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EXCHANGE RINGS WITH STABLE RANGE ONE

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Abstract. We characterize exchange rings having stable range one. An exchange ring \( R \) has stable range one if and only if for any regular \( a \in R \), there exist an \( e \in E(R) \) and a \( u \in U(R) \) such that \( a = e + u \) and \( aR \cap eR = 0 \) if and only if for any regular \( a \in R \), there exist \( e \in r.\text{ann}(a^+) \) and \( u \in U(R) \) such that \( a = e + u \) if and only if for any \( a, b \in R \), \( R/aR \cong R/bR \implies aR \cong bR \).

Keywords: exchange ring, stable range one, idempotent, unit

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1. Introduction

A right \( R \)-module \( A \) has the finite exchange property if for every right \( R \)-module \( Q \) and two decompositions \( Q = M \oplus N = \bigoplus_{i \in I} A_i \), where \( M_R \cong A \) and the index set \( I \) is finite, there exist submodules \( A_i' \subseteq A_i \) such that \( Q = M \oplus \left( \bigoplus_{i \in I} A_i' \right) \). We say that \( R \) is an exchange ring provided that \( R \) has the finite exchange property as a right \( R \)-module. By [14, Theorem 2.1], a ring \( R \) is an exchange ring if and only if for any \( x \in R \) there exists an idempotent \( e \in Rx \) such that \( 1 - e \in R(1 - x) \). It is well known in the literature that regular rings, \( \pi \)-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit \( C^* \)-algebras of real rank zero (cf. [3, Theorem 7.2]) are all exchange rings. In [1, Theorem 1.1], Ara proved that every purely infinite simple ring is an exchange ring.

Recall that a ring \( R \) has stable range one provided \( aR + bR = R \) with \( a, b \in R \) implies that there exists \( y \in R \) such that \( a + by \in U(R) \). This definition is right-left symmetric. Moreover, we know that a right \( R \)-module \( M \) can be cancelled from direct sums if and only if \( \text{End}_R M \) has stable range one. In this paper, we will characterize exchange rings having stable range one by various equivalent conditions.
An element $a \in R$ is regular if there exists an $x \in R$ such that $a = axa$. We say that $a \in R$ is unit-regular if it is the product of an idempotent and a unit. In [5, Theorem 3], Camillo and Yu proved that an exchange ring has stable range one if and only if every regular element in $R$ is unit-regular. Further, Yu proved that every exchange ring with artinian primitive factors has stable range one (cf. [17, Theorem 1]). In [2, Theorem 4], Ara proved that every strongly $\pi$-regular ring is an exchange ring having stable range one.

In parallel, $a \in R$ is clean if it is the sum of an idempotent and a unit. Camillo and Khurana (cf. [4, Theorem 1]) gave a characterization of unit regular rings. They showed that a ring $R$ is unit-regular if and only if for any $a \in R$ there exist an idempotent $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$.

Let $\mathbb{Z}$ be the ring of all integers. In [12, Example 4.5], Khurana and Lam showed that $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is not clean although it is unit-regular. In other words, a single unit-regular element in a ring may be not clean. This has inspired us to investigate clean property of unit-regular elements in an exchange ring having stable range one.

In this paper, we prove that an exchange ring $R$ has stable range one if and only if for any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$. This gives an affirmative answer to the problem in [8]. Furthermore, we prove that an exchange ring $R$ has stable range one if and only if for any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$. Additionally, we prove that an exchange ring $R$ has stable range one if and only if for any $a, b \in R$, $R/aR \cong R/bR \Rightarrow aR \cong bR$.

Throughout the paper, every ring is associative with an identity. A ring $R$ is (unit) regular provided every element in $R$ is (unit) regular. Let $r.ann(a) = \{r \in R; ar = 0\}$ and $l.ann(a) = \{r \in R; ra = 0\}$. We use $E(R)$ to denote the set of all idempotents in $R$ and $U(R)$ to denote the set of all units in $R$.

2. Clean property

**Theorem 2.1.** Let $R$ be an exchange ring. Then the following assertions are equivalent:

(1) $R$ has stable range one.

(2) For any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$.

(3) For any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $Ra \cap Re = 0$.

**Proof.** (1) $\Rightarrow$ (2) Let $a \in R$ be regular. Then we have $x \in R$ such that $a = axa$, and so $R = aR \oplus (1 - ax)R = xR \oplus r.ann(a)$. Clearly, $aR \cong axR$
has the finite exchange property. So there exist right $R$-modules $X_1, Y_1$ such that $R = aR \oplus X_1 \oplus Y_1$ with $X_1 \subseteq r.\text{ann}(a)$ and $Y_1 \subseteq xR$. It is easy to verify that $r.\text{ann}(a) = r.\text{ann}(a) \cap (X_1 \oplus aR \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = r.\text{ann}(a) \cap (aR \oplus Y_1)$. Likewise, we have a right $R$-module $Y_2$ such that $xR = Y_1 \oplus Y_2$. Obviously, $a \in R$ is unit-regular; hence, $r.\text{ann}(a) \cong R/aR$. Thus $X_1 \oplus X_2 = r.\text{ann}(a) \cong R/aR \cong X_1 \oplus Y_1$, and so we have an isomorphism $k: X_1 \oplus X_2 \to X_1 \oplus Y_1$. Furthermore, $X_1$ can be cancelled from direct sums, and hence we get a right $R$-module isomorphism $\psi: X_2 \to Y_1$.

Let $h: R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = R$ be given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v: R = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = R$ be given by $v(x_1 + y_1 + x_2 + y_2) = k^{-1}(x_1 + y_1) + \psi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we have

$$hv(x_1 + x_2 + y_1 + y_2) = hv(k(x_1 + x_2) + y_1)$$
$$= h(x_1 + x_2 + k^{-1}(y_1)) = k(x_1 + x_2) + y_1$$
$$= h(x_1 + x_2 + y_1 + y_2);$$

hence $h = hv$. Set $e = hv$. Then $e \in \text{End}_R(R)$ is an idempotent.

Assume that $(a - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. Then

$$a(y_1 + y_2) = x_1 + y_1 + \psi(x_2) \in aR \cap (X_1 \oplus Y_1) = 0,$$

and consequently $x_1 = -y_1 - \psi(x_2) \in X_1 \cap Y_1 = 0$. It follows from $a(y_1 + y_2) = 0$ that $y_1 + y_2 \in (X_1 \oplus X_2) \cap (Y_1 \oplus Y_2) = 0$; hence $y_1 + y_2 = 0$. This infers that $y_1 = -y_2 \in Y_1 \cap Y_2 = 0$, and so $y_1 = y_2 = 0$. Furthermore, we get $\psi(x_2) = -y_1 = 0$. As $\psi$ is an isomorphism, we have $x_2 = 0$. Thus $x_1 + y_1 + x_2 + y_2 = 0$. This means that $a - e \in R$ is a monomorphism.

Given any $t \in aR, x_1 \in X_1, y_1 \in Y_1$, we have $t \in aR = a(Y_1 \oplus Y_2)$. So we can find $y'_1 \in Y_1$ and $y'_2 \in Y_2$ such that $t = a(y'_1 + y'_2)$. Choose $x'_1 = -x_1$ and $x'_2 = -\psi^{-1}(y'_1 + y'_1)$. It is easy to verify that

$$(a - hv)(x'_1 + x'_2 + y'_1 + y'_2) = a(y'_1 + y'_2) - (x'_1 + y'_1 + \psi(x'_2))$$
$$= t - (-x_1 + y'_1 - y_1 - y'_1) = t + x_1 + y_1.$$
(2) ⇒ (1) Given any regular \( a \in R \), there exist an \( e \in E(R) \) and a \( u \in U(R) \) such that \( a = e + u \) and \( aR \cap eR = 0 \). As a result, \((au^{-1} - 1)a = eu^{-1}a \in aR \cap eR = 0\); hence, \( a = au^{-1}a \). According to [5, Theorem 3], we complete the proof.

(1) ⇔ (3) As stable range one condition is symmetric, we obtain the result by applying (1) ⇔ (2) to the opposite ring \( R^{op} \).

A ring \( R \) is said to have bounded index if there exists an integer \( n \) such that \( x^n = 0 \) for every nilpotent \( x \in R \). Let \( R \) be an exchange ring of bounded index. We claim that for any regular \( a \in R \), there exist an \( e \in E(R) \) and a \( u \in U(R) \) such that \( a = e + u \) and \( aR \cap eR = 0 \). In view of [17, Theorem Corollary 4], \( R \) has stable range one, and we are done by Theorem 2.1.

Recall that a ring \( R \) is clean provided that every element in \( R \) is clean. It is well known that every clean ring is an exchange ring. Now we give a new proof of [14, Proposition 1.8] as follows.

**Corollary 2.2.** Every exchange ring with all idempotents central is a clean ring.

**Proof.** Let \( R \) be an exchange ring with all idempotents central, and let \( a \in R \). By [14, Theorem 2.1], there exists an idempotent \( e \in R \) such that \( e = as \) and \( 1 - e = (1 - a)t \) for some \( s, t \in R \). Clearly, \( ea = (ea)s(ea) \) and \( (1 - e)a = ((1 - e)a)t((1 - e)a) \). In view of Theorem 2.1, we can find idempotents \( f_1, f_2 \in R \) and units \( u_1, u_2 \in R \) such that \( ea = f_1 + u_1 \) and \( (1 - e)(1 - a) = f_2 + u_2 \). It follows that

\[
a = ea + (1 - e)a = (ef_1 + eu_1) + ((1 - e) - (1 - e)f_2 - (1 - e)u_2)
= (ef_1 + (1 - e)(1 - f_2)) + (eu_1 - (1 - e)u_2).
\]

Let \( f = ef_1 + (1 - e)(1 - f_2) \) and \( u = eu_1 - (1 - e)u_2 \). Then \( f = f^2 \) and \( u^{-1} = eu_1^{-1} - (1 - e)u_2^{-1} \), and therefore \( R \) is a clean ring.

We note that an exchange ring plus stable range one is a Morita invariant. Using this fact, we derive

**Corollary 2.3.** Let \( R \) be an exchange ring and \( \frac{1}{2} \in R \). If \( R \) has stable range one, then every regular square matrix over \( R \) is the sum of three invertible matrices.

**Proof.** Since \( R \) is an exchange ring having stable range one, so is \( M_n(R) \). Let \( A \in M_n(R) \) be regular. In view of Theorem 2.1, there exist an idempotent \( E \in M_n(R) \) and an invertible \( U \in M_n(R) \) such that \( A = E + U \). As \( \frac{1}{2} \in R \), it follows that \( E = \text{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n \times n} + (E - \text{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n \times n}) \). One easily checks that \( (E - \text{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n \times n})(4E - \text{diag}(2, \ldots, 2)_{n \times n}) = I_n = (4E - \text{diag}(2, \ldots, 2)_{n \times n})(E - \text{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n \times n}) \). That is, \( E - \text{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n \times n} \in M_n(R) \) is invertible. Therefore \( A \) is the sum of three invertible matrices, as asserted. □
Example 2.4. Let $R$ be a $2 \times 2$ matrix over $\mathbb{F}(x^2)$, where $\mathbb{F}$ is a field. Clearly, $R$ is a strongly $\pi$-regular ring; hence, it is an exchange ring having stable range one. Take $a = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix}$. Then $a \in R$ is regular, while $a^2 \in R$ is not regular. In view of Theorem 2.1, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ with $aR \cap eR = 0$, while $a^2$ can not be written in this form.

Theorem 2.5. If $R$ is an exchange ring having stable range one, then every square matrix over $R$ is an algebraic sum of idempotent matrices and invertible matrices.

Proof. Let $R$ be an exchange ring having stable range one, and let $S = M_n(R)$. Then $S$ is an exchange ring having stable range one. Let $A \in S$. By [14, Theorem 2.1], there exists an idempotent $E \in S$ such that $E = AS$ and $I_n - E = (I_n - A)T$ for some $S, T \in S$. Analogously to Corollary 2.3, we see that both $EA$ and $(I_n - E)A$ are regular. In view of Theorem 2.1, we can find idempotents $F_1, F_2 \in S$ and invertible $U_1, U_2 \in S$ such that $EA = F_1 + U_1$ and $(I_n - E)(I_n - A) = F_2 + U_2$. So we deduce that $A = EA + (I_n - E)A = F_1 + U_1 + (I_n - E) - F_2 - U_2$. This means that $A$ is an algebraic sum of idempotent matrices and invertible matrices.

Let $I$ be an ideal of a ring $R$. We say that $I$ has stable range one provided $(1_R + a)R + bR = R$ with $a \in I, b \in R$ implies that there exists a $y \in R$ such that $1_R + a + by \in U(R)$.

Corollary 2.6. Let $R$ be a regular ring, and let $A = (a_{ij}) \in M_n(R)$. If each $Ra_{ij}R$ has stable range one, then $A$ is an algebraic sum of idempotent matrices and invertible matrices.

Proof. Let $I = \sum_{1 \leq i,j \leq n} Ra_{ij}R$. Given $(1 + \sum_{1 \leq i,j \leq n} r_{ij})x + b = 1$ with $x, b \in R$ and each $r_{ij} \in Ra_{ij}R$, then $(1 + r_{11})x + \left( \sum_{1 \leq i,j \leq n, i \neq 1} r_{ij} \right)x + b = 1$. As $Ra_{11}R$ has stable range one, we can find $y_{11} \in R$ such that $x + y_{11} \left( \sum_{1 \leq i,j \leq n, i \neq 1} r_{ij} \right)x + y_{11}b = u_1 \in U(R)$. Let $r'_{ij} = y_{11}r_{ij}$. Then $r'_{ij} \in Ra_{ij}R$ and $\left(1 + \sum_{1 \leq i,j \leq n, i \neq 1} r'_{ij}\right)(ux_1) + bu_1 = 1$. Likewise, we prove that $(1 + r'_{nn})ux_1u_2 \ldots u_{nn} + bu_1u_2 \ldots u_{nn} = 1$ for some $u_2, \ldots, u_n \in U(R)$. As $Ra_{nn}R$ has stable range, we have $z \in R$ such that $ux_1u_2 \ldots u_{nn} + zbu_1u_2 \ldots u_{nn} \in U(R)$. Thus $x + zb \in U(R)$, and so $I$ has stable range one. Clearly, each $a_{ij} \in I$. Furthermore, there exists an idempotent $e \in I$ such that each $a_{ij} \in eRe$; hence $A \in M_n(eRe)$. Clearly, $eRe$ is unit-regular. It follows by Theorem 2.5 that $A$ is an algebraic sum of idempotent matrices and invertible matrices over $eRe$. Let $U \in M_n(eRe)$ be invertible. Then we have $V \in M_n(eRe)$ such that $UV = \text{diag}(e, e, \ldots, e)_{n \times n}$. Hence $U + \text{diag}(1 - e, 1 - e, \ldots,$
\(1 - e)_{n \times n} (V + \text{diag}(1 - e, 1 - e, \ldots, 1 - e)_{n \times n}) = I_n.\) In other words, \(U + \text{diag}(1 - e, 1 - e, \ldots, 1 - e)_{n \times n} \in M_n(R)\) is invertible, and so \(U\) is an algebraic sum of an idempotent matrix and an invertible matrix over \(R.\) Therefore \(A\) is an algebraic sum of idempotent matrices and invertible matrices over \(R,\) as asserted. \(\square\)

Recall that an ideal \(I\) of a ring \(R\) is of bounded index if there is a positive integer \(n\) such that \(x^n = 0\) for any nilpotent \(x \in I.\)

**Corollary 2.7.** Let \(R\) be a regular ring, and let \(A = (a_{ij}) \in M_n(R).\) If each \(Ra_{ij}R\) is of bounded index, then \(A\) is an algebraic sum of idempotent matrices and invertible matrices.

**Proof.** For any idempotent \(e \in Ra_{ij}R\) we have \(eRe \subseteq Ra_{ij}R.\) Hence \(eRe\) is a regular ring of bounded index. In view of [9, Corollary 7.11], \(eRe\) is unit-regular. This shows that \(Ra_{ij}R\) has stable range one, and therefore we complete the proof by Corollary 2.6. \(\square\)

### 3. Extensions

Let \(I\) be a right ideal of a ring \(R.\) We say that \(a \in R\) is a right unit modulo \(I\) provided \(ab \equiv 1 (\text{mod } I).\) Now we extend this result as follows.

**Lemma 3.1.** Let \(R\) be an exchange ring. Then the following conditions are equivalent:
(1) \(R\) has stable range one.
(2) Every right unit lifts modulo \(I\) any right ideal of \(R.\)
(3) Every left unit lifts modulo \(I\) any left ideal of \(R.\)

**Proof.** (1) \(\Rightarrow\) (2) Let \(I\) be a right ideal of \(R,\) and let \(a \in R\) be a right unit modulo \(I.\) Then there exists \(b \in R\) such that \(ab \equiv 1 (\text{mod } I).\) Hence we can find an \(r \in I\) such that \(ab + r = 1.\) Since \(R\) has stable range one, we can find \(c \in R\) such that \(a + rc \in U(R).\) Set \(u = a + rc.\) Then \(a - u = r(-c) \in I.\) That is, \(a \equiv u (\text{mod } I),\) as desired.

(2) \(\Rightarrow\) (1) Given \(ab + c = 1\) in \(R,\) then \(ab - 1 \in cR.\) This means that \(ab \equiv 1 (\text{mod } cR).\) By hypothesis, there exists a right unit \(u \in R\) such that \(a - u \in cR.\) So we can find an \(r \in R\) such that \(a + cr = u \in R.\) As \(u \in R\) is a right unit, there is \(v \in R\) such that \(uv = 1.\) Since \(vu + (1 - vu) = 1,\) by the above consideration we have \(s \in R\) such that \(v + (1 - vu)s = t \in U(R)\) is a right unit. Clearly, \(ut = u(v + (1 - vu)s) = 1;\) hence, \(t \in R\) is a left unit. Thus \(t \in U(R).\) This implies that \(u \in U(R).\) That is, \(a + cr \in U(R).\) Therefore \(R\) has stable range one.

(1) \(\Leftrightarrow\) (3) is symmetric. \(\square\)
We say that $b \in R$ is a reflexive inverse of $a \in R$ if $a = aba$ and $b = bab$, and denote $b$ by $a^+$. Clearly, every regular element has a reflexive element. Using such elements, we give a new characterization of exchange rings having stable range one.

**Theorem 3.2.** Let $R$ be an exchange ring. Then the following conditions are equivalent:

1. $R$ has stable range one.
2. For any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.
3. For any regular $a \in R$, there exist $e \in l.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.

**Proof.** (1) $\Rightarrow$ (2) Given any regular $a \in R$, there exists $a^+$ such that $a = aa^+a$ and $a^+ = a^+aa^+$. Hence $a^+(aa^+ - 1) = 0$. That is, $aa^+ \equiv 1 \pmod{r.ann(a^+)}$. By virtue of Lemma 3.1, we can find a right unit $u \in R$ such that $a - u \in r.ann(a^+)$. Thus there exists $e \in r.ann(a^+)$ such that $a = e + u$. As $R$ has stable range one, it is directly finite. This infers that $u \in U(R)$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$ be regular. Then $a = aa^+a$ and $a^+ = a^+aa^+$. By assumption, there exist $e \in r.ann(a)$ and $u \in U(R)$ such that $a^+ = e + u$; hence, $a^+ - u \in r.ann(a)$. As a result, $a(a^+ - u) = 0$. This implies that $a = aa^+a = aua$. That is, $a \in R$ is unit-regular. Consequently, $R$ has stable range one by [5, Theorem 3].

(1) $\Leftrightarrow$ (3) Since $R$ is an exchange ring having stable range one if and only if so is the opposite ring $R^{op}$, the result follows by symmetry. 

**Corollary 3.3.** Let $R$ be an exchange ring of bounded index. Then the following assertions hold:

1. For any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.
2. For any regular $a \in R$, there exist $e \in l.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.

**Proof.** In view of [17, Corollary 4], $R$ has stable range one. So the proof follows by Theorem 3.2.

Recall that a ring $R$ is strongly $\pi$-regular provided that for any $a \in R$ there exists a positive integer $n(a)$ such that $a^{n(a)} \in a^{n(a)+1}R$. 

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**Corollary 3.4.** Let $R$ be a strongly $\pi$-regular ring. Then the following assertions hold:

(1) For any regular $a \in R$, there exist $e \in r.\text{ann}(a^+)\text{ and } u \in U(R)$ such that $a = e + u$.

(2) For any regular $a \in R$, there exist $e \in l.\text{ann}(a^+)\text{ and } u \in U(R)$ such that $a = e + u$.

**Proof.** In view of [2, Theorem 4], $R$ is an exchange ring having stable range one. Therefore we complete the proof by Theorem 3.2. □

A regular ring $R$ is abelian provided that every idempotent in $R$ is central.

**Corollary 3.5.** Let $R$ be a ring. Then the following assertions are equivalent:

(1) $R$ is an abelian regular ring.

(2) For any $a \in R$, there exist $e \in r.\text{ann}(a)\text{ and } u \in U(R)$ such that $a = e + u$.

(3) For any $a \in R$, there exist $e \in l.\text{ann}(a)\text{ and } u \in U(R)$ such that $a = e + u$.

**Proof.** (1) $\Rightarrow$ (2) Let $R$ be an abelian regular ring. Then it is an exchange ring having stable range one by [17, Theorem 6]. For any $a \in R$, there exists $a^+ \in R$ such that $a = aa^+a$ and $a^+ = a^+aa^+$. As every idempotent in $R$ is central, one checks that $r.\text{ann}(a^+) = r.\text{ann}(a)$. In view of Theorem 3.2, we can find $e \in r.\text{ann}(a)$ and $u \in U(R)$ such that $a = e + u$, as desired.

(2) $\Rightarrow$ (1) Given any $a \in R$, there exist $e \in r.\text{ann}(a)$ and $u \in U(R)$ such that $a = e + u$. Hence $a - u \in r.\text{ann}(a)$, and then $a(a - u) = 0$. This implies that $a = a^2u^{-1}$. According to [9, Theorem 3.5], $R$ is an abelian regular ring.

(1) $\Leftrightarrow$ (3) is obtained by symmetry. □

4. Cokernels

In [7, Theorem 14], the author proved that a regular ring $R$ is unit-regular if and only if whenever $aR \cong bR$, then there exist $u, v \in R$ such that $a = ubv$. In this section, we characterize exchange rings having stable range one by cokernels of their elements, which is also a generalization of [10, Theorem 2.1].

**Theorem 4.1.** Let $R$ be an exchange ring. Then the following conditions are equivalent:

(1) $R$ has stable range one.

(2) For any $a, b \in R$, $R/aR \cong R/bR$ implies that there exist $u, v \in U(R)$ such that $a = ubv$. 

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(3) For any \( a, b \in R \), \( R/Ra \cong R/Rb \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \).

**Proof.** (1) \( \Rightarrow \) (2) Since \( \varphi: R/aR \cong R/bR \), there exists a \( c \in R \) such that \( \varphi(1 + aR) = c + bR \). So \( R + bR = cR + bR \); hence, \( R = cR + bR \). Since \( R \) has stable range one, there exists a \( d \in R \) such that \( c + bd = u \in U(R) \). Clearly, \( bR = \varphi(aR) = \varphi(aR + aR) = caR + bR \), and then \( caR \subseteq bR \). Furthermore, \( uaR \subseteq bR \).

On the other hand, we have \( \varphi(1 + aR) = (c + bd) + bR = u + bR \). It follows that \( \varphi^{-1}(1 + bR) = u^{-1} + aR \). This implies that \( u^{-1}b + aR = (u^{-1} + aR)b = \varphi^{-1}(1 + bR)b = \varphi^{-1}(bR) = aR \). Hence \( u^{-1}bR \subseteq aR \), and then \( bR \subseteq uaR \). Thus we can find \( x, y \in R \) such that \( xa = bx \) and \( b = uay \). Since \( R \) has stable range one, it follows from \( xy + (1 - xy) = 1 \) that there exists a \( z \in R \) such that \( x + (1 - xy)z = v \in U(R) \). Thus we deduce that \( bx = b(x + (1 - xy)z) = bv \). As a result, we prove that \( a = u^{-1}bx = u^{-1}bv \), as desired.

(2) \( \Rightarrow \) (1) Given \( eR \cong fR \) with idempotents \( e, f \in R \), we have \( R/(1 - e)R \cong R/(1 - f)R \). By assumption, there exist \( u, v \in R \) such that \( 1 - e = u(1 - f)v \). Let \( y = u(1 - f)u^{-1} \). Then \( y(1 - e) = u(1 - f)u^{-1}(1 - e) = u(1 - f)v = 1 - e \) and \( y = u(1 - f)u^{-1} = u(1 - f)v(v^{-1}u^{-1}) = (1 - e)v^{-1}u^{-1} \). Hence \( (1 - e)y = y \). As a result, we prove that \( (e + y)^{-1} = 2 - e - y \). Set \( w = (e + y)u \). Then \( w \in U(R) \). Furthermore, one easily checks that

\[
w(1 - f)w^{-1} = (e + y)u(1 - f)u^{-1}(2 - e - y) = (e + y)y(2 - e - y) = y(2 - e - y) = y - ye = 1 - e.\]

This implies that \( e = wfw^{-1} \). In view of [17, Theorem 10], we prove that \( R \) has stable range one.

(1) \( \Leftrightarrow \) (3) is obtained by symmetry. \( \square \)

In the proof of Theorem 4.1, we prove that an exchange ring \( R \) has stable range one if and only if for any regular \( a, b \in R \), \( R/aR \cong R/bR \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \) if and only if for any regular \( a, b \in R \), \( R/Ra \cong R/Rb \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \). We note that the condition (1) and (2) above are not equivalent for some non-exchange rings. In [6, Example 6.7], Canfell supplied a principal ideal domain \( R \) which has elements \( a \) and \( b \) for which \( R/aR \cong R/bR \) but \( a \neq ubv \) for any \( u, v \in U(R) \).

**Corollary 4.2.** Let \( R \) be an exchange ring. Then the following assertions are equivalent:

(1) \( R \) has stable range one.
(2) For any regular \( a, b \in R \), \( r.\text{ann}(a) \cong r.\text{ann}(b) \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \).

(3) For any regular \( a, b \in R \), \( l.\text{ann}(a) \cong l.\text{ann}(b) \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( r.\text{ann}(a) \cong r.\text{ann}(b) \) with regular \( a, b \in R \). Then there exist \( x, y \in R \) such that \( a = axa \) and \( b = byb \). Hence \( (1 - xa)R = r.\text{ann}(a) \cong r.\text{ann}(b) = (1 - yb)R \). As \( 1 - xa, 1 - ya \in R \) are idempotents, it follows that \( R(1 - xa) \cong R(1 - yb) \). Clearly, \( R(1 - xa) \cong R/Rxa \) and \( R(1 - yb) \cong R/ybR \). As a result, \( R/Ra \cong R/Rb \). In view of Theorem 4.1, we can find \( u, v \in U(R) \) such that \( a = ubv \).

(2) \( \Rightarrow \) (1) Given \( eR \cong fR \) with idempotents \( e, f \in R \), we have \( r.\text{ann}(1 - e) \cong r.\text{ann}(1 - f) \). By assumption, we can find \( u, v \in U(R) \) such that \( 1 - e = u(1 - f)v \). Analogously to Theorem 4.1, we have a \( w \in U(R) \) such that \( 1 - e = w(1 - f)w^{-1} \).

Thus \( 1 - e = ab \) and \( 1 - f = ba \), where \( a = (1 - e)w(1 - f) \in (1 - e)R(1 - f) \) and \( b = (1 - f)w^{-1}(1 - e) \in (1 - f)R(1 - e) \). This implies that \( (1 - e)R \cong (1 - f)R \). Using [17, Theorem 10], we prove that \( R \) is unit-regular.

(1) \( \Leftrightarrow \) (3) is symmetric. \( \square \)

A regular ring is unit-regular if and only if it has stable range one (cf. [9, Proposition 4.12]). It follows by Corollary 4.2 that a regular ring is unit-regular if and only if \( r.\text{ann}(a) \cong r.\text{ann}(b) \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \) if and only if \( l.\text{ann}(a) \cong l.\text{ann}(b) \) implies that there exist \( u, v \in U(R) \) such that \( a = ubv \).

**Example 4.3.** Let \( V \) be an infinite-dimensional vector space over a division ring \( D \), and let \( R = \text{End}_D(V) \). Then \( R \) is an exchange ring but it has stable range \( \infty \). Using Corollary 4.1, the condition (2) above doesn’t hold. Let \( \{x_1, x_2, \ldots \} \) be a basis of \( V \). Define \( \sigma: V \to V \) by \( \sigma(x_i) = x_{i+1} \) for \( i = 1, 2, 3, \ldots \). Let \( \tau: V \to V \) be the identity map. Define \( \varrho: V \to V \) given by \( \tau(x_1) = 0 \) and \( \varrho(x_i) = x_{i-1}(i = 2, 3, \ldots, n, \ldots) \). Then \( \varrho \sigma = 1_V \) and \( \varrho \varrho \neq 1_V \). Thus \( \sigma \) and \( \tau \) are both regular and \( r.\text{ann}(\sigma) \cong r.\text{ann}(\tau) \), while \( \sigma \neq u\tau v \) for any automorphisms \( u \) and \( v \).

**Corollary 4.4.** Let \( R \) be an exchange ring having stable range one, and let \( a, b \in R \). Then the following conditions are equivalent:

1. \( \varphi: aR \cong bR \) and \( \varphi(a) = ua \) for \( a \in U(R) \).
2. There exist \( u, w \in U(R) \) such that \( a = vw \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that \( \varphi: aR \cong bR \) and \( \varphi(a) = ua \) for \( a \in U(R) \). Let \( \psi: R \to R \) be given by \( \psi(r) = ur \) for any \( r \in R \). Then \( \psi \) is an automorphism.
So we have \( \varphi: R/aR \to R/aR \) such that the following diagram commutes.

\[
\begin{array}{cccccc}
0 & \to & aR & \xrightarrow{\varphi} & R & \xrightarrow{\psi} & R/aR & \to & 0 \\
\downarrow & & \downarrow & & \varphi & & \downarrow \\
0 & \to & bR & \xrightarrow{\psi} & R & \xrightarrow{\varphi} & R/bR & \to & 0
\end{array}
\]

Since both \( \varphi \) and \( \psi \) are isomorphic, so is \( \varphi \). That is, \( R/aR \cong R/bR \). According to Theorem 4.1, we prove that \( a = vbw \) for some \( v, w \in U(R) \).

(2) \( \Rightarrow \) (1) Suppose that \( a = vbw \) with \( v, w \in U(R) \). Construct a map \( \varphi: aR \to bR \) given by \( \varphi(ar) = v^{-1}(ar) \) for any \( r \in R \). It is easy to verify that \( \varphi: aR \cong bR \). In addition, \( \varphi(a) = v^{-1}a \), and thus we complete the proof. \( \square \)

It is easy to check that a regular ring \( R \) is unit-regular if and only if for any \( a, b \in R, R/aR \cong R/bR \rightleftharpoons aR \cong bR \). In contrast to this fact, we derive

**Theorem 4.5.** Let \( R \) be an exchange ring. Then the following conditions are equivalent:

(1) \( R \) has stable range one.
(2) For any \( a, b \in R, R/aR \cong R/bR \rightleftharpoons aR \cong bR \).
(3) For any \( a, b \in R, R/Ra \cong R/Rb \rightleftharpoons Ra \cong Rb \).

**Proof.** (1) \( \Rightarrow \) (2) Given \( R/aR \cong R/bR \), it follows by Theorem 4.1 that there exist \( u, v \in U(R) \) such that \( a = ubv \). Construct a map \( \varphi: aR \to bR \) given by \( \varphi(ar) = u^{-1}(ar) \) for any \( r \in R \). Then \( \varphi: aR \cong bR \), as asserted.

(2) \( \Rightarrow \) (1) Given \( eR \cong fR \) with idempotents \( e, f \in R \), then \( R/(1-e)R \cong R/(1-f)R \). By hypothesis, we get \( (1-e)R \cong (1-f)R \). Using [17, Theorem 10], we prove that \( R \) is unit-regular.

(1) \( \iff \) (3) is symmetric. \( \square \)

As an immediate consequence of Theorem 4.5, we deduce that an exchange ring \( R \) has stable range one if and only if for any regular \( a, b \in R, R/aR \cong R/bR \rightleftharpoons aR \cong bR \) if and only if for any regular \( a, b \in R, R/Ra \cong R/Rb \rightleftharpoons Ra \cong Rb \).

**Corollary 4.6.** Let \( R \) be a regular ring. Then the following conditions are equivalent:

(1) \( R \) is unit-regular.
(2) For any \( a, b \in R, R/aR \cong R/bR \iff aR \cong bR \).
(3) For any \( a, b \in R, R/Ra \cong R/Rb \iff Ra \cong Rb \).

**Proof.** (1) \( \Rightarrow \) (2) For any \( a, b \in R, R/aR \cong R/bR \rightleftharpoons aR \cong bR \) by Theorem 4.5. Conversely, assume that \( aR \cong bR \). Then we can find idempotents
e, f ∈ R such that aR = eR and bR = fR. In view of [9, Theorem 4.14], we have
(1−e)R ∼ (1−f)R; hence, R/eR ∼ R/fR. As a result, we prove that R/aR ∼ R/bR.
(2) ⇒ (1) is clear by Theorem 4.5.
(1) ⇔ (3) is symmetric.

□

References


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