2004)

Abstract. The complete tripartite graph $K_{n,n,n}$ has $3n^2$ edges. For any collection of positive integers $x_1, x_2, \ldots, x_m$ with $\sum\limits_{i=1}^{m} x_i = 3n^2$ and $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, we exhibit an edge-disjoint decomposition of $K_{n,n,n}$ into closed trails (circuits) of lengths $x_1, x_2, \ldots, x_m$.

Keywords: cycles, decomposing complete tripartite graphs

MSC 2000: 05C70, 05C38

1. Introduction

All graphs discussed here will be simple, with no loops or multiple edges, and no isolated vertices. A *complete equipartite* graph $K_{m(n)}$ has mn vertices partitioned into m parts of size n, where each pair of vertices in different parts is joined by an edge, while each pair of vertices in the same part is never joined by an edge. More generally, a *complete partite* graph $K(n_1, n_2, \ldots, n_m)$ has $\sum\limits_{i=1}^{m} n_i$ vertices, partitioned into m parts of size n_i, $1 \leqslant i \leqslant m$, where the only edges are those joining any pair of vertices in different parts. Thus in an equipartite graph, $n_i = n$ for $1 \leqslant i \leqslant n$. A graph is called *even* if all its degrees are even.

Several recent papers have considered edge-disjoint decompositions of complete partite graphs into cycles, and factorisations into 2-factors (necessarily cycles). Liu [11] solves the 2-factorisation problem for an equipartite graph in the case that each 2-factor is a set of cycles of some one fixed length. In other papers such as [8], the present authors have considered decompositions of complete partite graphs into cycles of some even length (not into 2-factors, and not necessarily equipartite).

Supported by Ministry of Education of the Czech Republic as project LN00A056.

In this paper we restrict ourselves to tripartite graphs. Some recent decomposition work into cycles for tripartite graphs includes [5] (equipartite, but arbitrary cycle length); [3] (not equipartite; mixed cycles of lengths 3 and 4); [7], [8], [6] (not equipartite; cycles of length 5). In fact the problem of finding necessary and sufficient conditions for $K(n_1, n_2, n_3)$ to have an edge-disjoint decomposition into 5-cycles, originally mentioned in [12], remains open for the case when all three parts have different odd sizes.

In this paper we consider closed trails rather than cycles. A closed trail of length x in a tripartite graph (or indeed in any simple graph) may be regarded as a set of cycles of lengths 3 or greater, forming a connected simple graph, the sum of whose lengths is x. Equivalently, a closed trail may be regarded as any connected even graph.

A connected even graph G is said to be *arbitrarily decomposable into closed trails*, or ADCT for short, if given any (multi)set $\{x_1, x_2, \ldots, x_m\}$ of positive integers greater than 2, with the property that G contains a closed trail of length x_i, $1 \leqslant i \leqslant m$, and also satisfying $\sum_{i=1}^{m} x_i = |E(G)|$, then the graph G has an edge-disjoint decomposition into closed trails of lengths $x_1, x_2, \ldots, x_m$.

Necessary and sufficient conditions for a graph to be ADCT are not known in general. Balister [1] showed that K_n (n odd) or $K_n - F$ (F a 1-factor and n even) are both ADCT. Also in [2] Balister showed that any sufficiently large and sufficiently dense even graph G is ADCT. The paper [9] (Theorem 2) shows that $K_{a,b}$ is ADCT if a and b are positive even integers; here of course the trail lengths x_i are necessarily all even. Surprisingly, the oft-quoted result of Sotteau [13] giving necessary and sufficient conditions for existence of an edge-disjoint decomposition of a bipartite graph $K_{a,b}$ into cycles of some fixed, necessarily even, length is not used in [9].

The first complete multipartite graph with the possibility of being ADCT, with no restriction on the cycle lengths other than being length at least 3, is a complete tripartite graph; this is the motivation behind this paper. It has been shown [10], and can also be easily verified, that if the tripartite graph K_{r_1,r_2,r_3} is ADCT, then the partite sizes are $1, 1, 3$, or $1, 1, 5$ (since this latter graph contains no 5-cycle), or else all parts have the same size. Moreover, [10] also showed that $K_{n,n,n}$ is ADCT if $n \in \{1, 2, 3, 4\}$ or $n = 5 \cdot 2^i$ where $i \geqslant 0$, and conjectured that every complete tripartite graph with all parts the same size is ADCT. The aim of this paper is to prove this conjecture. Our main result is the following.

Theorem 1.1. *The tripartite graph $K_{n,n,n}$ is arbitrarily decomposable into closed trails (ADCT) for all positive integers n.*

The main technique enabling the proof of this theorem is similar to that of a "latin representation", using ideas from [3], first appearing in [7], and described in the survey paper [4]. We outline in the next section what we need here, and give some crucial but simple "trades", relating subsections of latin squares to cycles of lengths 3, 4 and 5 (and sometimes 7). Then in Section 3 we deal with the case n even, and in Section 4 with the case n odd. Section 5 contains concluding remarks, including a conjecture concerning closed trails in arbitrary equipartite graphs.

2. Some useful trades

Let $R(n) = \{r_1, r_2, \ldots, r_n\}$, $C(n) = \{c_1, c_2, \ldots, c_n\}$ and $N(n) = \{1, 2, \ldots, n\}$. A partial latin square P is a set of ordered triples of the form (r_i, c_j, k), where $r_i \in R(n)$, $c_j \in C(n)$ and $k \in N(n)$, with the following properties:

- if $(r_i, c_j, k) \in P$ and $(r_i, c_j, k') \in P$, then $k = k'$;
- if $(r_i, c_j, k) \in P$ and $(r_i, c_{j'}, k) \in P$, then $j = j'$; and
- if $(r_i, c_j, k) \in P$ and $(r_{i'}, c_j, k) \in P$, then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array of integers such that if $(r_i, c_j, k) \in P$, then the entry k occurs in row r_i and column c_j (i.e. in cell (r_i, c_j)). We define S_P to be the set of filled cells in a partial latin square P.

A partial latin square has the property that each element of $N(n)$ occurs at most once in each row and at most once in each column. A cell is termed "empty" if it contains no entry and "filled" otherwise. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a latin square of order n is an $n \times n$ array with entries chosen from the set $N(n)$ such that each entry occurs precisely once in each row and precisely once in each column.

We say that two partial latin squares P and P' are *isotopic* if there exist permutations α (on $R(n)$), β (on $C(n)$) and γ (on $N(n)$) such that $(r_i, c_j, k) \in P$ if and only if $(\alpha(r_i), \beta(c_j), \gamma(k)) \in P'$.

It is well-known that a latin square of order n is in a sense "equivalent" to an edge-disjoint decomposition of a complete tripartite graph, $K_{n,n,n}$, into triangles. One way to see this is to take the vertex set of the tripartite graph to be $\{r_1, r_2, \ldots, r_n\} \cup \{c_1, c_2, \ldots, c_n\} \cup \{1, 2, \ldots, n\}$. Then entry k in row r_i and column c_j of the latin square will correspond to the triangle with vertices r_i, c_j and k in the tripartite graph decomposition. Thus each entry in the latin square corresponds to a triangle in $K_{n,n,n}$.

Any latin square of order n may be chosen to represent the complete tripartite graph $K_{n,n,n}$. For our purposes here it is convenient to take the back circulant latin square B_n; with rows, columns and entries all taken from $\{1, 2, \ldots, n\}$, this has the entry in cell (i, j) equal to $i + j - 1 \pmod{n}$. An example appears in Table 1.

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

Table 1. Back circulant latin square of order 7, B_7.

By judicious choice of sets of four entries at a time, or five entries, in such an order n latin square, we may replace or *trade* four triangles with three 4-cycles, or five triangles with three 5-cycles. We may also trade three carefully chosen entries (three triangles) with a 4-cycle and a 5-cycle. We start with some simple lemmata which show these ideas.

The graph-theoretic properties of the following trades are preserved under isotopisms; thus, when we use these trades, the rows, columns and entries may be relabelled. However, adjacency of rows and columns modulo n will in general be preserved.

We remark on some notation. As usual, $(x_1, x_2, \dots, x_n)$ refers to a cycle with vertices x_1, $x_2, \dots, x_n$ and edges $\{x_i, x_{i+1 \pmod n}\}$, $1 \leqslant i \leqslant n$. The notation $|P|$ will denote the number of filled cells in the partial latin square P. Moreover, when we write $\{m, m, m\}$, we intend the *multiset* with repetitions of symbol m retained.

In the next five lemmata, the notation $T_{x.i}$ will refer to a trade involving x triangles (entries in the latin square). The proofs are given in tabular form, listing a suitable decomposition using precisely the edges from the triangles corresponding to the filled cells in $T_{x.i}$.

Lemma 2.1. *The three triangles corresponding to the partial latin squares* $T_{3.1}$ *and* $T_{3.2}$

	c_1	c_2
r_1	1	2
r_2	2	

$T_{3.1}$

	c_1	c_2
r_1		1
r_2	1	2

$T_{3.2}$

are equivalent to one 4*-cycle and one* 5*-cycle.*

Proof.

$T_{3.1}$	$(c_1, r_1, c_2, 2)$, $(c_1, r_2, 2, r_1, 1)$
$T_{3.2}$	$(c_1, r_2, c_2, 1)$, $(r_1, c_2, 2, r_2, 1)$

□

Lemma 2.2. *The four triangles corresponding to the four entries in each of the partial latin squares $T_{4.1}$, $T_{4.2}$ or $T_{4.3}$*

	c_1	c_2
r_1	1	2
r_2	2	3

$T_{4.1}$

	c_1	c_2	c_3
r_1		1	2
r_2	1	2	

$T_{4.2}$

	c_1	c_2
r_1		1
r_2	1	2
r_3	2	

$T_{4.3}$

are equivalent to three 4-cycles.

Proof.

$T_{4.1}$	$(c_2, 3, r_2, 2),\ (c_1, r_1, c_2, r_2),\ (r_1, 1, c_1, 2)$
$T_{4.2}$	$(r_2, 1, r_1, 2),\ (r_1, c_2, 2, c_3),\ (r_2, c_1, 1, c_2)$
$T_{4.3}$	$(c_2, 1, c_1, 2),\ (c_1, r_2, 2, r_3),\ (c_2, r_1, 1, r_2)$

□

Lemma 2.3. *The five triangles corresponding to the five entries in the subarrays $T_{5.1}$, $T_{5.2}$, $T_{5.3}$, $T_{5.4}$, $T_{5.5}$, $T_{5.6}$, $T_{5.7}$, $T_{5.8}$ and $T_{5.9}$ are each equivalent to three 5-cycles.*

Furthermore: array $T_{5.5}$ is also equivalent to two 4-cycles and one 7-cycle; in this case the trade is called $T_{5.5A}$; array $T_{5.9}$ is also equivalent to two 4-cycles and one 7-cycle; in this case the trade is called $T_{5.9A}$.

	c_1	c_2
r_1	1	2
r_2	2	3
r_3	3	

$T_{5.1}$

	c_1	c_2	c_3
r_1	1	2	3
r_2	2	3	

$T_{5.2}$

	c_1	c_2
r_1		2
r_2	2	3
r_3	3	4

$T_{5.3}$

	c_1	c_2	c_3
r_1		2	3
r_2	2	3	4

$T_{5.4}$

	c_1	c_2	c_3
r_1			5
r_2	4	5	6
r_3	5		

$T_{5.5}$

	c_1	c_2	c_3	c_4
r_1			1	2
r_2		1		
r_3	1	2		

$T_{5.6}$

	c_1	c_2	c_3	c_4
r_1			1	2
r_2			2	
r_3	1	2		

$T_{5.7}$

	c_1	c_2	c_3
r_1		1	2
r_2	1	2	
r_3	2		

$T_{5.8}$

	c_1	c_2	c_3
r_1	1	2	3
r_2	2		
r_3	3		

$T_{5.9}$

P r o o f.

$T_{5.4}$	$(r_1, c_3, 4, r_2, 3), (c_2, 3, c_3, r_2, 2), (r_1, c_2, r_2, c_1, 2)$
$T_{5.5}$	$(r_1, c_3, 6, r_2, 5), (r_2, 4, c_1, 5, c_3), (r_3, c_1, r_2, c_2, 5)$
$T_{5.5A}$	$(r_2, c_3, 5, c_2), (r_3, 5, r_2, c_1), (r_1, c_3, 6, r_2, 4, c_1, 5)$
$T_{5.6}$	$(r_1, c_4, 2, r_3, 1), (r_1, c_3, 1, c_2, 2), (r_2, 1, c_1, r_3, c_2)$
$T_{5.7}$	$(r_3, c_1, 1, r_1, 2), (r_3, c_2, 2, c_3, 1), (r_2, 2, c_4, r_1, c_3)$
$T_{5.8}$	$(r_1, c_3, 2, r_2, 1), (c_2, r_1, 2, c_1, r_2), (2, r_3, c_1, 1, c_2)$
$T_{5.9}$	$(r_1, c_2, 2, c_1, 3), (r_3, c_1, r_1, c_3, 3), (r_2, c_1, 1, r_1, 2)$
$T_{5.9A}$	$(r_1, c_1, r_2, 2), (r_1, 1, c_1, 3), (r_1, c_2, 2, c_1, r_3, 3, c_3)$

□

Lemma 2.4. *The triangles corresponding to the entries in the subarrays $T_{8.1}$, $T_{8.2}$ and $T_{8.3}$*

	c_1	c_2	c_3
r_1	1	2	3
r_2	2	3	4
r_3	3	4	

$T_{8.1}$

	c_1	c_2	c_3
r_1			2
r_2	1	2	3
r_3	2	3	4
r_4	3		

$T_{8.2}$

	c_1	c_2	c_3
r_1		1	2
r_2	1	2	3
r_3	2	3	4

$T_{8.3}$

are each equivalent to six 4-cycles.

P r o o f.

$T_{8.1}$	$(r_1, 1, c_1, 2), (r_1, c_1, r_2, c_2), (r_1, 3, r_2, c_3), (r_2, 2, c_2, 4), (r_3, c_1, 3, c_2), (r_3, 3, c_3, 4)$
$T_{8.2}$	$(c_1, r_4, 3, r_2), (2, r_1, c_3, r_3), (r_2, 1, c_1, 2), (r_3, 4, c_3, 3), (r_3, c_1, 3, c_2), (r_2, c_2, 2, c_3)$
$T_{8.3}$	$(r_1, 1, c_1, 2), (r_1, c_2, 2, c_3), (r_2, 1, c_2, 3), (r_2, c_1, r_3, 2), (r_3, 4, c_3, 3), (r_3, c_3, r_2, c_2)$

□

Lemma 2.5. *The triangles corresponding to the entries in the subarrays $T_{12.1}$ and $T_{12.2}$*

	c_1	c_2	c_3
r_1	1	2	3
r_2	2	3	4
r_3	3	4	5
r_4	4	5	6

$T_{12.1}$

	c_1	c_2	c_3
r_1	1	2	3
r_2	2	3	4
r_3	3	4	5
r_4	4	5	
r_5	5		

$T_{12.2}$

are each equivalent to nine 4-cycles.

P r o o f.

$T_{12.1}$	$(r_1, 1, c_1, 2), (r_1, c_1, r_2, c_2), (r_2, 2, c_2, 4), (c_1, 3, c_3, 4), (r_1, 3, r_2, c_3),$ $(r_3, c_2, r_4, c_1), (r_3, 4, r_4, c_3), (r_3, 5, c_2, 3), (r_4, 5, c_3, 6)$
$T_{12.2}$	$(r_1, 1, c_1, 2), (r_1, c_1, r_2, c_2), (r_1, c_3, r_2, 3), (r_2, 2, c_2, 4), (r_3, c_1, 5, c_3),$ $(r_3, 3, c_2, 5), (r_3, c_2, r_4, 4), (r_4, c_1, r_5, 5), (c_1, 3, c_3, 4)$

□

In what follows, let $P = \{x_1, x_2, \ldots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Our aim in Sections 3 and 4 is to display an edge-disjoint decomposition of $K_{n,n,n}$ into closed trails (or circuits) of lengths $x_1, x_2, \ldots x_m$. We can partition P into subsets of size at most three of the x_i at a time so that:

(1) the sum of the numbers in each subset of P is divisible by 3; and
(2) each subset is minimal in size with respect to (1).

In addition we may ensure that at most two of the subsets have size equal to 2. We state this formally in the following lemma.

Lemma 2.6. *Let $P = \{x_1, x_2, \ldots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Then there exists a partition of P into disjoint subsets $P_1, P_2, \ldots, P_l$ for some l such that for $1 \leqslant j \leqslant l$,*

(1) $\sum_{x \in P_j} x$ *is divisible by* 3;
(2) $\sum_{x \in P'_j} x$ *is not divisible by* 3 *for each* $\emptyset \subset P'_j \subset P_j$;
(3) $|P_j| \leqslant 3$; *and*
(4) *there exist at most two subsets P_j such that $|P_j| = 2$.*

P r o o f. First, form subsets of size one from each element of P divisible by 3. Next, form subsets of size three from elements of P congruent to 1 modulo 3, until there are at most two such integers remaining. Similarly, form subsets of size three from elements of P congruent to 2 modulo 3, again with at most two remaining. Now there are at most four elements of P remaining.

The condition $\sum_{i=1}^{m} x_i$ is divisible by 3 dictates three possibilities: There are two integers congruent to 1 modulo 3 and two integers congruent to 2 modulo 3; or there is one integer congruent to 1 modulo 3 and one integer congruent to 2 modulo 3; or there are no integers remaining. In each case we can partition any remaining integers into subsets of size 2 so that the sum of the integers in each subset is divisible by 3. The resulting partition of P satisfies the conditions of the lemma. □

Corollary 2.7. *Let P_1, $P_2, \dots P_l$ be a partition of P as in the previous lemma. Then for each P_j, $1 \leqslant j \leqslant l$:*
(1) $P_j = \{\alpha\}$ *where* $3 \mid \alpha$; *or*
(2) $P_j = \{\alpha, \beta\}$ *where* $\alpha \equiv 1 \pmod 3$ *and* $\beta \equiv 2 \pmod 3$; *or*
(3) $P_j = \{\alpha, \beta, \gamma\}$ *where* $\alpha \equiv \beta \equiv \gamma \equiv 1 \pmod 3$; *or*
(4) $P_j = \{\alpha, \beta, \gamma\}$ *where* $\alpha \equiv \beta \equiv \gamma \equiv 2 \pmod 3$.

In addition, there are at most two subsets P_j with $|P_j| = 2$.

The following lemma is also immediate.

Lemma 2.8. *A collection of triangles arising from entries in a (partial) latin square L forms a connected circuit provided the entries can be ordered so that adjacent entries are*
(i) *in the same row of L, or*
(ii) *in the same column of L, or*
(iii) *are the same symbol.*

Subsequently we shall exhibit: a circuit of length $0 \pmod 3$ as a collection of connected triangles; a circuit of length $1 \pmod 3$ as a 4-cycle (or occasionally a 7-cycle) connected to a collection of connected triangles; and a circuit of length $2 \pmod 3$ as a 5-cycle connected to a collection of connected triangles. Provided that the cycles are connected, it is immediate that such an even connected graph has an euler tour, that is, forms a closed trail or circuit.

3. The case n even

Throughout this section we assume that $n \geqslant 6$, since the cases $n = 2$ and 4 appeared in [10].

We use the back circulant latin square B_n on the set $\{1, 2, \dots, n\}$, and since n is even, we can partition $L = B_n$ into pairs of adjacent columns. As in the previous section, let $P = \{x_1, x_2, \dots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Let P_1, $P_2, \dots P_l$ be a partition of P satisfying the conditions of Lemma 2.6.

The general idea is that we "trade" triangles with sets of closed trails with integer lengths from sets P_j, $1 \leqslant j \leqslant l$. Condition 1 from Lemma 2.6 gives a necessary condition for this to be possible. The triangles correspond to entries in B_n. We "delete" entries from B_n once we have traded the corresponding triangles in $K_{n,n,n}$. Our aim is that after each deletion, the resulting partial latin square will have the same generic form, which we call a *proper form* (see definition below). If we can show that the partial latin square retains this proper form after each deletion, then our result will follow recursively.

Definition 3.1. Let n be an even integer. Let Q be a partial latin square such that $Q \subseteq B_n$. We say that Q is in *proper form* if there exists $m = m_Q$, $0 \leqslant m \leqslant \frac{1}{2}n$ such that

(1) Q is empty in columns 1 to $2m-2$;
(2) Q is filled in columns $2m+1$ to n; and
(3) in columns $2m-1$ and $2m$, there exist integers $z = z_Q$, $y = y_Q$, $z \neq y$, $1 \leqslant z, y \leqslant n$ such that rows $r_{z+i \pmod n}$, (where $0 \leqslant i \leqslant y-z-1 \pmod n$) are empty, rows $r_{y+i \pmod n}$, (where $1 \leqslant i \leqslant z-y-1 \pmod n$) are filled, cell $(y, 2m-1)$ is empty and cell $(y, 2m)$ is either empty or filled.

So in columns $2m-1$ and $2m$: rows z, $z+1$, …, $z+(y-z-1) = y-1$ are empty, and rows $y+1$, $y+2$, …, $y+(z-y-1) = z-1$ are filled (numbers modulo n).

Observe that both B_n and a completely empty partial latin square are in proper form. Table 2(a) gives an example of a partial latin square Q of order 8 in proper form. Observe that $m_Q = 2$, $z_Q = 6$, $y_Q = 3$ and cell $(y_Q, 2m_Q)$ is filled.

					5	6	7	8
$y-1$					6	7	8	1
y				6	7	8	1	2
$y+1$			6	7	8	1	2	3
$z-1$			7	8	1	2	3	4
z					2	3	4	5
$z+1$					3	4	5	6
					4	5	6	7

(a)

				14	15	32	33
				16	17	34	35
			1	18	19	36	37
		2	3	20	21	22	23
		4	5	6	7	24	25
				8	9	26	27
				10	11	28	29
				12	13	30	31

(b)

Table 2. (a) A partial latin square of order 8 in proper form.
(b) The standard ordering for this partial latin square.

We now describe how to "pack" the partition of P (as described in Corollary 2.7) into B_n. The subsets of the form $\{4,4,4\}$ must be placed carefully, to avoid awkward overlap between pairs of columns.

Lemma 3.2. *Let l be an integer such that $12l \leqslant 3n^2$. Then there exists a partial latin square Q in proper form such that the triangles corresponding to the entries in the partial latin square $B_n \setminus Q$ may be traded with $3l$ 4-cycles.*

Proof. Let $l = a(\frac{1}{2}n) + b$, where $a \geqslant 0$ and $0 \leqslant b < \frac{1}{2}n$. Since n is even, each pair of columns partitions into $\frac{1}{2}n$ trades of type $T_{4.1}$. We may use the first $2a$ columns of B_n in this way. Then in columns $2a+1$ and $2a+2$ we use the first $2b$

rows with trades of type $T_{4.1}$. Removing these entries from B_n, we have a partial latin square Q in proper form as per Definition 3.1. up 2×2 blocks, □

Next we need a definition and a lemma about the structure of an arbitrary partial latin square Q in standard form.

Definition 3.3. Suppose Q is a partial latin square of order n in proper form. We denote the following ordering $f\colon \{1, 2, \ldots, |Q|\} \to S_Q$ as the *standard ordering* of Q. If the cell $(y_Q, 2m_Q)$ is filled, let $f(1) = (y_Q, 2m_Q)$. Otherwise, for convenience we temporarily define $f(0) = (y_Q, 2m_Q)$ (although in our final ordering, $f(0)$ is not defined!). Then we define $f(i+1)$ in terms of $f(i)$ recursively as follows. Let $f(i) = (a, b)$. If b is even and $(a+1 \pmod n, b-1)$ is both non-empty and not equal to $f(i')$ for any $i' < i$, then $f(i+1) = (a+1 \pmod n, b-1)$. Otherwise, we let $f(i+1) = (a, b+1)$.

See Table 2(b) for an example of the standard ordering of a partial latin square in proper form.

The next lemma follows directly from the definition of the standard ordering f.

Lemma 3.4. *Let Q be a partial latin square of order n in proper form and let f be the standard ordering of Q as in the previous definition. Then:*

(1) *f is a bijection;*
(2) *for any integer i, if the entries from cells $f(i')$ for each $i' < i$ are deleted, the resulting partial latin square Q' is in proper form;*
(3) *if $f(i)$ and $f(i+1)$ are both defined, then $f(i)$ and $f(i+1)$ share either a common row or a common entry;*
(4) *if $f(i)$ and $f(i+2)$ are both defined, then $f(i)$ and $f(i+2)$ share either a common row or a common column.*

Lemma 3.5. *Let Q be a partial latin square of order n in proper form. Let Let $\alpha \leqslant 3|Q|$ be an integer divisible by 3. Then there exist $\frac{1}{3}\alpha$ entries in Q such that:*

- *the corresponding triangles form a closed trail of length α; and*
- *the removal of these entries from Q leaves a partial latin square Q' in proper form.*

Proof. Let f be a standard ordering of Q. Then Conditions 2 and 3 of Lemma 3.4 give the desired result. □

Lemma 3.6. *Let Q be a partial latin square of order n in proper form and let $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$ be integers with $\eta_1 + \eta_2 \leqslant |Q|$. Let f be the standard ordering of Q. Then there exists a partial latin square Q' (with $Q' \subseteq Q$), also in proper form, such that the triangles corresponding to the entries in $Q \setminus Q'$ may be traded with two closed trails of length $3\eta_1$ and $3\eta_2$. In addition, the triangles corresponding to the entries in cells $f(1)$ and $f(2)$ are included in the trails of length $3\eta_1$ and $3\eta_2$, respectively.*

Proof. Let $\varepsilon = \min(\eta_1, \eta_2)$. We form two "trails" of triangles, one from $f(2), f(4), \ldots, f(2\varepsilon)$, the second from $f(1), f(3), \ldots, f(2\varepsilon - 1)$. Condition 4 of Lemma 3.4 ensures that both these sets of triangles form a connected graph. Then, depending on which of η_1 and η_2 is greater, we attach the triangles corresponding to cells $f(2\varepsilon + 1)$ up to cell $f(\eta_1 + \eta_2)$ to one of these trails, and we have the desired result. □

Now we give a lemma that ultimately deals with subsets of size 2 in the partition of P.

Lemma 3.7. *Let $\alpha \equiv 1 \pmod 4$ and $\beta \equiv 2 \pmod 4$ be integers such that $\alpha \geqslant 4$ and $\beta \geqslant 5$. Let Q be a partial latin square of order n in proper form with $|Q| \geqslant \frac{1}{3}(\alpha + \beta)$. Suppose, in addition, that there are either at least three non-empty cells in columns $2m_Q$ and $2m_Q - 1$, or $\alpha + \beta \geqslant 15$.*

Then there exists a partial latin square Q' with $Q' \subseteq Q$ such that Q' is in proper form, $|Q'| = |Q| - \frac{1}{3}(\alpha + \beta)$, and the triangles corresponding to the elements of $Q \setminus Q'$ may be traded with exactly one closed trail of length α and one closed trail of length β.

Proof. Let f be a standard ordering of Q as in Definition 3.3. First suppose that there are at least three non-empty cells in columns $2m_Q$ and $2m_Q - 1$. Then we may place either a trade of type $T_{3.1}$ or $T_{3.2}$ on the cells $f(1)$, $f(2)$ and $f(3)$.

If $\alpha = 4$ and $\beta = 5$ we are done. Otherwise from Lemma 3.6 we can form two trails of lengths $\alpha - 4$ and $\beta - 5$ that trade with triangles $f(4), f(5), \ldots, f(\frac{1}{3}(\alpha + \beta))$, such that the trail of length $\alpha - 4$ uses the triangle corresponding to $f(4)$, and the closed trail of length $\beta - 5$ uses the triangle corresponding to $f(5)$. (If either $\alpha = 4$ or $\beta = 5$, we construct just one trail from Lemma 3.5.)

If we used a trade of type $T_{3.1}$, then $f(1) = (a, b)$, $f(2) = (a, b+1)$ and $f(3) = (a+1 \pmod n, b)$ for some a and b. Also, from Lemma 2.1, the 4-cycle includes the vertices c_b and c_{b+1}, while the 5-cycle includes the vertices c_b and $r_{a+1 \pmod n}$.

Since $f(4) = (a + 1 \pmod n, b + 1)$, the triangle corresponding to this cell is connected to both the four cycle and the five-cycle in the trade $T_{3.1}$. Also the cell $f(5)$ will either involve column c_b or row $r_{a+1 \pmod n}$, depending on whether cell

$f(5)$ is in the same pair of columns $\{c_b, c_{b+1}\}$, or the next pair of columns. Thus the triangle corresponding to the entry in this cell is connected to the five cycle in the trade $T_{3.1}$. So in this case we can make closed trails of length α and β as needed.

Otherwise assume we have used a trade of type $T_{3.2}$. Then we have $f(1) = (a, b)$, $f(2) = (a + 1 \pmod n, b - 1)$ and $f(3) = (a + 1 \pmod n, b)$ for some a and b. In the trade $T_{3.2}$ the 4-cycle includes the vertices c_{b-1}, c_b and $r_{a+1 \pmod n}$, while the 5-cycle includes the vertices r_a, $r_{a+1 \pmod n}$, c_b and $a + b$. If $f(4) = (a + 2 \pmod n, b - 1)$, then the triangle corresponding to this cell contains the vertices c_{b-1} and $a + b$, and is thus connected to the 4-cycle and the 5-cycle. Additionally, the triangle corresponding to $f(5)$ includes the vertex c_b and is thus connected to the 4-cycle and the 5-cycle. Otherwise $f(4) = (a + 1 \pmod n, b + 1)$ and $f(5) = (a + 1 \pmod n, b + 2)$. So the triangles corresponding to $f(4)$ and $f(5)$ each contain the vertex $r_{a+1 \pmod n}$, and are thus each connected to both the 4-cycle and the 5-cycle. In any case we can make closed trails of length α and β as needed.

Next, suppose that there are at most two non-empty cells in columns $2m_Q$ and $2m_Q - 1$. Then from the conditions of this lemma, $\alpha + \beta \geqslant 15$. We place a latin trade of type $T_{3.1}$ on the first three cells in the sequence $f(1), f(2), \ldots, f(5)$ that occur in columns $2m_Q + 1$ and $2m_Q + 2$. Both the 4-cycle and the 5-cycle in this copy of $T_{3.1}$ intersect the row containing cells $f(1)$ and $f(2)$, so we may attach these triangles to either the 4-cycle or the 5-cycle. Any remaining triangles may be attached just as we did above. □

Now all that remains is to deal with any sets of size three not equal to $\{4, 4, 4\}$. We first need the next definition and two lemmata which show us how to trail our sets of triangles.

Definition 3.8. Suppose Q is a partial latin square of order n in proper form. We denote the following ordering $f'\colon \{1, 2, \ldots, |Q|\} \to S_Q$ as the *alternate ordering* of Q. If the cell $(y_Q, 2m_Q)$ is filled, let $f'(1) = (y_Q, 2m_Q)$. Otherwise for convenience we temporarily define $f'(0) = (y_Q, 2m_Q)$ (although $f'(0)$ is undefined in the final alternate ordering). Then we define $f'(i+1)$ in terms of $f'(i)$ recursively as follows. Let $f'(i) = (a, b)$. If b is even and $(a + 1, b - 1)$ is both non-empty and not equal to $f'(i')$ for any $i' < i$, then $f'(i + 1) = (a + 1 \pmod n, b - 1)$. If b is even and either $(a + 1, b - 1)$ is empty or equal to $f'(i')$ for some $i' < i$, then $f'(i + 1) = (a - 1 \pmod n, b + 1)$. Otherwise, b is odd and we let $f'(i + 1) = (a, b + 1)$.

Table 3 gives two partial latin squares with the cells showing the *ordering* (rather than the cell entries), standard on the left and alternate on the right. The only difference is in the way the cells are ordered as a pair of columns is completed.

standard ordering

				8	9
				10	11
				12	13
			1	14	15
		2	3	16	17
		4	5	6	7

alternate ordering

				10	11
				12	13
				14	15
			1	16	17
		2	3	6	7
		4	5	8	9

Table 3. Standard and alternate ordering illustration, case $n = 6$.

Lemma 3.9. *Let Q be a partial latin square of order n in proper form and let f' be the alternate ordering of Q as in the previous definition. Then:*

(1) *f' is a bijection;*

(2) *for any integer i, if the entries from cells $f'(i')$ for each $i' < i$ are deleted, the resulting partial latin square Q' is in proper form;*

(3) *if $f'(i)$ and $f'(i+1)$ are both defined, then $f'(i)$ and $f'(i+1)$ share either a common row or a common entry;*

(4) *if $f'(i)$ and $f'(i+4)$ are both defined, then $f'(i)$ and $f'(i+4)$ share a common row or a common column.*

Proof. Conditions 1, 2, 3 and 4 follow from the definition of f'. This can be observed in Table 3 above. □

The following lemma enables three trails of arbitrary lengths congruent to 0 (mod 3) to be formed at the same time without upsetting the "proper form" of our matrix.

Lemma 3.10. *Let Q be a partial latin square of order n in proper form and let $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$, $\eta_3 \geqslant 1$ be integers with $\eta_1 + \eta_2 + \eta_3 \leqslant |Q|$. Let f' be the alternate ordering of Q. Then there exists a partial latin square Q', also in proper form, with $Q' \subseteq Q$, such that the triangles corresponding to the entries in $Q \setminus Q'$ may be traded with three closed trails of lengths $3\eta_1$, $3\eta_2$ and $3\eta_3$. In addition, the triangles corresponding to the entries in cells $f'(1)$, $f'(2)$ and $f'(3)$ are included in the trails of length $3\eta_1$, $3\eta_2$ and $3\eta_3$, in some one-to-one correspondence.*

Proof. Let $\eta_1 \leqslant \eta_2 \leqslant \eta_3$. We shall take cells in the alternate ordering of Q and use them in closed trails CT_i of lengths $3\eta_i$, $i = 1, 2, 3$, starting as follows:

$f'(1)$ is used in CT_1, $f'(2)$ is used in CT_2, $f'(3)$ is used in CT_3,
$f'(4)$ is used in CT_3, $f'(5)$ is used in CT_1, $f'(6)$ is used in CT_2,
$f'(7)$ is used in CT_2, $f'(8)$ is used in CT_3, $f'(9)$ is used in CT_1, ...

Note that this is placing triangles from cells into the three trails in the order (with period 9)

$$1, 2, 3, 3, 1, 2, 2, 3, 1, 1, 2, 3, 3, 1, 2, 2, 3, 1, \ldots$$

In this way, consecutive cells forming part of CT_1 are either adjacent or 4 apart in the alternate ordering, and likewise for CT_2 and CT_3. So conditions 3 and 4 of Lemma 3.9 ensure that these three (partial) trails of length (so far) $3\eta_1$ are connected trails.

Now if $\eta_1 = \eta_2 = \eta_3$, we are done.

Next, if $\eta_1 = \eta_2 < \eta_3$, removing cells $f'(1), f'(2), \ldots, f'(3\eta_1)$ from Q leaves a partial latin square Q_1 still in proper form. Using a *standard* ordering of Q_1, we can form a trail of length $3(\eta_3 - \eta_1)$ (Lemma 3.5) which is connected to at least one of the trails of length $3\eta_1$ formed so far; this can become the trail of length $3\eta_3$.

Finally, if $\eta_1 < \eta_2 < \eta_3$, we form three trails of length $3\eta_1$, leaving a partial latin square Q_2 in proper form. Applying Lemma 3.6 to Q_2 we can obtain two trails of lengths $3(\eta_2 - \eta_1)$ and $3(\eta_3 - \eta_1)$. It remains to show that these are connected to two of the three trails of length $3\eta_1$ so far formed.

Since we only need that the entries in cells $f'(1)$, $f'(2)$ and $f'(3)$ are in some one-to-one correspondence with the trails of length $3\eta_1$, $3\eta_2$ and $3\eta_3$, we may assume without loss of generality that $f'(3\eta_1 - 1)$ is used in CT_2 and $f'(3\eta_1)$ is used in CT_3. There are three cases to consider: $f'(3\eta_1)$ is the final cell in a pair of columns, the penultimate cell in a pair of columns, or otherwise.

Consider the first case. Here Q_2 can be partitioned exactly into an even number of completely filled pairs of columns, so in a standard ordering f of Q_2 we can begin at any row. We therefore let $f(1)$ and $f(2)$ be in the same row as $f'(3\eta_1 - 1)$ and $f'(3\eta_1)$.

Consider the second case. If $f'(3\eta_1)$ is the penultimate cell in a pair of columns, then in a standard ordering of Q_2, $f(1)$ will be in the same column as $f'(3\eta_1 - 1)$ and $f(2)$ will be in the same row as $f'(3\eta_1)$.

In the last case, $f(1)$ will be in the same column as $f'(3\eta_1 - 1)$ and $f(2)$ will be in the same column as $f'(3\eta_1)$.

So in all cases we form closed trails of required lengths. □

Lemma 3.11. *Let α, β and γ be integers such that $\alpha \equiv \beta \equiv \gamma \equiv 1 \pmod 3$ and $4 \leqslant \alpha \leqslant \beta \leqslant \gamma$. Furthermore, suppose that $\gamma > 4$. Let Q be a partial latin square of order n in proper form such that $|Q| \leqslant \frac{1}{3}(\alpha + \beta + \gamma)$. Then there exist $\frac{1}{3}(\alpha + \beta + \gamma)$ entries in Q such that:*

- *the corresponding triangles can be decomposed into closed trails of length α, β and γ; and*
- *the removal of these entries from Q leaves a partial latin square Q' in proper form.*

P r o o f. Let f' be an alternate ordering of Q. Suppose first that $f'(1)$, $f'(2)$, $f'(3)$, …, $f'(7)$ are in the same pair of columns. Then we can use either $T_{4.1}$ or $T_{4.3}$ to trade the four triangles in $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$ with three 4-cycles. From Lemma 2.2, one such four cycle intersects c_2, a second 4-cycle intersects c_1, and a third intersects both c_1 and c_2. Thus the triangles corresponding to the cells $f'(5)$, $f'(6)$ and $f'(7)$ may be attached each to one of our four-cycles. So from Lemma 3.5 (if $\beta = 4$), Lemma 3.6 (if $\alpha = 4 < \beta$) or Lemma 3.10 (if $\alpha > 4$), we can construct trails of length α, β and γ from the triangles from entries in cells $f'(1), f'(2), \ldots$, $f'(\frac{1}{3}(\alpha+\beta+\gamma))$.

Next $f'(1), f'(2), f'(3), \ldots, f'(6)$ are in the same pair of columns, but $f'(7)$ lies in the next pair of columns. Then we use $T_{4.1}$ on cells $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$. But from Lemma 2.2, one 4-cycle intersects c_2, a second 4-cycle intersects c_1, and a third intersects r_2. Since $f'(7)$ lies in the same row as $f'(3)$ and $f'(4)$, we may proceed as in the previous paragraph.

Next $f'(1)$, $f'(2)$, $f'(3)$, …, $f'(5)$ are in the same pair of columns, but $f'(6)$ and $f'(7)$ lie in the next pair of columns. Then we use $T_{4.3}$ on cells $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$. But from Lemma 2.2, one 4-cycle intersects r_2 (and thus $f'(7)$), a second 4-cycle also intersects r_2 (and thus $f'(6)$), and a third intersects c_2 (and thus $f'(5)$). We proceed as before.

If $f'(4)$ is in a different pair of columns to $f'(5)$, we use $T_{4.1}$ on cells $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$. Then observe that one 4-cycle intersects r_1 (and thus $f'(5)$), a second 4-cycle intersects r_2 (and thus $f'(7)$), and a third intersects r_1 and r_2 (and thus $f'(6)$).

If $f'(3)$ is in a different pair of columns to $f'(4)$, we use $T_{4.2}$ on cells $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$. Again we can attach our trails as required.

If $f'(2)$ is in a different pair of columns to $f'(3)$, there are some special cases to consider. If $\alpha, \beta, \gamma \leqslant 7$, we use $T_{4.2}$ on cells $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. Now, the entries in cells $f'(1)$, $f'(6)$ and $f'(7)$ attach in 1–1 correspondence with the 4-cycles in $T_{4.2}$. Therefore these entries may be attached (where needed) to construct the closed trails of length at most 7. Otherwise assume, without loss of generality, that $\gamma \geqslant 10$. In this case, we place $T_{4.1}$ on cells $f'(3)$, $f'(4)$, $f'(5)$ and $f'(6)$. As above, we can attach trails of length $\alpha - 4$, $\beta - 4$ and $\gamma - 10$ (where non-zero) to these 4-cycles. The 6-circuit formed by the entries in cells $f'(1)$ and $f'(2)$ may then be attached to any of the 4-cycles in $T_{4.1}$. So we form closed trails of the required lengths.

Finally, when $f'(1)$ is in a different pair of columns to $f'(2)$, we again use $T_{4.2}$, but this time on cells $f'(1)$, $f'(2)$, $f'(3)$ and $f'(4)$. Once again the trails are attached using one of Lemmata 3.5, 3.6 or 3.10. □

Lemma 3.12. *Let α, β, and γ be integers such that $\alpha \equiv \beta \equiv \gamma \equiv 2 \pmod 3$, and $5 \leqslant \alpha \leqslant \beta \leqslant \gamma$. Let Q be a partial latin square of order n in proper form such that $|Q| \geqslant \frac{1}{3}(\alpha+\beta+\gamma)$. Then there exist $\frac{1}{3}(\alpha+\beta+\gamma)$ entries in Q such that:*

- *the corresponding triangles can be decomposed into closed trails of length α, β and γ; and*
- *the removal of these entries from Q leaves a partial latin square Q' in proper form.*

Proof. In our proof we will not always delete entries according to the alternate ordering of Q; however we will always have left a partial latin square Q' in proper form.

Let f' be the alternate ordering of Q. Suppose first that $f'(1)$, $f'(2)$, $f'(3)$, …, $f'(8)$ are in the same pair of columns. Suppose also that $f'(7)$ and $f'(8)$ lie in the same row. Then we can use $T_{5.1}$ to trade the five triangles in $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$ with three 5-cycles. If $\alpha=\beta=\gamma=5$ we are done.

Otherwise, from Lemma 2.3, one such 5-cycle intersects c_1, and the remaining intersect c_2. Thus the triangles corresponding to the cells $f'(6)$, $f'(7)$ and $f'(8)$ may be attached each to one of our five-cycles. So from Lemma 3.5 (if $\beta=5$), Lemma 3.6 (if $\alpha=5<\beta$) or Lemma 3.10 (if $\alpha>5$), we can construct trails of length α, β and γ from the triangles from entries in cells $f'(1), f'(2), \ldots, f'(\frac{1}{3}(\alpha+\beta+\gamma))$. If $f'(7)$ and $f'(8)$ are in different rows, we use the trade $T_{5.3}$, and the rest follows similarly.

Next $f'(1)$, $f'(2)$, $f'(3)$, …, $f'(7)$ are in the same pair of columns, but $f'(8)$ lies in the next pair of columns. Then we use $T_{5.3}$ on cells $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. But from Lemma 2.3, one 5-cycle intersects c_1, a second 5-cycle intersects c_2, and a third intersects r_3. Also $f'(8)$ lies in the same row as $f'(4)$ and $f'(5)$. Thus we may proceed as in the previous paragraph.

Next $f'(1)$, $f'(2)$, $f'(3)$, …, $f'(6)$ are in the same pair of columns, but $f'(7)$ and $f'(8)$ lie in the next pair of columns. Then we use $T_{5.1}$ on cells $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. From Lemma 2.3, one 5-cycle intersects r_3 (and thus $f'(6)$), and the remaining two intersect r_2 (and thus $f'(7)$ and $f'(8)$). We proceed as before.

If $f'(5)$ is in a different pair of columns to $f'(6)$, we use $T_{5.3}$ on cells $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. Then observe that one 5-cycle intersects r_3 (and thus $f'(8)$, and the remaining 5-cycles intersects r_2 (and thus $f'(7)$ and $f'(6)$).

If $f'(4)$ is in a different pair of columns to $f'(5)$, we use $T_{5.2}$ on cells $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. Then one 5-cycle intersects r_1 while the remaining 5-cycles intersect r_2. Again we can attach our trails as required.

If $f'(3)$ is in a different pair of columns to $f'(4)$, we must do something a little different. Let the cells $f'(1)$, $f'(2)$ and $f'(3)$ be (a,b), $(a+1 \pmod n, b-1)$ and $(a+1 \pmod n, b)$ respectively. Then we may place the trade $T_{5.6}$ on these cells together with cells $(a-1 \pmod n), b+1)$ and $(a-1 \pmod n), b+2)$. The deletion of these cells leaves a partial latin square Q_1 still in proper form. Moreover, the triangles corresponding to the first three cells in an alternate ordering of Q_1 may be attached in one-to-one correspondence with the three 5-cycles in $T_{5.6}$. One 5-cycle in $T_{5.6}$ intersects c_4 and r_3, a second intersects c_3 while a third intersects r_2 and r_3.

If $f'(2)$ is in a different pair of columns to $f'(3)$, this is similar to the previous paragraph except we use the trade $T_{5.7}$. Finally when $f'(1)$ is in a different pair of columns to $f'(2)$, we use $T_{5.4}$ on cells $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$ and $f'(5)$. Once again the trails are attached (if needed at all) with one of Lemmata 3.5, 3.6 or 3.10. □

Theorem 3.13. *Let n be an even integer and let $P = \{x_1, x_2, \ldots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Then it is possible to decompose the edges of $K_{n,n,n}$ into trails of lengths $x_1, x_2, \ldots, x_m$.*

Proof. Let $P_1, P_2, \ldots, P_L$ be a partition of P satisfying the conditions of Lemma 2.6. We will "trade" triangles with sets of closed trails with integer lengths from sets P_j, $1 \leqslant j \leqslant L$.

Without loss of generality, let $P_1 = P_2 = \ldots = P_l = \{4, 4, 4\}$, for some integer l, such that for all P_j with $l < j \leqslant L$, $P_j \neq \{4, 4, 4\}$. (Note that we may have $l = 0$.) Then from Lemma 3.2 there exists a partial latin square Q_1 in proper form such that the partial latin square $B_n \setminus Q_1$ can be partitioned into l copies of $T_{4.1}$.

Next assume, without loss of generality, that $P_{l+1} = \{x_{l+1}\}$, $P_{l+2} = \{x_{l+2}\}$, …, $P_{l'} = \{x_{l'}\}$, and for all P_j with $j > l'$, P_j is not a subset of size one. (It is possible that $l = l'$.) Then, from repeated applications of Lemma 3.5, there is a partial latin square Q_2 in proper form such that the triangles in $Q_1 \setminus Q_2$ can be exchanged with circuits of lengths $x_{l+1}, \ldots, x_{l'}$.

Similarly, let $P_{l'+1}, \ldots, P_{l''}$ be the remaining subsets of size 3, where $l'' \geqslant l'$. Applying Lemmata 3.11 and 3.12 repeatedly leaves a partial latin square Q_3 in proper form, such that the triangles in $Q_2 \setminus Q_3$ may be exchanged with circuits of lengths from the sets $P_{l'+1}, \ldots, P_{l''}$.

If there are no remaining sets P_j, we are done. Otherwise, there are at most two remaining sets P_j, each of size 2. First consider each set of the form $\{\alpha, \beta\}$ with the property that $\alpha + \beta \geqslant 15$. We apply Lemma 3.7 to each such set, and are left with a partial latin square Q_4 in proper form. Finally, there are at most two remaining subsets of the form $\{\alpha, \beta\}$ such that $9 \leqslant \alpha + \beta \leqslant 12$. However, since $n \geqslant 6$, this implies that there are at most two non-empty columns in Q_4. Moreover for each such subset, there are at least three non-empty cells in the final two columns. So again we may apply Lemma 3.7. Thus our decomposition is complete. □

Example 3.14. $K_{6,6,6}$ decomposed into closed trails of lengths $\{4,4,4,4,4,4,4,4,5,5,5,7,8,9,10,10,17\}$.

We partition $P = \{4,4,4,4,4,4,4,4,5,5,5,7,8,9,10,10,17\}$ as given below, where the cell numbers indicate the part of B_6 (in Table 4) which we use to trade cells (triangle edges) to form closed trails of appropriate lengths.

Partition	Cells	Structure of trails
$\{4,4,4\}$	**1–4**	C_4, C_4, C_4
$\{4,4,4\}$	**5–8**	C_4, C_4, C_4
$\{9\}$	**9–11**	$C_3 \cup C_3 \cup C_3$
$\{4,4,10\}$	**12**,**21**,**22**,**23**,**24**,**13**	C_4, C_4, $C_3 \cup C_3 \cup C_4$
$\{5,5,5\}$	**14–18**	C_5, C_5, C_5
$\{7,8\}$	**19**,**20**,**31**,**32**,**33**	$C_3 \cup C_4$ $C_3 \cup C_5$
$\{10,17\}$	**25–30**, **34–36**	$C_3 \cup C_3 \cup C_4$, $C_3 \cup C_3 \cup C_3 \cup C_3 \cup C_5$

The latin square B_6 in Table 4 has the cells numbered (in bold subscripts) according to order as described above.

$1_{\mathbf{1}}$	$2_{\mathbf{2}}$	$3_{\mathbf{13}}$	$4_{\mathbf{14}}$	$5_{\mathbf{25}}$	$6_{\mathbf{26}}$
$2_{\mathbf{3}}$	$3_{\mathbf{4}}$	$4_{\mathbf{15}}$	$5_{\mathbf{16}}$	$6_{\mathbf{27}}$	$1_{\mathbf{28}}$
$3_{\mathbf{5}}$	$4_{\mathbf{6}}$	$5_{\mathbf{17}}$	$6_{\mathbf{18}}$	$1_{\mathbf{29}}$	$2_{\mathbf{30}}$
$4_{\mathbf{7}}$	$5_{\mathbf{8}}$	$6_{\mathbf{19}}$	$1_{\mathbf{20}}$	$2_{\mathbf{31}}$	$3_{\mathbf{32}}$
$5_{\mathbf{9}}$	$6_{\mathbf{10}}$	$1_{\mathbf{21}}$	$2_{\mathbf{22}}$	$3_{\mathbf{33}}$	$4_{\mathbf{34}}$
$6_{\mathbf{11}}$	$1_{\mathbf{12}}$	$2_{\mathbf{23}}$	$3_{\mathbf{24}}$	$4_{\mathbf{35}}$	$5_{\mathbf{36}}$

Table 4: Latin square B_6, with cells numbered for reference.

4. The case n odd

In this case we take the first three columns of the back circulant latin square L, and then pair all the remaining columns, as in the previous section. Luckily once we have dealt with the first three columns the proof becomes almost identical to the previous section.

Throughout here we assume that $n \geqslant 7$ is odd. (See [10] for $n \leqslant 5$.)

Definition 4.1. Let n be an odd integer. Let Q be a partial latin square such that $Q \subseteq B_n$. We say that Q is in *early odd form* if there exist integers y_Q, $1 \leqslant y_Q \leqslant n-1$, and $\delta_Q(=\delta)$, $\delta_Q \geqslant 5$, such that

(1) Q is filled in columns 4 to n;
(2) in columns 1, 2 and 3, rows $1, 2, \ldots, y_Q - 1$ are empty and rows $y_Q + 1, \ldots, n$ are filled;

(3) cell $(y_Q, 1)$ is empty;
(4) if cell $(y_Q, 3)$ is empty then cell $(y_Q, 2)$ is empty; and
(5) there are $\delta_Q \geqslant 5$ non-empty cells in the first 3 columns.

See Table 5(a) for an example of a partial latin square in early odd form.

			4	5	6	7
			5	6	7	1
			6	7	1	2
		6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

(a)

		1				
2	3	4				
5	7	8				
6	9	10				

(b)

Table 5. (a) A partial latin square of order 7 in early odd form.
(b) The early odd ordering (see Definition 4.4) for this partial latin square.

Definition 4.2. Let n be an odd integer. Let Q be a partial latin square such that $Q \subseteq B_n$. We say that Q is in *proper odd form* if there exists $m = m_Q$, $1 \leqslant m \leqslant \frac{1}{2}(n-1)$ such that

(1) Q is empty in columns 1 to $2m-1$;
(2) Q is filled in columns $2m+2$ to n; and
(3) in columns $2m$ and $2m+1$, there exist integers $z = z_Q$, $y = y_Q$, $z \neq y$, $1 \leqslant z, y \leqslant n$ such that rows $r_{z+i \pmod n}$, (where $0 \leqslant i \leqslant y - z - 1 \pmod n$) are empty, rows $r_{y+i \pmod n}$, (where $1 \leqslant i \leqslant z - y - 1 \pmod n$) are filled, cell $(y, 2m)$ is empty and cell $(y, 2m+1)$ is either empty or filled.

The above definition is almost identical to Definition 3.1, and thus once we have taken care of either the first column or the first three columns, our proof proceeds almost identically to the previous section. Note that, in particular, the concepts of standard ordering and alternate ordering extend in the obvious way to a partial latin square in *proper* odd form.

As in the previous section subsets of the form $\{4, 4, 4\}$ are dealt with first.

Lemma 4.3. *Let $3n^2 > 4l$ for some integer $l \geqslant 6$ divisible by 3. Then there exists a partial latin square Q in either early odd form or proper odd form, such that the triangles corresponding to the entries in $B_n \setminus Q$ may be exchanged with l 4-cycles.*

P r o o f. The first case we consider is when $4l \geqslant 9n - 9$.

Suppose $n \equiv 1 \pmod 4$. Then we fill the first 3 columns with trades of type $T_{12.1}$ (from Lemma 2.5) until there are exactly five non-empty rows in the first three columns. Then we place a trade of type $T_{12.2}$ in the final five rows, so that only the cells $(n-1,3)$, $(n,2)$ and $(n,3)$ remain non-empty. Then all the remaining non-empty cells constitute a partial latin square in proper odd form. Also we have used $\frac{1}{4}(9n-9)$ 4-cycles. The remaining 4-cycles are exchanged with triangles using trades of type $T_{4.3}$ within a pair of columns, and trades of type $T_{4.2}$ between columns. This is possible as there will always be an odd overlap between pairs of columns.

Otherwise $n \equiv 3 \pmod 4$. Then we place trades of type $T_{12.1}$ until there are exactly three rows left in the first three columns. In the last three rows we use a trade of type $T_{8.1}$. The remaining partial latin square is in proper odd form. We then proceed as in the previous paragraph.

The remaining case is $4l \leqslant 9n-15$ (since 3 divides l and n is odd). Then we place trades of type $T_{12.1}$ in the first three columns until six, twelve or no 4-cycles remain. If no 4-cycles remain we are done. If there are six remaining 4-cycles, we use a trade $T_{8.1}$. Finally if there are twelve remaining 4-cycles, then we use trades $T_{8.1}$ and $T_{8.2}$ combined. In each case the resulting partial latin square is in early odd form with at least 5 non-empty cells in the first three columns. ☐

So from the above lemma, if there are at least two triples $\{4,4,4\}$ in the partition of P (so $l \geqslant 6$) we have dealt with all such triples $\{4,4,4\}$.

The following definition gives an ordering of elements in the first three columns of a partial latin square in early odd form. Note the irregularity in the ordering of the final two rows. This is to aid the awkward transition from the first three columns to the following pair of columns.

Definition 4.4. Let Q be a partial latin square of order n in early odd form. We denote the following ordering $g\colon N(\delta_Q) \to S_Q$, where $N(\delta_Q) = \{1,2,\ldots,\delta_Q\}$, as the *early odd ordering* of Q. First, $g(\delta) = (n,3)$, $g(\delta-1) = (n,2)$, $g(\delta-2) = (n-1,3)$, $g(\delta-3) = (n-1,2)$ and $g(\delta-4) = (n,1)$. Then, assume $\delta > 5$. Let $g(1) = (y_Q+1,1)$ if cell $(y_Q,3)$ is empty; otherwise let $g(1) = (y_Q,a)$, where a is the least column with a non-empty cell in row y_Q. Then we define $g(i+1)$ in terms of $g(i)$ recursively as follows for $i < \delta-5$. Let $g(i) = (a,b)$. If $b < 3$, $g(i+1) = (a,b+1)$. If $b = 3$, $g(i+1) = (a+1,1)$.

See Table 5(b) for an example illustrating this early odd ordering. Table 6 below exhibits the early odd ordering of Q in general, in the final three rows.

$\delta-8$	$\delta-7$	$\delta-6$			
$\delta-5$	$\delta-3$	$\delta-2$			
$\delta-4$	$\delta-1$	δ			

Table 6. Early odd ordering, last three rows.

Lemma 4.5. *Let Q be a partial latin square of order n in early odd form and let g be the early odd ordering of Q as in the previous definition. Then:*

(1) *for any integer $i \leqslant \delta - 5$, if the entries from cells $g(i')$ for each $i' \leqslant i$ are deleted, then the resulting partial latin square is in early odd form;*

(2) *for each integer $\delta - 4 \leqslant i \leqslant \delta$, if the entries from cells $g(i')$ for each $i' \leqslant i$ are deleted, then the resulting partial latin square is in proper odd form;*

(3) *if $i < \delta - 3$ and $i \neq \delta - 7$, then $g(i)$ and $g(i+3)$ share a common entry, column or row;*

(4) *except when both $g(i)$ is in column 3 and $i \leqslant \delta - 6$, the cells $g(i)$ and $g(i+1)$ share a common row, column or entry;*

(5) *if $g(i)$ is in column 3 and $i \leqslant \delta - 6$, then both $g(i-2)$ and $g(i-1)$ (where defined) share a common row, column or entry with both $g(i)$ and $g(i+1)$;*

(6) *unless $i = \delta - 4$, the cells $g(i)$ and $g(i+2)$ share a common row, column or entry.*

Lemma 4.6. *Let Q be a partial latin square in early odd form. Let $l \neq 6$ be an integer divisible by 3 such that $\frac{1}{3}l \leqslant |Q|$. Then there exists a partial latin square Q' such that the triangles corresponding to the entries in $Q \setminus Q'$ may be exchanged with a circuit of length l. In addition, Q' is either in early odd form or in proper odd form.*

Proof. Let g be an early odd ordering of Q. First suppose that $\frac{1}{3}l \leqslant \delta_Q$. Then it is not difficult to check that the triangles corresponding to cells $g(1), g(2), \ldots, g(\frac{1}{3}l)$ form a connected closed trail of length l. The removal of these cells gives the desired Q'.

Otherwise $\delta_Q < \frac{1}{3}l$. Now, let Q'' be the partial latin square in proper odd form, remaining after the removal of cells $g(1), g(2), \ldots, g(\delta_Q)$. Let f be a standard ordering of Q'' with $f(1) = (n, 4)$. Then, clearly, the triangles corresponding to the above cells, together with $f(1), \ldots, f(\frac{1}{3}l - \delta_Q)$ form a closed trail of length l, and the deletion of all these cells leaves a partial latin square Q' in proper odd form. □

Lemma 4.7. *Let Q be a partial latin square in early odd form. Then there exists a partial latin square Q' such that the triangles corresponding to the entries in $Q \setminus Q'$ may be exchanged with two circuits of length 6. In addition, Q' is either in early odd form or in proper odd form.*

Proof. Let g be an early odd ordering of Q. Let Q' be the partial latin square formed by deleting from Q the entries in cells $g(1)$, $g(2)$, $g(3)$ and $g(4)$. From Lemma 4.5, Q' is in either early odd form or proper odd form. So it remains to chech that we can partition the four cells into two 2-sets so that the cells in each 2-set share an adjacent row, column or entry. If $\delta \leqslant 8$, it is straightforward to check this condition in each case. If $\delta > 8$, then either $g(1)$ and $g(4)$ share the same column and $g(2)$ and $g(3)$ share the same row, or $g(1)$ and $g(2)$ lie in one row and $g(3)$ and $g(4)$ lie in the next row. □

Lemma 4.8. *Let Q be a partial latin square of order n in early odd form and let $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$ be integers with $\eta_1 + \eta_2 \leqslant |Q|$. Let g be the early odd ordering of Q. Then there exists a partial latin square Q' (with $Q' \subseteq Q$), such that Q' is in either early odd form or proper odd form, and the triangles corresponding to the entries in $Q \setminus Q'$ may be traded with two closed trails of lengths $3\eta_1$ and $3\eta_2$.*

In addition, the triangles corresponding to the entries in cells $g(1)$ and $g(2)$ may be put in some $1-1$ correspondence with the trails of lengths $3\eta_1$ and $3\eta_2$. Moreover, if $\eta_1 + \eta_2 \geqslant 4$, either $1 - 1$ correspondence is possible.

Proof. Let $\eta_m = \min\{\eta_1, \eta_2\}$. We initially aim to construct two closed trails, each of length $3\eta_m$.

First consider when δ is even. For each i, $i \leqslant \delta - 4$ and $i \leqslant 2\eta_m$, let the triangle corresponding to $g(i)$ be part of a closed trail CT_1 or CT_2, for i odd or even, respectively. If $2\eta_m \geqslant \delta - 2$, we take a standard ordering f of the remaining partial latin square (with $f(1) = (n-1, 2)$, $f(2) = (n-1, 3)$, and so forth). Then we proceed in a similar fashion to Lemma 3.6, with $f(i)$ being assigned to CT_1 and CT_2, for i even or odd, respectively, for $i \leqslant 2\eta_m$.

Next consider when δ is odd. If $\delta = 5$, we assign the entries in $g(1)$ and $g(2)$ to CT_1 and CT_2 respectively. In a standard ordering f of the remaining partial latin square, we assign $f(i)$ to CT_1 and CT_2, for i even or odd, respectively, for $i \leqslant 2\eta_m$. Otherwise $\delta > 5$. Then for each i, $i \leqslant \delta - 5$ and $i \leqslant 2\eta_m$, let the triangle corresponding to $g(i)$ be part of a closed trail CT_1 or CT_2, for i odd or even, respectively. If $\delta - 3 \leqslant 2\eta_m$, we assign the entries in $g(\delta-4)$ and $g(\delta-3)$ to CT_2 and CT_1 respectively. If $\delta - 1 \leqslant 2\eta_m$, we take the standard ordering f of the remaining partial latin square (so $f(1) = (n-1, 3)$, $f(2) = (n, 2)$, etc.). Then we proceed in a similar fashion to Lemma 3.6, with $f(i)$ being assigned to CT_1 and CT_2, for i odd or even, respectively.

Lemma 4.5 ensures that CT_1 and CT_2 are both connected trails of length $3\eta_m$. If $\eta_1 = \eta_2$ we are done. Otherwise suppose that $\eta_1 \neq \eta_2$. If $2\eta_m \geqslant \delta - 4$, then as in Lemma 3.6 we take a standard ordering f' of the remaining partial latin square. We form a connected closed trail of length $3|\eta_1 - \eta_2|$ from the first $|\eta_1 - \eta_2|$ entries in this standard ordering. If $\eta_1 + \eta_2 \geqslant 4$, this closed trail will connect to either of the trails CT_1 or CT_2. In any case this closed trail will connect to at least one of CT_1 or CT_2.

Otherwise, $2\eta_m \leqslant \delta - 5$. Take an early odd ordering g' of the remaining cells in the first three columns. From Lemma 4.6, if either $|\eta_2 - \eta_1| \neq 2$ or $g'(1)$ is not in column 3, we can construct a trail of length $3|\eta_2 - \eta_1|$, so that the partial latin square that is left is in either early odd form or proper odd form. Conditions 4 and 5 of Lemma 4.5 ensure that if $\eta_1 + \eta_2 \geqslant 4$, this trail may be attached to either one of the trails of length $3\eta_m$. In any case, we can attach the trail to at least one of the trails of length $3\eta_m$. When $|\eta_2 - \eta_1| = 2$ and $g'(1)$ is in column 3, the cells $g'(1)$ and $g'(2)$ do not share a common row, column or entry. However, in this case, $f(2\eta_m - 1)$ and $f(2\eta_m)$ are each connected to the entries in $g'(1)$ and $g'(2)$. So in all cases we obtain the required connected closed trails. □

Lemma 4.9. *Let Q be a partial latin square in early odd form. Let α, β be integers, such that $\alpha \equiv 1$(modulo 3), $\beta \equiv 2$(modulo 3), $\alpha \geqslant 4$, $\beta \geqslant 5$ and $10 \leqslant \frac{1}{3}(\alpha + \beta) \leqslant |Q|$. Then there exists a partial latin square Q' such that the triangles corresponding to the entries in $Q \setminus Q'$ may be exchanged with circuits of lengths α and β. In addition, Q' is either in early odd form or proper odd form.*

Proof. Let $\delta \geqslant 5$ as usual be the number of non-empty cells in the first three columns of Q. Let g be an early odd ordering of Q. We split our proof into three cases, according to the congruency of δ modulo 3.

First suppose that δ is divisible by 3. If $\delta \geqslant 9$, we place the trade $T_{3.1}$ on cells $g(1)$, $g(2)$ and $g(4)$. The triangle corresponding to the entry in $g(3)$ can be attached to either the 4-cycle or the 5-cycle from $T_{3.1}$. We then apply Lemma 4.8 on Q after removing these 5 entries to construct either trails of length $\alpha - 7$ and $\beta - 5$, or $\alpha - 4$ and $\beta - 8$ (whichever ensures that both values are greater than zero). From the condition $\alpha + \beta - 12 > 5$, we can ensure that these trails intersect $g(5)$ and $g(6)$, respectively. Since the entry in $g(5)$ attaches to the 4-cycle and the entry in $g(6)$ attaches to the 5-cycle, we are done. If $\delta = 6$, we place the trade $T_{3.1}$ on cells $g(1)$, $g(2)$ and $g(3)$. What remains is in proper odd form, and $g(4)$ and $g(5)$ attach to the 4-cycle and 5-cycle, respectively, from $T_{3.1}$.

Next let $\delta \equiv 1$ (modulo 3). If $\delta \geqslant 10$, we place the trade $T_{3.1}$ on cells $g(2)$, $g(3)$ and $g(5)$. Then the entries in $g(1)$ and $g(4)$ each attach to either the 4-cycle or the 5-cycle from $T_{3.1}$. If $\delta = 7$, we use the trade $T_{3.1}$ on cells $g(2)$, $g(3)$ and $g(4)$. Then the entries in cells $g(1)$ and $g(5)$ each attach to either the 4-cycle or the 5-cycle from $T_{3.1}$. The rest is similar to before.

Finally, assume that $\delta \equiv 2$ (modulo 3). If $\delta \geqslant 11$, we use the trade $T_{3.2}$ on cells $g(2)$, $g(4)$ and $g(5)$. Here the triangles corresponding to the entries in $g(1)$ and $g(3)$ each attach to either the 4-cycle or the 5-cycle. Moreover, the entries in $g(6)$ and $g(7)$ attach to the 4-cycle and the 5-cycle, respectively. If $\delta = 8$, we use $T_{3.2}$ again, this time on cells $g(2)$, $g(5)$ and $g(6)$. Then each of the triangles corresponding to entries in $g(1)$, $g(3)$ and $g(4)$ attach to either the 4-cycle or the 5-cycle in $T_{3.2}$. Also $g(7)$ and $g(8)$ attach to the 4-cycle and 5-cycle respectively. (Note that in this case we need the full strength of the condition $\frac{1}{3}(\alpha+\beta) \geqslant 10$. This ensures that two trails of total length $\alpha+\beta-18 \geqslant 12$ can be constructed in any specified correspondence with $g(7)$ and $g(8)$, from the previous lemma. This strong condition is needed because the 4-cycle and the 5-cycle are not indistinguishable.) If $\delta = 5$, we place $T_{3.2}$ on cells $g(3)$, $g(4)$ and $g(5)$. Each of the triangles corresponding to entries in $g(1)$ and $g(2)$ attach to either the 4-cycle or the 5-cycle in $T_{3.2}$. Also the entries in cells $(n, 4)$ and $(n, 5)$ attach to the 4-cycle and 5-cycle respectively. □

Lemma 4.10. *Let Q be a partial latin square of order n in early odd form and let $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$, $\eta_3 \geqslant 1$ be integers with $\eta_1 + \eta_2 + \eta_3 \leqslant |Q|$. Let g be the early odd ordering of Q. Then there exists a partial latin square Q', also in proper form, with $Q' \subseteq Q$, such that the triangles corresponding to the entries in $Q \setminus Q'$ may be traded with three closed trails of lengths $3\eta_1$, $3\eta_2$ and $3\eta_3$. In addition, the triangles corresponding to the entries in cells $g(1)$, $g(2)$ and $g(3)$ are included in the trails of length $3\eta_1$, $3\eta_2$ and $3\eta_3$, in some one-to-one correspondence.*

Proof. Assume without loss of generality that $\eta_1 \leqslant \eta_2 \leqslant \eta_3$. For each i, $i \leqslant \delta - 6$ and $i \leqslant 3\eta_1$, let the entry in cell $g(i)$ be part of a closed trail CT_j, where $j \in \{1, 2, 3\}$ and $j \equiv i$ (modulo 3). Clearly each CT_j is so far connected, since $f(i)$ and $f(i + 3)$ lie in the same column when $i \leqslant \delta - 6$.

Now if $3\eta_1 > \delta - 6$, we continue forming the trails CT_j, $j \in \{1, 2, 3\}$ as follows. If $\delta \equiv 0$ (modulo 3) and $3\eta_1 \geqslant \delta - 3$, we let the entries in $g(\delta - 5)$, $g(\delta - 4)$ and $g(\delta - 3)$ be part of closed trails CT_2, CT_1 and CT_3 respectively.

Next let $\delta \equiv 2$ (modulo 3). If $3\eta_1 \geqslant \delta - 5$, we let the entry in $g(\delta - 5)$ be part of closed trail CT_1. Furthermore if $3\eta_1 \geqslant \delta - 2$, we let the entries in $g(\delta - 4)$, $g(\delta - 3)$ and $g(\delta - 2)$ be part of closed trails CT_3, CT_2 and CT_1 respectively.

Thirdly, let $\delta \equiv 1$ (modulo 3). If $3\eta_1 \geqslant \delta - 3$, we let the entries in $g(\delta - 5)$ and $g(\delta - 4)$ be part of closed trails CT_1 and CT_2 respectively. Furthermore if $3\eta_1 \geqslant \delta - 1$, we let $g(\delta - 3)$, $g(\delta - 2)$ and $g(\delta - 1)$ be part of closed trails CT_3, CT_1 and CT_2 respectively.

Now suppose that the three closed trails so far formed are still not each of length $3\eta_1$. Let Q_1 be the partial latin square in proper odd form that remains after the removal of the entries so far used. Let f' be an alternate ordering of Q_1. We shall take cells in Q_1 and use them in CT_j, $j \in \{1, 2, 3\}$ in the following pattern (with period 9).

$$\begin{gathered} f'(1) \text{ is used in } CT_2,\ f'(2) \text{ is used in } CT_3,\ f'(3) \text{ is used in } CT_1, \\ f'(4) \text{ is used in } CT_1,\ f'(5) \text{ is used in } CT_2,\ f'(6) \text{ is used in } CT_3, \\ f'(7) \text{ is used in } CT_3,\ f'(8) \text{ is used in } CT_1,\ f'(9) \text{ is used in } CT_2,\ \ldots \end{gathered}$$

Note that this is essentially the same pattern as used in Lemma 3.10. We continue until three closed trails, each of length $3\eta_1$ are formed.

Let Q_2 be the remaining partial latin square, either in early odd form or proper odd form. If Q_2 is in proper odd form, then the remainder of the proof follows as in the proof of Lemma 3.10. So suppose that Q_2 is in early odd form. If $\eta_1 = \eta_2 = \eta_3$ we are done. If $\eta_1 = \eta_2 < \eta_3$ and $\eta_3 - \eta_1 \neq 2$, we can create a trail of length $3(\eta_3 - \eta_1)$ as in Lemma 4.6, leaving a partial latin square in either early odd form or proper odd form. This trail will connect to at least one of the closed trails of length $3\eta_1$, giving closed trails of required lengths. If $\eta_3 - \eta_1 = 2$ then let g' be an early odd ordering of Q_2. From Condition 5 of Lemma 4.5 both $g'(1)$ and $g'(2)$ connect to at least one of the three trails of length $3\eta_1$.

Otherwise $\eta_1 < \eta_2 < \eta_3$. Then we create trails of lengths $3(\eta_3 - \eta_1)$ and $3(\eta_2 - \eta_1)$ as in Lemma 4.8. The conditions of Lemma 4.5 ensure that we can attach these two trails to at least two of our closed trails of length $3\eta_1$. □

Lemma 4.11. *Let Q be a partial latin square in early odd form. Let α, β, γ be integers, such that $\alpha \equiv \beta \equiv \gamma \equiv 1$ (modulo 3), $4 \leqslant \alpha \leqslant \beta \leqslant \gamma$ and $5 \leqslant \frac{1}{3}(\alpha+\beta+\gamma) \leqslant |Q|$. Then there exists a partial latin square Q' such that the triangles corresponding to the entries in $Q \setminus Q'$ may be exchanged with circuits of length α, β and γ. In addition, Q' is either in early odd form or proper odd form.*

P r o o f. Let δ as usual be the number of non-empty cells in the first three columns of Q and let g be an early odd ordering of Q. If $\delta = 5$, then we can place $T_{4.1}$ on cells $g(\delta)$, $g(\delta-1)$, $g(\delta-2)$ and $g(\delta-3)$, and adjoin the triangle from cell $g(\delta-4)$ to any of the 4-cycles from $T_{4.1}$; then using an alternate ordering of the remaining cells in Q, we can adjoin the remaining triangles to trails, as in the previous section.

If $\delta = 6$, we place $T_{4.1}$ on cells $g(\delta-5)$, $g(\delta-3)$, $g(\delta-4)$ and $g(\delta-1)$, attach the triangle from $g(\delta-2)$ to any of the 4-cycles, and then adjoin remaining triangles with an alternate ordering of Q minus these 5 cells.

If $\delta = 7$, use trade $T_{5,5A}$ on cells $g(\delta-6)$, $g(\delta-5)$, $g(\delta-4)$, $g(\delta-3)$ and $g(\delta-2)$. Note that the triangles corresponding to cells $g(\delta-1)$, $g(\delta)$ and $(n-1,4)$ may be attached in any $1-1$ correspondence with the two 4-cycles and one 7-cycle. This follows since in $T_{5.5A}$, r_2 occurs in both 4-cycles and the 7-cycle (so the triangle corresponding to cell $(n-1,4)$ is attached to all these three cycles), the triangle (on vertices r_3, c_2 and 6) corresponding to cell $g(\delta-1)$ is adjacent to all three cycles in $T_{5.5A}$ while the triangle corresponding to cell $f(\delta) = (r_3, c_3)$ is also adjacent to all three cycles in $T_{5.5A}$. So we use an alternate ordering of Q minus the cells from the trade $T_{5.5A}$.

If $\delta = 8$, place the trade $T_{4.2}$ on cells $g(\delta-7)$, $g(\delta-6)$, $g(\delta-5)$ and $g(\delta-3)$. The triangle corresponding to cell $g(\delta-4)$ attaches to any of the 4-cycles in this trade. Also, the cells corresponding to $g(\delta-2)$, $g(\delta-1)$ and $g(\delta)$ may be put in $1-1$ correspondence with the three 4-cycles. So we proceed as before.

If $\delta = 9$ and $\{\alpha, \beta, \gamma\} = \{4, 4, 7\}$, we place the trade $T_{5.9A}$ on cells $g(\delta-8)$, $g(\delta-7)$, $g(\delta-6)$, $g(\delta-5)$ and $g(\delta-4)$. Otherwise $\alpha+\beta+\gamma \geqslant 18$. Then we place $T_{4.3}$ on cells $g(\delta-7)$, $g(\delta-5)$, $g(\delta-4)$ and $g(\delta-3)$. The triangles corresponding to cells $g(\delta-8)$ and $g(\delta-6)$ are adjacent to any of the three 4-cycles in this trade. Also the cells corresponding to $g(\delta-2)$, $g(\delta-1)$ and $g(\delta)$ may be put in $1-1$ correspondence with the three 4-cycles.

Otherwise $\delta \geqslant 10$. If $\delta \equiv 1$ (modulo 3), use the trade $T_{5.5A}$ on cells $g(1), \ldots, g(5)$. If $\alpha = \beta = 4$ and $\gamma = 7$ we are done. If either $\alpha = 4$ and $\beta = \gamma = 7$ or $\alpha = \beta = 4$ and $\gamma = 10$, then we can attach the triangle corresponding to cell $g(6)$ to either one of the 4-cycles or the 7-cycle, as needed. If $\alpha = \beta = 4$ and $\gamma = 13$, then Conditions 4 and 5 from Lemma 4.5 ensure that $g(6)$ and $g(7)$ form a closed trail of length 13 with the 7-cycle from the trade $T_{5.5A}$. Otherwise, the 7-cycle in the trade attaches to each of $g(6)$, $g(7)$ and $g(8)$, and the 4-cycles in the trade may then be attached in some fashion to the remaining two cells from $g(6)$, $g(7)$ and $g(8)$. So from Lemma 4.6 (if $\alpha = \beta = 4$), or Lemma 4.8 (if $\alpha = 4$ and $\beta > 4$) or Lemma 4.10 (if $\alpha > 4$), we can construct trails of lengths α, β and γ as required.

If $\delta \equiv 2$ (modulo 3), use the trade $T_{4.2}$ on cells $g(1), \ldots, g(4)$. If $\gamma = 4$ we are done. If $\beta = 4$ and $\gamma = 10$ then Conditions 4 and 5 from Lemma 4.5 ensure that $g(5)$ and $g(6)$ form a closed trail of length 10 with a 4-cycle from the trade $T_{4.2}$. Otherwise $g(5)$, $g(6)$ and $g(7)$ may be attached in $1-1$ correspondence with the 4-cycles from the trade. (Note this includes the case $\delta = 11$ when $g(7)$ and $g(6)$ lie in the same column.) The rest is similar to the previous paragraph.

If $\delta \equiv 0$ (modulo 3), use the trade $T_{4.2}$ on cells $g(2), g(3), g(4)$ and $g(5)$. The triangle from cell $g(1)$ attaches to any of these 4-cycles. As in the previous case $g(6)$, $g(7)$ and $g(8)$ may be attached in $1-1$ correspondence with the 4-cycles. □

Lemma 4.12. *Let Q be a partial latin square in early odd form. Let α, β, γ be integers, such that $\alpha \equiv \beta \equiv \gamma \equiv 2$ (modulo 3), $5 \leqslant \alpha \leqslant \beta \leqslant \gamma$ and $\frac{1}{3}(\alpha+\beta+\gamma) \leqslant |Q|$. Then there exists a partial latin square Q' such that the triangles corresponding to the entries in $Q \setminus Q'$ may be exchanged with circuits of lengths α, β and γ. In addition, Q' is either in early odd form or proper odd form.*

Proof. Let $\delta = \delta_Q$ and let g be an early odd ordering of Q.

If $\delta = 5$, then we can place $T_{5.4}$ on cells $g(1), \ldots, g(5)$. Let Q_1 be the partial latin square formed after deleting the entries from these five cells in Q. We then take an alternative ordering f' on Q_1, with $f'(1) = (n-1, 4)$. Then the entries in cells $f'(1)$, $f'(2)$ and $f'(3)$ may be put in some $1-1$ correspondence with the 5-cycles from the trade. Thus we may apply one of Lemmata 3.5, 3.6 or 3.10 from the previous section to create any non-zero trails of lengths $\alpha-5$, $\beta-5$ and $\gamma-5$ (from entries in cells $f'(1), \ldots, f'(\frac{1}{3}(\alpha+\beta+\gamma)-5))$ to attach in $1-1$ correspondence with our 5-cycles. The further deletion of the entries in these cells gives the desired Q'.

For $\delta = 6$, 7, 8 or 9 we place the trades $T_{5.2}$, $T_{5.5}$, $T_{5.8}$ or $T_{5.9}$, respectively, on cells $g(1), \ldots, g(5)$. In each case we place an alternative ordering f' on Q minus these five cells, and the rest follows similarly to the previous paragraph.

Otherwise $\delta \geqslant 10$. Then for $\delta \equiv 0, 1$ or 2 (mod 3), we place trades $T_{5.2}$, $T_{5,5}$ or $T_{5.4}$, respectively, on cells $g(1), \ldots, g(5)$. If $\gamma = 5$ we are done. If $\beta = 5$ and $\gamma = 11$ then Conditions 4 and 5 from Lemma 4.5 ensure that we can form a closed trail of length 11 using triangles from $g(6)$ and $g(7)$ and one of the 5-cycles. Otherwise, in each case $g(6)$, $g(7)$ and $g(8)$ may be put into some $1-1$ correspondence with the 5-cycles from the trade used. So from Lemma 4.6 (if $\beta = 5 < \gamma$ and $\gamma \neq 11$), or Lemma 4.8 (if $\alpha = 5$ and $\beta > 5$) or Lemma 4.10 (if $\alpha > 5$), we can construct trails of lengths α, β and γ as required. □

Theorem 4.13. *Let n be an odd integer and let $P = \{x_1, x_2, \ldots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Then it is possible to decompose the edges of $K_{n,n,n}$ into trails of lengths $x_1, x_2, \ldots, x_m$.*

Proof. Let $P_1, P_2, \ldots, P_L$ be a partition of P satisfying the conditions of Lemma 2.6. As in the case n even, we shall "trade" triangles with sets of closed trails with integer lengths from sets P_j, $1 \leqslant j \leqslant L$.

If there are two or more subsets $\{4, 4, 4\}$, i.e. $l \geqslant 2$, let $P_1 = P_2 = \ldots = P_l = \{4, 4, 4\}$, where $P_j \neq \{4, 4, 4\}$ for all $l < j \leqslant L$. From Lemma 4.3, there exists a partial latin square Q_1, in either early odd form or proper odd form, such that $B_n \setminus Q_1$ can have its entries traded with $3l$ 4-cycles. Otherwise let $Q_1 = B_n$. If there is *one* subset $\{4, 4, 4\}$, we let $P_L = \{4, 4, 4\}$, and let $l = 0$. If there are no subsets $\{4, 4, 4\}$, then let $l = 0$.

Next, assume, without loss of generality, that $P_{l+1} = \{x_{l+1}\}$, $P_{l+2} = \{x_{l+2}\}, \ldots$, $P_{l'} = \{x_{l'}\}$, and that these are the only subsets in the partition of P of size one. Now from Lemma 3.2 (if Q_1 is in proper odd form) or Lemmata 4.7 and 4.6 (if Q_1 is in early odd form), repeatedly, there exists a partial latin square Q_2 in either early odd form or proper odd form such that the entries in $Q_1 \setminus Q_2$ may be exchanged with circuits of lengths $x_{l+1}, \ldots, x_{l'}$ (with possibly one circuit remaining of length 6).

Next let $P_{l'+1}, \ldots, P_{l''}$ be the remaining subsets of size 3, not of the form $\{4, 4, 4\}$. Applying Lemmata 3.11 and 3.12 (if Q_2 is in proper odd form) or Lemmata 4.11 and

4.12 (if Q_2 is in early odd form), repeatedly, leaves a partial latin square Q_3 in either early odd form or proper odd form, with entries in $Q_2 \setminus Q_3$ which can be exchanged with circuits of lengths from subsets $P_{l'+1}, \ldots, P_{l''}$.

Next consider the subsets of size 2. If Q_3 is in early odd form, we can conclude from $n \geqslant 7$ and careful counting that there remains a set $\{\alpha_1, \beta_1\}$ such that $\frac{1}{3}(\alpha_1 + \beta_1) \geqslant 10$. (We also ensure that $\alpha_1 + \beta_1 \geqslant \alpha_2 + \beta_2$ for any other remaining set of the form $\{\alpha_2, \beta_2\}$.) So from Lemma 4.9, there exists a partial latin square Q_4 in *proper odd form*, with entries in $Q_3 \setminus Q_4$ which can be exchanged with circuits of lengths α_1 and β_1. If Q_3 is in proper odd form, let $Q_4 = Q_3$.

Next, we apply Lemma 3.7 to each set of the form $\{\alpha, \beta\}$ with the property that $\alpha + \beta \geqslant 15$. We are left with a partial latin square Q_5 in proper odd form. Since $n \geqslant 7$, and since there are at most two remaining subsets of the form $\{\alpha, \beta\}$ such that $9 \leqslant \alpha + \beta \leqslant 12$ *plus* at most one subset of the form $\{4, 4, 4\}$ *plus* at most one subset of the form $\{6\}$, there are at most 2 non-empty columns in Q_5.

Thus we can apply Lemma 3.7 to any remaining 2-sets and there are at most 6 entries left in our partial latin square. It is straightforward to exchange any remaining circuit lengths with any remaining entries. Our decomposition is complete. □

5. Concluding remarks

Combining the results in the previous two sections yields the following.

Theorem 5.1. *Let $P = \{x_1, x_2, \ldots, x_m\}$ be a set of integers such that $x_i \geqslant 3$ for $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} x_i = 3n^2$. Then there exists an edge-disjoint decomposition of $K_{n,n,n}$ into closed trails of lengths $x_1, x_2, \ldots, x_m$.*

This means that all regular tripartite graphs (equi-tripartite graphs) are ADCT, proving the conjecture mentioned in [10]. Since closed trails of lengths 3, 4 or 5 must be cycles we have the following Oberwolfach-type corollary.

Corollary 5.2. *Let α, β, γ be non-negative integers such that $3\alpha + 4\beta + 5\gamma = 3n^2$. Then there exists an edge-disjoint decomposition of $K_{n,n,n}$ into α triangles, β 4-cycles and γ 5-cycles.*

It is perhaps appropriate to make a further conjecture at this stage.

Conjecture. *The equipartite graph $K_{m(n)}$ with m parts of size n, is ADCT for all $m \geqslant 3$, provided n is even or m is odd.*

The requirement that n be even or m be odd merely ensures that the degree of each vertex is even. The case $m = 3$ is our result above; the case $m \geqslant 4$ remains open. The case $n = 2$ appears in [1], since the graph $K_{m(2)}$ is the same as K_{2m} with a 1-factor removed.

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