Milan Demko

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Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 2, 607–629

Persistent URL: http://dml.cz/dmlcz/128193

Czechoslovak Mathematical Journal, 57 (132) (2007), 607–629

LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF HALF LINEARLY ORDERED LOOPS

Milan Demko, Prešov

(Received February 25, 2005)

Abstract. In this paper we prove for an hl-loop Q an assertion analogous to the result of Jakubík concerning lexicographic products of half linearly ordered groups. We found conditions under which any two lexicographic product decompositions of an hl-loop Q with a finite number of lexicographic factors have isomorphic refinements.

Keywords: half linearly ordered quasigroup, half linearly ordered loop, lexicographic product, isomorphic refinements

MSC 2000: 20N05, 06F99

1. Introduction

The notion of a half linearly ordered group has been introduced by Giraudet and Lucas [6]. Lexicographic products of half linearly ordered groups were discussed by Jakubík in [9].

In the present paper we define the Φ-lexicographic product of half linearly ordered loops. This definition includes as a particular case the lexicographic product of half linearly ordered groups and also the lexicographic product of linearly ordered loops. Here we will prove the following assertion analogous to [9; Theorem 4.5]: Let Q be a half linearly ordered loop and let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two lexicographic product decompositions of Q with a finite number of lexicographic factors have isomorphic refinements.

The analogous theorem for lexicographic product decompositions of linearly ordered groups was proved by Maltsev [10]; this result was generalized by Fuchs [5]. Further, lexicographic product decompositions of some types of ordered algebraic structures were dealt with in the papers [2], [3], [7], [8].

2. Preliminaries

General information concerning quasigroups can be found in [1]. Recall that a quasigroup Q is defined as an algebra having a binary operation "$\cdot$" which satisfies the condition that for any $a, b \in Q$ the equations $ax = b$ and $ya = b$ have unique solutions x and y. A quasigroup Q having an identity element 1 (i.e., such that $1.x = x.1 = x$ for each $x \in Q$) is called a loop. If Q is a quasigroup, then we define $a/b = c$ if and only if $a = cb$; in this case we also put $c \setminus a = b$.

Let Q be a quasigroup. An equivalence relation θ on Q is called a normal congruence relation on Q, if it satisfies the following conditions

$$a\theta b \Leftrightarrow ac\theta bc \Leftrightarrow ca\theta cb.$$

A subquasigroup (subloop) H of a quasigroup (loop) Q is called a normal subquasigroup (subloop) of Q if H is a class with respect to some normal congruence relation on Q. If Q is a loop, then a subloop H is normal in Q (see [1]) if and only if $xH = Hx$, $xy \cdot H = x \cdot yH$, $H \cdot xy = Hx \cdot y$ for all $x, y \in Q$. It is routine to verify that for loops the following assertion (analogous to that for groups) is valid.

2.1. Lemma. *Let H be a normal subloop of a loop Q. Then a relation θ on Q defined by the rule*

$$x\theta y \Leftrightarrow x/y \in H$$

is a normal congruence relation on Q.

Now, let Q be a quasigroup and at the same time let $\leqslant$ be a partial order on Q. We denote by $Q\uparrow$ (or $Q\downarrow$) the set of all $x \in Q$ such that whenever $y, z \in Q$, then $y \leqslant z$ if and only if $xy \leqslant xz$ (or $y \leqslant z$ if and only if $xy \geqslant xz$, respectively).

2.2. Definition. Q is said to be a half linearly ordered quasigroup (hl-quasigroup) if the following conditions are satisfied:

(i) the partial order $\leqslant$ on Q is nontrivial;
(ii) if $x, y, z \in Q$, then $y \leqslant z$ if and only if $yx \leqslant zx$;
(iii) $Q = Q\uparrow \cup\ Q\downarrow$;
(iv) $Q\uparrow$ is a linearly ordered set.

In particular, if Q is a loop, then Q is called a half linearly ordered loop (hl-loop).

Let Q be an hl-quasigroup. If $Q\downarrow = \emptyset$, then Q is a linearly ordered quasigroup. If Q is a group under the binary operation, then, by the definition in [6], Q is a half linearly ordered group (hl-group). In this case, $Q\downarrow \neq \emptyset$ yields that $Q\uparrow$ is a normal subgroup of Q with index 2 (see [6]). The situation is different if we consider quasigroups instead of groups. There exists an hl-quasigroup Q such that $Q\uparrow$ is not a subquasigroup of Q (for an example see [4]). On the other hand there exist hl-quasigroups Q such that $Q\uparrow$ is normal in Q and the number of classes modulo $Q\uparrow$ is greater than 2; moreover, their number can be infinite (see [4; Theorem 2]). We will apply the following results which were proved in [4].

2.3. Proposition. *Let Q be an hl-loop with the identity 1, $Q\downarrow \neq \emptyset$. Then*
(i) *$p \in Q\uparrow$ if and only if p and 1 are comparable;*
(ii) *if $p \in Q\uparrow$, $q \in Q\downarrow$, then p and q are incomparable.*

2.4. Proposition. *Let Q be an hl-loop. Then $Q\uparrow$ is a normal subloop of Q.*

Let Q be an hl-quasigroup. For each $a, b \in Q$ we put

(1) $$a\varrho b \Leftrightarrow a, b \text{ are comparable.}$$

2.5. Proposition (Cf. [4]). *Let Q be an hl-quasigroup such that $Q\uparrow$ is a subquasigroup of Q. Then ϱ is a normal congruence relation on Q and $Q\uparrow$ is normal in Q.*

2.6. Notation. *Let Q be an hl-quasigroup. Let ϱ be a congruence relation on a quasigroup Q defined by (1). For each $a \in Q$ we denote $T_a = \{x \in Q\colon x\varrho a\}$. Since ϱ is a normal congruence relation on Q, the sets T_a are elements of the quotient-quasigroup Q/ϱ with an operation defined by $T_a \cdot T_b = T_{ab}$ (cf. [1]). The cardinal* card Q/ϱ *will be called the index of an hl-quasigroup Q.*

If Q is an hl-loop, then, by 2.3, $T_1 = Q\uparrow$. For the quotient-loop Q/ϱ we will use the notation $Q/Q\uparrow$.

2.7. Definition. Let Q_1 and Q_2 be hl-quasigroups and ϱ_i be a normal congruence relation on Q_i, $i = 1, 2$, defined by (1). We say that hl-quasigroups Q_1 and Q_2 are h-equivalent, written $Q_1 \sim_h Q_2$, if Q_1/ϱ_1 and Q_2/ϱ_2 are isomorphic quasigroups.

2.8. Remark. The relation $\sim_h$ is obviously reflexive, symmetric and transitive.

2.9. Remark. All hl-quasigroups Q with $Q\downarrow = \emptyset$ are h-equivalent and their index is 1. All hl-groups G with $G\downarrow \neq \emptyset$ are h-equivalent and they have index 2.

3. The lexicographic product of hl-loops

Let $I = \{1, 2, \ldots, n\}$. Let Q_i be an hl-loop for each $i \in I$. We denote by $Q^{(1)}$ the direct product of the loops Q_i. The elements of $Q^{(1)}$ will be expressed as $\overline{g} = (g_1, g_2, \ldots, g_n)$; g_i is the component of $\overline{g}$ in Q_i. For the components of an identity $\overline{1} \in Q^{(1)}$ we will use the unit notation 1. By $A^{(1)}$ (or $B^{(1)}$) we denote the set of all elements $\overline{g} \in Q^{(1)}$ such that for each $i \in I$ $g_i \in Q_i\uparrow$ (or $g_i \in Q_i\downarrow$, respectively).

Let H be a subset of $Q^{(1)}$. We say that a relation $\leqslant$ on H is a lexicographic order on H if for arbitrary elements $\overline{g}, \bar{r} \in H$ we have $\overline{g} \leqslant \bar{r}$ if and only if $\overline{g} = \bar{r}$ or $g_i < r_i$ for the least $i \in I$ with $g_i \neq r_i$. It is easy to verify that $\leqslant$ is a partial order on H.

Finally, let us denote by $\mathcal{L}_{Q^{(1)}}$ the set of all H such that

(i$_0$) H is a subloop of $Q^{(1)}$;

(ii$_0$) $A^{(1)} \subseteq H$;

(iii$_0$) under the lexicographic order $\leqslant$, H is an hl-loop.

3.1. Lemma. *Let $H \in \mathcal{L}_{Q^{(1)}}$. Then $H\uparrow = A^{(1)}$ and $H\downarrow \subseteq B^{(1)}$.*

Proof. Assume that $\overline{g} \in H$ has components $g_j \in Q_j\uparrow$ and $g_k \in Q_k\downarrow$ for some $j, k \in I$. There exist elements $\bar{r}, \bar{s} \in A^{(1)}$ such that $r_j < s_j$ and $r_i = s_i = 1$ for each $i \in I$, $i \neq j$. By (ii$_0$) $\bar{r}, \bar{s} \in H$. Clearly $\bar{r} < \bar{s}$ and $\overline{g} \cdot \bar{r} < \overline{g} \cdot \bar{s}$. Thus $\overline{g} \in H\uparrow$. Now, let $\bar{r'}$, $\bar{s'}$ be the elements of $A^{(1)}$ such that $r'_k < s'_k$ and $r'_i = s'_i = 1$ for each $i \in I$, $i \neq k$ (such elements exist and belong to H). Since $\bar{r'} < \bar{s'}$ and $g_k \in Q_k\downarrow$, we have $\overline{g} \cdot \bar{r'} > \overline{g} \cdot \bar{s'}$. Hence $\overline{g} \in H\downarrow$, which contradicts the fact that $\overline{g} \in H\uparrow$. So, either $\overline{g} \in A^{(1)}$ or $\overline{g} \in B^{(1)}$ and this yields that $H\uparrow = A^{(1)}$ and $H\downarrow \subseteq B^{(1)}$. □

In view of 2.6 we will use the notations $T_{\overline{g}} = \{\bar{x} \in H \colon \bar{x}, \overline{g}$ are comparable$\}$ (or $T_{g_i} = \{x \in Q_i \colon x, g_i$ are comparable$\}$) for elements of $H/H\uparrow$ (or $Q_i/Q_i\uparrow$, respectively).

3.2. Lemma. *Let $H \in \mathcal{L}_{Q^{(1)}}$, $\overline{g}, \bar{r} \in H$. Let there exist an index $j \in I$ such that $T_{g_j} = T_{r_j}$. Then $T_{g_i} = T_{r_i}$ for each $i \in I$.*

Proof. Assume that $\overline{g}, \bar{r} \in H$, $T_{g_j} = T_{r_j}$, $T_{g_k} \neq T_{r_k}$. There exists $\bar{y} \in H$ such that $\bar{r} \cdot \bar{y} = \overline{1}$. Denote $\bar{s} = \overline{g} \cdot \bar{y}$. Obviously $\bar{s} \in H$ and

$$T_{s_j} = T_{g_j y_j} = T_{g_j} T_{y_j} = T_{r_j} T_{y_j} = T_{r_j y_j} = T_1 = Q_j\uparrow.$$

Thus $s_j \in Q_j\uparrow$. At the same time

$$T_{s_k} = T_{g_k} T_{y_k} \neq T_{r_k} T_{y_k} = Q_k\uparrow,$$

therefore $s_k \in Q_k\downarrow$, which contradicts 3.1. □

3.3. Lemma. *Let $H \in \mathcal{L}_{Q^{(1)}}$, $\overline{g}, \bar{r} \in H$. Then $T_{\overline{g}} = T_{\bar{r}}$ if and only if $T_{g_i} = T_{r_i}$ for each $i \in I$.*

P r o o f. From $T_{\overline{g}} = T_{\bar{r}}$ it follows that $\overline{g}, \bar{r}$ are comparable. Therefore there exists $k \in I$ such that g_k and r_k are comparable, i.e., $T_{g_k} = T_{r_k}$. Then, by 3.2, $T_{g_i} = T_{r_i}$ for each $i \in I$. Conversely, $T_{g_i} = T_{r_i}$ yields that g_i, r_i are comparable. Thus $\overline{g}$ and $\bar{r}$ are comparable, i.e., $T_{\overline{g}} = T_{\bar{r}}$. □

In the remaining part of the present section we assume that for each $i, j \in I$, Q_i and Q_j are h-equivalent hl-loops. This means that for each $i \in I$ there exists an isomorphism (with respect to the loop operation)

$$\varphi_i \colon\ Q_1/Q_1{\uparrow} \to Q_i/Q_i{\uparrow}. \tag{1}$$

Let $\Phi = (\varphi_i, i \in I)$ be a system of isomorphisms (1) such that $\varphi_1 = id$, where id is the identity transformation of $Q_1/Q_1{\uparrow}$. We denote by $Q^{(0)}$ the subset of $Q^{(1)}$ such that

$$\overline{g} \in Q^{(0)} \text{ if and only if } \quad T_{g_i} = \varphi_i(T_{g_1}) \text{ for each } i \in I.$$

3.4. Lemma. *$Q^{(0)}$ is a subloop of $Q^{(1)}$.*

P r o o f. Obviously $\bar{1} \in Q^{(0)}$. Let $\overline{g}, \bar{r} \in Q^{(0)}$ and $\bar{s} = \overline{g} \cdot \bar{r}$. Since $\varphi_i \in \Phi$ is an isomorphism with respect to the loop operation, we have (for each $i \in I$)

$$\varphi_i(T_{s_1}) = \varphi_i(T_{g_1 r_1}) = \varphi_i(T_{g_1} T_{r_1}) = \varphi_i(T_{g_1})\varphi_i(T_{r_1}) = T_{g_i} T_{r_i} = T_{s_i}.$$

Thus $\bar{s} \in Q^{(0)}$. Analogously $\overline{g}/\bar{r} \in Q^{(0)}$ and $\bar{r} \setminus \overline{g} \in Q^{(0)}$. □

3.5. Lemma. *Under the lexicographic order $\leqslant$, $Q^{(0)}$ is an hl-loop.*

P r o o f. By 3.4, $Q^{(0)}$ is a loop. Clearly, under $\leqslant$, $Q^{(0)}$ is a partially ordered set. Since Q_1 is an hl-loop, there exists $p \in Q_1{\uparrow}$, $p > 1$. Let $\bar{r}$ be an element of $Q^{(1)}$ such that $r_1 = p$ and $r_i = 1$ for each $i \in I$, $i \neq 1$. It is obvious that $\bar{r} \in Q^{(0)}$ and $\bar{1} < \bar{r}$. Thus $\leqslant$ is a nontrivial partial order on $Q^{(0)}$. Likewise, it is trivial to see that if $\overline{g}, \bar{r}, \bar{s} \in Q^{(0)}$, then $\overline{g} \leqslant \bar{r}$ if and only if $\overline{g} \cdot \bar{s} \leqslant \bar{r} \cdot \bar{s}$. We are going to show that $Q^{(0)} = Q^{(0)}{\uparrow} \cup\ Q^{(0)}{\downarrow}$. Evidently $Q^{(0)}{\uparrow} \cup\ Q^{(0)}{\downarrow} \subseteq Q^{(0)}$. Assume that $\overline{g} \in Q^{(0)}$. If $g_1 \in Q_1{\uparrow}$, then for each $i \in I$ we have $T_{g_i} = \varphi_i(T_{g_1}) = \varphi_i(Q_1{\uparrow}) = Q_i{\uparrow}$. This yields that $g_i \in Q_i{\uparrow}$ for each $i \in I$ and therefore $\overline{g} \in Q^{(0)}{\uparrow}$. Similarly, if $g_1 \in Q_1{\downarrow}$, then $\overline{g} \in Q^{(0)}{\downarrow}$. Therefore $Q^{(0)} = Q^{(0)}{\uparrow} \cup\ Q^{(0)}{\downarrow}$. Further, it is easy to see that $Q^{(0)}{\uparrow}$ is a linearly ordered set, thus we can conclude that $Q^{(0)}$ is an hl-loop. □

3.6. Theorem. *Let $Q^{(0)}$ be as above. Then $Q^{(0)} \in \mathcal{L}_{Q^{(1)}}$ and $Q^{(0)}$, Q_i are h-equivalent hl-loops for each $i \in I$.*

Proof. By 3.4 and 3.5 (i_0) and (iii_0) hold. Also, it is easy to verify that $\bar{p} \in A^{(1)}$ implies $\bar{p} \in Q^{(0)}$, thus (ii_0) is valid. We have that $Q^{(0)} \in \mathcal{L}_{Q^{(1)}}$ and we are going to show that $Q^{(0)} \sim_h Q_i$. Define

$$\psi\colon\ Q^{(0)}/Q^{(0)}\!\uparrow\ \to Q_1/Q_1\!\uparrow;\quad \psi(T_{\overline{g}}) = T_{g_1}.$$

In view of 3.2 and 3.3 we have

$$T_{\overline{g}} = T_{\bar{r}} \ \text{ if and only if } \ T_{g_1} = T_{r_1}.$$

Hence ψ is an injective map. To prove that ψ is a surjection take $T_r \in Q_1/Q_1\!\uparrow$. For each $i \in I$, $i \neq 1$ there exists $r_i \in Q_i$ such that $T_{r_i} = \varphi_i(T_r)$, where $\varphi_i \in \Phi$. Let $\overline{g}$ be an element of $Q^{(1)}$ such that $g_1 = r$ and $g_i = r_i$ for each $i \in I$, $i \neq 1$. Clearly $\overline{g} \in Q^{(0)}$ and $\psi(T_{\overline{g}}) = T_r$. Thus ψ is a surjection. Evidently ψ preserves the loop operation, therefore ψ is an isomorphism of $Q^{(0)}/Q^{(0)}\!\uparrow$ onto $Q_1/Q_1\!\uparrow$. We have shown that $Q^{(0)} \sim_h Q_1$. Now, since $Q_1 \sim_h Q_i$ for all $i \in I$, we can conclude, by 2.8, that $Q^{(0)} \sim_h Q_i$ for each $i \in I$. $\square$

3.7. Definition. Let Q_i ($i \in I$) and $Q^{(0)}$ be as above. Then $Q^{(0)}$ is said to be the Φ-lexicographic product of hl-loops Q_i and we express this fact by writing

$$Q^{(0)} = (\Phi) \mathop{\Gamma}_{i=1}^{n} Q_i$$

or

$$Q^{(0)} = (\Phi)(Q_1 \circ Q_2 \circ \ldots \circ Q_n).$$

The hl-loops Q_i are called lexicographic factors of $Q^{(0)}$.

3.8. Remark. The Φ-lexicographic product of hl-loops Q_i depends on the system Φ. There exist hl-loops Q_i and systems of isomorphisms Φ and Ψ such that $\Phi \neq \Psi$ and hl-loops $(\Phi) \mathop{\Gamma}_{i=1}^{n} Q_i$ and $(\Psi) \mathop{\Gamma}_{i=1}^{n} Q_i$ are not isomorphic (see Example 3.9). If Q_i are hl-groups, then there exists exactly one system of isomorphisms (1) and $Q^{(0)}$ is the lexicographic product of hl-groups Q_i (cf. [9]). If Q_i are linearly ordered loops (or groups), then $Q^{(0)} = Q^{(1)}$ and $Q^{(0)}$ is the lexicographic product of linearly ordered loops (or groups, respectively) Q_i.

3.9. Example. Let $(\mathbb{Z}_4, \oplus)$ be the additive group of residues modulo 4. Let $Q = \mathbb{Z}_4 \times \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers) and let $\leqslant$ be the relation on Q defined by

$$(i, x) \leqslant (j, y) \Leftrightarrow i = j \text{ and } x \leqslant y.$$

Put

$$(i,x)\cdot(j,y)=\begin{cases}(i\oplus j, x+y) & \text{if } i=0,\\ \\ (i\oplus j, x-iy) & \text{if } i\neq 0.\end{cases}$$

It is routine to verify that $(Q,\cdot,\leqslant)$ is an hl-loop and

1. $Q\uparrow=\{(0,x);\ x\in\mathbb{R}\}$ and $Q\downarrow=\{(i,x);\ i\in\mathbb{Z}_4, i\neq 0, x\in\mathbb{R}\}$;
2. $Q\uparrow$ is normal in Q and $Q/Q\uparrow=\{T_{(0,0)},T_{(1,0)},T_{(2,0)},T_{(3,0)}\}$.

We take a map $\psi\colon\ Q/Q\uparrow\rightarrow Q/Q\uparrow$ such that

$$\psi\colon\ T_{(0,0)}\mapsto T_{(0,0)};\ T_{(1,0)}\mapsto T_{(3,0)};\ T_{(2,0)}\mapsto T_{(2,0)};\ T_{(3,0)}\mapsto T_{(1,0)}.$$

It is trivial to see that ψ is an isomorphism of $Q/Q\uparrow$ onto $Q/Q\uparrow$ with respect to the loop operation. Let us put

$$Q^{(0)}=(\Phi)(Q\circ Q),\ \text{where } \Phi=\{\mathrm{id},\mathrm{id}\},$$
$$G^{(0)}=(\Psi)(Q\circ Q),\ \text{where } \Psi=\{\mathrm{id},\psi\}.$$

Clearly $Q^{(0)}=\{((0,x),(0,y)),((1,x),(1,y)),((2,x),(2,y)),(3,x),(3,y))\colon\ x,y\in\mathbb{R}\}$ and $G^{(0)}=\{((0,x),(0,y)),((1,x),(3,y)),((2,x),(2,y)),((3,x),(1,y))\colon\ x,y\in\mathbb{R}\}$.

Now we consider the following condition for hl-loops.

(C) There exists T_a such that for each $b\in T_a$ the assertion $a\cdot a=b\cdot b$ holds.

Since $Q^{(0)}$ satisfies (C) (taking $a=((1,x),(1,y))$ for any $x,y\in\mathbb{R}$) and for $G^{(0)}$ (C) fails to hold, the hl-loops $Q^{(0)}$ and $G^{(0)}$ are not isomorphic.

Let Q be an hl-loop. The isomorphism

$$\alpha\colon\ Q\rightarrow(\Phi)\mathop{\Gamma}_{i=1}^{n}Q_i$$

with respect to the loop operation and the partial order is said to be a Φ-lexicographic product decomposition of Q.

3.10. Remark. Let $\alpha_0\colon\ Q\rightarrow Q_1$ be an isomorphism of the hl-loop Q onto the hl-loop Q_1. We regard α_0 as a lexicographic product decomposition of Q and Q_1 as a Φ-lexicographic product with one factor Q_1, where Φ contains only the identity transformation of $Q_1/Q_1\uparrow$.

Let

$$\beta\colon\ Q\rightarrow(\Psi)\mathop{\Gamma}_{i=1}^{m}G_i$$

be a Ψ-lexicographic product decomposition of an hl-loop Q. We say that α,β are isomorphic decompositions if $m=n$ and Q_i, G_i are isomorphic hl-loops for each $i=1,2,\ldots,n$.

4. Two-factor Φ-lexicographic product decompositions

The lexicographic product decompositions of a partially ordered quasigroup with an idempotent element h were discussed in [3]. Putting $h = 1$ we can apply these results to the linearly ordered loops, especially for $Q\uparrow$, where Q is an hl-loop. We start this section by recalling some notions from [3], formulated for the case of Q a linearly ordered loop.

Let Q be a linearly ordered loop and let A be a subloop of Q. A linear order on Q induces a linear order on A under which A is again a linearly ordered loop; A will be called a linearly ordered subloop of Q.

Let A, B be the linearly ordered subloops of Q such that (cf. [3, Section 4], where we put $h = 1$):

(C1) for each $p \in Q$ there exists exactly one pair (a, b) such that $a \in A, b \in B$ and $p = ab$;

(C2) if $p_1, p_2 \in Q$, $p_1 = a_1b_1$, $p_2 = a_2b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$, then

$$p_1p_2 = (a_1a_2) \cdot (b_1b_2);$$

(C3) under the notation as in (C2), the relation $p_1 \leqslant p_2$ is valid if and only if either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leqslant b_2$.

Then we write

$$Q = A \circ B.$$

From [3, Section 4] we have that if $Q = A \circ B$, then Q is isomorphic to the lexicographic product of A and B (with respect to the loop operation and the linear order). Conversely, if Q is a lexicographic product of linearly ordered loops Q_1, Q_2, then there exist linearly ordered subloops A, B of Q such that $Q = A \circ B$. We say that $Q = A \circ B$ defines the lexicographic product decomposition of Q.

Now, let Q be an hl-loop. We take one element from every class $T_r \in Q/\,Q\uparrow$; from $T_1 = Q\uparrow$ we choose an identity element 1. We denote by R the set of all elements chosen from the respective T_r; R will be called the set of representatives of an hl-loop Q. In what follows we assume that R is any fixed set of representatives of Q.

4.1. Lemma. *If $Q\uparrow = A \circ B$, then each element $x \in Q$ can be uniquely written in the form $x = ab \cdot r$, where $a \in A, b \in B$ and $r \in R$.*

Proof. For each element $x \in Q$ there exists exactly one element $r \in R$ such that $r \in T_x$. Since $x/r \in Q\uparrow$, by (C1) there exists exactly one pair of elements $a \in A, b \in B$ such that $x = ab \cdot r$. □

In view of 4.1 we employ the following notation.

4.2. Notation. *Let $Q\uparrow = A \circ B$ and let R be a set of representatives of Q. For each $x \in Q$ we denote $a_x \in A$, $b_x \in B$ and $r_x \in R$ the elements which fulfil $x = a_x b_x \cdot r_x$. By* 4.1 *these elements are uniquely determined* (*for a fixed set R*).

Obviously $r_x = r_y$ if and only if $T_x = T_y$ (i.e., x and y are comparable).

4.3. Lemma. *Let $Q\uparrow = A \circ B$, where A, B are normal subloops of Q. Then for each $x, y, z \in Q$ the following conditions are satisfied:*

(i) $r_{xz} = r_{yz} \Leftrightarrow r_x = r_y$;

(ii) *if* $r_x = r_y$, *then* $b_x \leqslant b_y \Leftrightarrow b_{xz} \leqslant b_{yz}$.

Proof. (i) This is obvious. (ii) Put $r = r_x = r_y$. Since A, B are normal subloops of Q, there exist $a_x^{(1)}, a_x^{(2)}, a_x^{(3)} \in A$ and $b_x^{(1)}, b_x^{(2)} \in B$ such that:

$$\begin{aligned} xz &= (a_x b_x \cdot r)z = (a_x^{(1)} \cdot b_x r)z = a_x^{(2)} \cdot (b_x r \cdot z) = a_x^{(2)}(b_x^{(1)} \cdot rz) \\ &= a_x^{(2)}[b_x^{(1)} \cdot (a_{rz} b_{rz} \cdot r_{rz})] = a_x^{(2)}[(b_x^{(2)} \cdot a_{rz} b_{rz}) \cdot r_{rz}]. \end{aligned}$$

Hence, applying (C2), we obtain

$$xz = a_x^{(2)}[(a_{rz} \cdot b_x^{(2)} b_{rz}) \cdot r_{rz}] = [a_x^{(3)}(a_{rz} \cdot b_x^{(2)} b_{rz})]r_{rz} = (a_x^{(3)} a_{rz} \cdot b_x^{(2)} b_{rz})r_{rz}.$$

Analogously

$$\begin{aligned} yz &= (a_y b_y \cdot r)z = (a_y^{(1)} \cdot b_y r)z = a_y^{(2)} \cdot (b_y r \cdot z) = a_y^{(2)}(b_y^{(1)} \cdot rz) \\ &= a_y^{(2)}[b_y^{(1)} \cdot (a_{rz} b_{rz} \cdot r_{rz})] = a_y^{(2)}[(b_y^{(2)} \cdot a_{rz} b_{rz}) \cdot r_{rz}] \\ &= a_y^{(2)}[(a_{rz} \cdot b_y^{(2)} b_{rz}) \cdot r_{rz}] = [a_y^{(3)}(a_{rz} \cdot b_y^{(2)} b_{rz})]r_{rz} = (a_y^{(3)} a_{rz} \cdot b_y^{(2)} b_{rz})r_{rz}. \end{aligned}$$

By 4.1, we have

$$b_{xz} = b_x^{(2)} b_{rz}, \quad b_{yz} = b_y^{(2)} b_{rz}.$$

Using 2.2 (ii) and the above equations we can conclude:

$$\begin{aligned} b_x \leqslant b_y &\Leftrightarrow b_x r \cdot z \leqslant b_y r \cdot z \\ &\Leftrightarrow b_x^{(1)} \cdot rz \leqslant b_y^{(1)} \cdot rz \Leftrightarrow (b_x^{(2)} \cdot a_{rz} b_{rz})r_{rz} \leqslant (b_y^{(2)} \cdot a_{rz} b_{rz})r_{rz} \\ &\Leftrightarrow b_x^{(2)} \leqslant b_y^{(2)} \Leftrightarrow b_x^{(2)} b_{rz} \leqslant b_y^{(2)} b_{rz} \Leftrightarrow b_{xz} \leqslant b_{yz}. \end{aligned}$$

□

Using similar methods as in the proof of 4.3 we obtain

4.4. Lemma. *Let $Q\!\uparrow = A \circ B$, where A, B are normal subloops of Q. Then for each $x, y, z \in Q$ the following conditions are satisfied:*

(i) $r_{zx} = r_{zy} \Leftrightarrow r_x = r_y$;

(ii) *if $z \in Q\!\uparrow$ and $r_x = r_y$, then $b_x \leqslant b_y \Leftrightarrow b_{zx} \leqslant b_{zy}$;*

(iii) *if $z \in Q\!\downarrow$ and $r_x = r_y$, then $b_x \leqslant b_y \Leftrightarrow b_{zx} \geqslant b_{zy}$.*

Let $Q\!\uparrow = A \circ B$. For $x = a_x b_x$ and $y = a_y b_y$ from $Q\!\uparrow$ we put

(2) $$x \; \tau_1 \; y \Leftrightarrow a_x = a_y, \;\; x \; \tau_2 \; y \Leftrightarrow b_x = b_y.$$

It is routine to verify that τ_1, τ_2 are normal congruence relations on $Q\!\uparrow$. For $i = 1, 2$ and $x \in Q\!\uparrow$ we set $\tau_i[x] = \{y \in Q\!\uparrow\colon\; y \; \tau_i \; x\}$. Clearly $\tau_1[1] = B$ and $\tau_2[1] = A$.

Now, for $i = 1, 2$ we define a relation Θ_i on Q:

(3) $$x \; \Theta_i \; y \Leftrightarrow x/y \in Q\!\uparrow \text{ and } x/y \; \tau_i \; 1.$$

If A and B (i.e., $\tau_2[1]$ and $\tau_1[1]$) are normal subloops of Q, then, by 2.1, Θ_1, Θ_2 are normal congruence relations on Q. Analogously as above, for each $i = 1, 2$ and $x \in Q$ we set $\Theta_i[x] = \{y \in Q\colon\; y \; \Theta_i \; x\}$.

4.5. Lemma. *Let $Q\!\uparrow = A \circ B$, where A, B are normal subloops of Q. Let $x, y \in Q$. Then*

$$x \; \Theta_1 \; y \Leftrightarrow r_x = r_y \text{ and } a_x = a_y,$$
$$x \; \Theta_2 \; y \Leftrightarrow r_x = r_y \text{ and } b_x = b_y.$$

P r o o f. Assume that $x \; \Theta_1 \; y$. Then $x/y \; \tau_1 \; 1$, i.e., $x = by, b \in B$. Using the notations from 4.2 and the assumption that B is a normal subloop of Q we can write:

$$a_x b_x \cdot r_x = b(a_y b_y \cdot r_y) = (b' \cdot a_y b_y) r_y,$$

where $b' \in B$. By (C2) we obtain $a_x b_x \cdot r_x = (a_y \cdot b' b_y) r_y$ and hence, in view of 4.1, we get $r_x = r_y$ and $a_x = a_y$. Conversely, let x, y be elements of Q such that $a_x = a_y, r_x = r_y$. From $r_x = r_y$ we have $x/y \in Q\!\uparrow$. Therefore $x = py$, where $p \in Q\!\uparrow$. Thus $a_x b_x \cdot r_x = p(a_x b_y \cdot r_x)$. Since $Q\!\uparrow$ is a normal subloop of Q, there exists $z \in Q\!\uparrow$ such that

(4) $$a_x b_x \cdot r_x = p(a_x b_y \cdot r_x) = (z \cdot a_x b_y) r_x.$$

Hence $a_x b_x = z \cdot a_x b_y$. From $z \in Q\!\uparrow$ we have $z = ab$, where $a \in A, b \in B$. Then $a_x b_x = ab \cdot a_x b_y$ and hence, in view of (C2) and (C1), we get $a_x = a a_x$. Thus $a = 1$, and therefore $z \in B$. Since B is a normal subloop of Q and $z \in B$, we have, by (4), $p \in B$ $(= \tau_1[1])$. Hence $x/y \; \tau_1 \; 1$, i.e., $x \; \Theta_1 \; y$. The proof for Θ_2 is analogous. □

4.6. Lemma.

(i) *If* $A = \{1\}$, *then* $(x\ \Theta_1\ y \Leftrightarrow T_x = T_y)$ *and* $(x\ \Theta_2\ y \Leftrightarrow x = y)$.

(ii) *If* $B = \{1\}$, *then* $(x\ \Theta_2\ y \Leftrightarrow T_x = T_y)$ *and* $(x\ \Theta_1\ y \Leftrightarrow x = y)$.

Proof. This is a consequence of 4.5. □

In what follows we assume that $Q\!\uparrow = A \circ B$, A, B are normal subloops of Q and $A, B \neq \{1\}$. For each $i = 1, 2$ we denote

$$\overline{Q}_i = \{\Theta_i[x]\colon\ x \in Q\}.$$

Since Θ_i is a normal congruence relation on Q, $\overline{Q}_i$ with the operation $\Theta_i[x] \cdot \Theta_i[y] = \Theta_i[xy]$ is a loop. Put

(5) $$\Theta_1[x] \leqslant \Theta_1[y] \Leftrightarrow r_x = r_y \text{ and } a_x \leqslant a_y,$$

and

(6) $$\Theta_2[x] \leqslant \Theta_2[y] \Leftrightarrow r_x = r_y \text{ and } b_x \leqslant b_y.$$

It is easy to verify that the relation $\leqslant$ is correctly defined on $\overline{Q}_i$ ($i = 1, 2$), i.e., it does not depend on the choice of the elements from $\Theta_i[x]$. Further, we immediately obtain

4.7. Lemma. *The relation* $\leqslant$ *is a partial order on* $\overline{Q}_i$, $i = 1, 2$.

4.8. Lemma.

(i) $\Theta_1[x]$ *and* $\Theta_1[y]$ *are comparable* (*by the relation* $\leqslant$) *if and only if* $\Theta_2[x]$ *and* $\Theta_2[y]$ *are comparable;*

(ii) $x = y$ *if and only if* $\Theta_i[x] = \Theta_i[y]$ *for* $i = 1, 2$.

Proof. Since arbitrary two elements of $Q\!\uparrow$ are comparable, (i) follows from (5) and (6). The assertion (ii) is an immediate consequence of 4.5. □

4.9. Lemma.

(i) *If* $\Theta_1[x] < \Theta_1[y]$, *then* $x < y$.

(ii) *If* $x \leqslant y$, *then* $\Theta_1[x] \leqslant \Theta_1[y]$.

Proof. (i) From 4.5 and (5) it follows that $\Theta_1[x] < \Theta_1[y]$ implies $r_x = r_y$, $a_x < a_y$. Thus, by (C3), $x < y$. (ii) $x \leqslant y \Rightarrow r_x = r_y$, $a_x b_x \leqslant a_y b_y \Rightarrow r_x = r_y$, $a_x \leqslant a_y \Rightarrow \Theta_1[x] \leqslant \Theta_1[y]$. □

Now, for each $i = 1, 2$ we denote

$$H_i^{\uparrow} = \{\Theta_i[x];\ x \in Q{\uparrow}\}, \quad H_i^{\downarrow} = \{\Theta_i[x];\ x \in Q{\downarrow}\}.$$

Clearly $\overline{Q}_i = H_i^{\uparrow} \cup H_i^{\downarrow}$.

4.10. Lemma. *For each $i = 1, 2$, the loop $\overline{Q}_i$ under the relation* (5) (*or* (6), *respectively*) *is an hl-loop with $\overline{Q}_i{\uparrow} = H_i^{\uparrow}$ and $\overline{Q}_i{\downarrow} = H_i^{\downarrow}$.*

Proof. We are going to prove that $\overline{Q}_1$ fulfills the conditions (i)–(iv) from 2.2. By 4.7, under the relation $\leqslant$, $\overline{Q}_1$ is a partially ordered set. Since $A \neq \{1\}$, there exists $x \in A$ such that $x < 1$. Then $\Theta_1[x] < \Theta_1[1]$, thus $\leqslant$ is a nontrivial partial order on $\overline{Q}_1$; hence (i) is valid. Let $x, y, z \in Q$. Clearly

$$\Theta_1[x] = \Theta_1[y] \Leftrightarrow \Theta_1[xz] = \Theta_1[yz]$$

and, in view of 4.9,

$$\begin{aligned}\Theta_1[x] < \Theta_1[y] &\Leftrightarrow x < y,\ \Theta_1[x] \neq \Theta_1[y] \Leftrightarrow \\ &\Leftrightarrow xz < yz,\ \Theta_1[xz] \neq \Theta_1[yz] \Leftrightarrow \Theta_1[xz] < \Theta_1[yz] \\ &\Leftrightarrow \Theta_1[x] \cdot \Theta_1[z] < \Theta_1[y] \cdot \Theta_1[z],\end{aligned}$$

thus (ii) is valid. Using a similar method as above we can prove that $\overline{Q}_1{\uparrow} = H_1^{\uparrow}$ and $\overline{Q}_1{\downarrow} = H_1^{\downarrow}$. Hence $\overline{Q}_1 = \overline{Q}_1{\uparrow} \cup \overline{Q}_1{\downarrow}$; thus (iii) holds. Finally, since $\overline{Q}_1{\uparrow}$ is obviously a linearly ordered set, we have that $\overline{Q}_1$ is an hl-loop.

The proof that (6) is a nontrivial partial order on $\overline{Q}_2$ is analogous to that for $\overline{Q}_1$. Let $x, y, z \in Q$. From (6) and 4.3 we obtain

$$\begin{aligned}\Theta_2[x] \leqslant \Theta_2[y] &\Leftrightarrow r_x = r_y,\ b_x \leqslant b_y \Leftrightarrow r_{xz} = r_{yz}, b_{xz} \leqslant b_{yz} \\ &\Leftrightarrow \Theta_2[x]\Theta_2[z] \leqslant \Theta_2[y]\Theta_2[z],\end{aligned}$$

thus 2.2(ii) is valid. We are going to show that $\overline{Q}_2{\downarrow} = H_2^{\downarrow}$. Let $\Theta_2[z] \in \overline{Q}_2{\downarrow}$. By way of contradiction, suppose that $z \in Q{\uparrow}$. Since $\leqslant$ is a nontrivial partial order on $\overline{Q}_2$, there exist $x, y \in Q$ such that $\Theta_2[x] < \Theta_2[y]$. Then $\Theta_2[zx] > \Theta_2[zy]$, and thus $b_{zx} > b_{zy}$, $r_{zx} = r_{zy}$ Hence, by 4.4, $b_x > b_y$, $r_x = r_y$, which contradicts the fact that $\Theta_2[x] < \Theta_2[y]$. Therefore $z \in Q{\downarrow}$, i.e., $\Theta_2[z] \in H_2^{\downarrow}$. To prove the converse inclusion take $\Theta_2[z] \in H_2^{\downarrow}$ (this means that $z \in Q{\downarrow}$). Then

$$\begin{aligned}\Theta_2[x] \leqslant \Theta_2[y] &\Leftrightarrow r_x = r_y,\ b_x \leqslant b_y \\ &\Leftrightarrow r_{zx} = r_{zy},\ b_{zx} \geqslant b_{zy} \Leftrightarrow \Theta_2[zx] \geqslant \Theta_2[zy].\end{aligned}$$

Thus $\Theta_2[z] \in \overline{Q}_2{\downarrow}$. We have $\overline{Q}_2{\downarrow} = H_2^{\downarrow}$. To prove that $\overline{Q}_2{\uparrow} = H_2^{\uparrow}$ we proceed similarly. Now it is easy to see that $\overline{Q}_2 = \overline{Q}_2{\uparrow} \cup \overline{Q}_2{\downarrow}$, and since $\overline{Q}_2{\uparrow}$ is a linearly ordered set, we can conclude that $\overline{Q}_2$ is an hl-loop. □

The hl-loops $\overline{Q}_1$, $\overline{Q}_2$ are h-equivalent. Indeed, let

$$\varphi\colon\ \overline{Q}_1/\overline{Q}_1{\uparrow} \to \overline{Q}_2/\overline{Q}_2{\uparrow};\ T_{\Theta_1[x]} \mapsto T_{\Theta_2[x]}.$$

By 4.8 (i), $T_{\Theta_2[x]} = T_{\Theta_2[y]}$ if and only if $T_{\Theta_1[x]} = T_{\Theta_1[y]}$, thus φ is an injective mapping. Moreover, it is easy to see that φ is a surjection and φ preserves the loop operation. Thus $\overline{Q}_1 \sim_h \overline{Q}_2$.

Since φ is an isomorphism (with respect to the loop operation), we can construct Φ-lexicographic product

$$\overline{G} = (\Phi)(\overline{Q}_1 \circ \overline{Q}_2),\ \text{where } \Phi = \{\mathrm{id}, \varphi\}.$$

4.11. Lemma. *$(\Theta_1[x], \Theta_2[y]) \in \overline{G}$ if and only if $T_x = T_y$.*

Proof. $(\Theta_1[x], \Theta_2[y]) \in \overline{G} \Leftrightarrow \varphi(T_{\Theta_1[x]}) = T_{\Theta_2[y]} \Leftrightarrow T_{\Theta_2[x]} = T_{\Theta_2[y]} \Leftrightarrow$ $\Theta_2[x], \Theta_2[y]$ are comparable $\Leftrightarrow T_x = T_y$. □

Let us put

$$\psi\colon\ Q \to \overline{G};\ \psi(x) = (\Theta_1[x], \Theta_2[x]).$$

4.12. Lemma. *ψ is an isomorphism of the hl-loop Q onto the hl-loop $\overline{G}$.*

Proof. By 4.11, $(\Theta_1[x], \Theta_2[x]) \in \overline{G}$ for each $x \in Q$. Using 4.8 (ii) it is easy to see that ψ is an injective mapping. We are going to show that ψ is a surjection. Let $(\Theta_1[x], \Theta_2[y]) \in \overline{G}$. By 4.11, $T_x = T_y$, and thus there exists $r \in R$ (R is the set of representatives of Q) such that $x = a_x b_x \cdot r$ and $y = a_y b_y \cdot r$. Put $z = a_x b_y \cdot r$. Since $\Theta_1[z] = \Theta_1[x]$ and $\Theta_2[z] = \Theta_2[y]$, we have $\psi(z) = (\Theta_1[x], \Theta_2[y])$. Thus ψ is a surjection. It is routine to verify that ψ preserves the loop operation. Finally,

$$\begin{aligned}\psi(x) \leqslant \psi(y) &\Leftrightarrow \Theta_1[x] < \Theta_1[y] \text{ or } (\Theta_1[x] = \Theta_1[y],\ \Theta_2[x] \leqslant \Theta_2[y])\\ &\Leftrightarrow (r_x = r_y,\ a_x < a_y) \text{ or } (r_x = r_y,\ a_x = a_y,\ b_x \leqslant b_y)\\ &\Leftrightarrow a_x b_x \cdot r_x \leqslant a_y b_y \cdot r_y \Leftrightarrow x \leqslant y.\end{aligned}$$

Thus ψ is an isomorphism with respect to the loop operation and the partial order. □

Summarizing, we have

4.13. Theorem. *Let Q be an hl-loop and let A, B be nontrivial normal subloops of Q such that $Q\uparrow = A \circ B$. Then ψ is a Φ-lexicographic product decomposition of Q.*

5. Finite-factor Φ-lexicographic product decompositions

The finite-factor lexicographic product decomposition of a partially ordered quasigroup Q with an idempotent element h has been studied by author in [3]. Analogously as in Section 4, putting $h = 1$, we can apply these results to a linearly ordered loop $Q\uparrow$ in case Q is an hl-loop.

Firstly, assume that Q is a linearly ordered loop. Let A_1, A_2, A_3 be linearly ordered subloops of Q. Then (cf. [3, Lemma 4.5]) $Q = (A_1 \circ A_2) \circ A_3$ if and only if $Q = A_1 \circ (A_2 \circ A_3)$. Hence, by induction, we can conclude that the finite-factor lexicographic product decomposition of Q does not depend on the setting of parentheses. Moreover, putting $h = 1$ in [3; (4.4)] we immediately obtain

5.1. Lemma. *Let $Q = A_1 \circ A_2 \circ A_3$. Then $a^{(1)} \cdot (a^{(2)} \cdot a^{(3)}) = (a^{(1)} \cdot a^{(2)}) \cdot a^{(3)}$ for arbitrary elements $a^{(i)} \in A_i, i = 1, 2, 3$.*

For the lexicographic product decomposition of the linearly ordered loop Q with lexicographic factors $A_1, A_2, \dots, A_n$ we use the notation

$$Q = A_1 \circ A_2 \circ \ldots \circ A_n.$$

By 5.1, provided $Q = A_1 \circ A_2 \circ \ldots \circ A_n$ the parentheses in the product $a^{(1)} a^{(2)} \cdot \ldots \cdot a^{(n)}$ of elements $a^{(i)} \in A_i$ can be omitted. Moreover, by (C1), arbitrary elements $x, y \in Q$ can be uniquely written in the form $x = a^{(1)} a^{(2)} \ldots a^{(n)}$, $y = b^{(1)} b^{(2)} \ldots b^{(n)}$, where $a^{(i)}, b^{(i)} \in A_i$ and, by (C2), $xy = (a^{(1)} b^{(1)}) \cdot (a^{(2)} b^{(2)}) \cdot \ldots \cdot (a^{(n)} b^{(n)})$.

Now, let Q be an hl-loop, R be a set of representatives of Q. Suppose that

$$Q\uparrow = A_1 \circ A_2 \circ \ldots \circ A_n \tag{1}$$

is a lexicographic product decomposition of the linearly ordered loop $Q\uparrow$. It is easy to verify that the generalization of 4.1 is valid, i.e., each element $x \in Q$ can be uniquely written in the form $(a^{(1)} a^{(2)} \ldots a^{(n)}) \cdot r$, where $a^{(i)} \in A_i$ and $r \in R$. In view of this fact we will employ the notations $x = a_x^{(1)} a_x^{(2)} \ldots a_x^{(n)} \cdot r_x$, $a_x^{(i)} \in A_i$, $r_x \in R$, $y = a_y^{(1)} a_y^{(2)} \ldots a_y^{(n)} \cdot r_y$, $a_y^{(i)} \in A_i$, $r_y \in R$, $xy = a_{xy}^{(1)} a_{xy}^{(2)} \ldots a_{xy}^{(n)} \cdot r_{xy}$, $a_{xy}^{(i)} \in A_i$, $r_{xy} \in R$, etc. (we recall that the relations $a_{xy}^{(i)} = a_x^{(i)} a_y^{(i)}$, $r_{xy} = r_x r_y$ don't hold in general).

5.2. Lemma. *Let Q be an hl-loop. Let A_i, $i = 1, 2, \ldots, n$, be normal subloops of Q such that $Q\uparrow = A_1 \circ A_2 \circ \ldots \circ A_n$. Then $B = A_{i_1} \circ A_{i_2} \circ \ldots \circ A_{i_k}$ is a normal subloop of Q for arbitrary $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, $i_1 < i_2 < \ldots < i_k$.*

Proof. The assertion of the lemma is trivial for $k = 1$. Let $k \in N$, $1 < k \leqslant n$. We are going to show that if $B^* = A_{i_1} \circ A_{i_2} \circ \ldots \circ A_{i_{k-1}}$ is normal in Q, then B is a normal subloop of Q. It is routine to verify that B is a subloop of Q. Let $x \in Q$, $b \in B$. Clearly $b = ca$, where $c \in B^*$ and $a \in A_{i_k}$. Since A_{i_k}, B^* are normal subloops of Q, there exist $a' \in A_{i_k}$, $c' \in B^*$ such that $xb = x \cdot ca = c'a' \cdot x$. Therefore $xb \in Bx$. Analogously we obtain $bx \in xB$, thus we have $xB = Bx$ for each $x \in Q$. Similarly, if $x, y \in Q$, then $xy \cdot B = x \cdot yB$ and $B \cdot xy = Bx \cdot y$. Therefore B is a normal subloop of Q. □

Suppose that (1) is valid and A_i is normal in Q for each $i = 1, 2, \ldots, n$. We define a relation τ_i $(i = 1, 2, \ldots, n)$ on $Q\uparrow$:

$$x \, \tau_i \, y \Leftrightarrow a_x^{(i)} = a_y^{(i)}. \tag{2}$$

It is easy to see that τ_i is a normal congruence relation on $Q\uparrow$. By 5.2, $\tau_i[1] = \{x \in Q\uparrow\colon \ x \, \tau_i \, 1 \ \}$ is a normal subloop of Q. Now, for $x, y \in Q$ and for each $i = 1, 2, \ldots, n$ we put

$$x\Theta_i y \Leftrightarrow x/y \in Q\uparrow \text{ and } x/y \, \tau_i \, 1. \tag{3}$$

In view of 2.1 Θ_i is a normal congruence relation on Q. Analogously as in Section 4 we denote $\Theta_i[x] = \{z \in Q\colon \ z\Theta_i x\}$ and $\overline{Q}_i = \{\Theta_i[x]\colon \ x \in Q\}$. Recall that under the operation $\Theta_i[x] \cdot \Theta_i[y] = \Theta_i[xy]$, $\overline{Q}_i$ is a loop.

5.3. Lemma. *For each $i = 1, 2, \ldots, n$ the following holds*

$$x\Theta_i y \Leftrightarrow a_x^{(i)} = a_y^{(i)} \text{ and } r_x = r_y.$$

Proof. In view of 5.2 it suffices to apply the same method as in the proof of 4.5. □

Let us denote (for each $i = 1, 2, \ldots, n$)

$$H_i^{\uparrow} = \{\Theta_i[x];\ x \in Q\uparrow\}, \quad H_i^{\downarrow} = \{\Theta_i[x];\ x \in Q\downarrow\}.$$

Clearly $\overline{Q}_i = H_i^{\uparrow} \cup H_i^{\downarrow}$. For $\Theta_i[x]$, $\Theta_i[y] \in \overline{Q}_i$ we set

$$\Theta_i[x] \leqslant \Theta_i[y] \Leftrightarrow r_x = r_y \text{ and } a_x^{(i)} \leqslant a_y^{(i)}. \tag{4}$$

It is easy to see that the relation $\leqslant$ is a partial order on $\overline{Q}_i$.

5.4. Lemma. *Let Q be an hl-loop. Let $Q{\uparrow} = A_1 \circ A_2 \circ A_3$, where A_1, A_2, A_3 are nontrivial normal subloops of Q. Then for each $i = 1, 2, 3$ $\overline{Q}_i$ is an hl-loop and $\overline{Q}_i{\uparrow} = H_i^{\uparrow}$, $\overline{Q}_i{\downarrow} = H_i^{\downarrow}$.*

Proof. Denote $B = A_1 \circ A_2$. By 5.2, B is normal in Q. Clearly $Q{\uparrow} = B \circ A_3$, thus each element $x \in Q$ can be uniquely written in the form $x = ba \cdot r$, where $b \in B$, $a \in A_3$ and $r \in R$. Let η be a relation defined on Q by the rule

$$(ba \cdot r)\ \eta\ (b'a' \cdot r') \Leftrightarrow a = a',\ r = r'$$

and let

$$\eta[ba \cdot r] \leqslant' \eta[b'a' \cdot r'] \Leftrightarrow a \leqslant a',\ r = r'.$$

Denote $\overline{G} = \{\eta[x] \colon x \in Q\}$. By 4.10, under the relation $\leqslant'$, $\overline{G}$ is an hl-loop. But it is easy to see that $\overline{G} = \overline{Q}_3$ and

$$\eta[x] \leqslant' \eta[y] \Leftrightarrow \Theta_3[x] \leqslant \Theta_3[y],$$

where $\leqslant$ is the relation defined by (4). Therefore we can conclude that $\overline{Q}_3$ is an hl-loop and $\overline{Q}_3{\uparrow} = H_3^{\uparrow}$, $\overline{Q}_3{\downarrow} = H_3^{\downarrow}$. Analogously $\overline{Q}_1$ is an hl-loop and $\overline{Q}_1{\uparrow} = H_1^{\uparrow}$, $\overline{Q}_1{\downarrow} = H_1^{\downarrow}$. We are going to show that $\overline{Q}_2$ is an hl-loop. As in the proof of 4.10 it can be seen that $\leqslant$ is a nontrivial partial order on $\overline{Q}_2$. For completing the proof we verify (ii)–(iv) from 2.2. Denote $B = A_2 \circ A_3$. Then $Q{\uparrow} = A_1 \circ B$. Any elements $x, y \in Q$ can be uniquely expressed as $x = a_x^{(1)} b_x \cdot r_x$ and $y = a_y^{(1)} b_y \cdot r_y$, where b_x, b_y are elements of B, which are uniquely determined by $b_x = a_x^{(2)} a_x^{(3)}$ and $b_y = a_y^{(2)} a_y^{(3)}$. Let $z \in Q$. The elements xz, yz can be uniquely written in the form

$$xz = a_{xz}^{(1)} a_{xz}^{(2)} a_{xz}^{(3)} \cdot r_{xz} = a_{xz}^{(1)} b_{xz} \cdot r_{xz},$$

where $a_{xz}^{(i)} \in A_i$, $r_{xz} \in R$, $b_{xz} = a_{xz}^{(2)} a_{xz}^{(3)} \in B$, and

$$yz = a_{yz}^{(1)} a_{yz}^{(2)} a_{yz}^{(3)} \cdot r_{yz} = a_{yz}^{(1)} b_{yz} \cdot r_{yz},$$

where $a_{yz}^{(i)} \in A_i$, $r_{yz} \in R$, $b_{yz} = a_{yz}^{(2)} a_{yz}^{(3)} \in B$. Clearly

$$\Theta_2[x] = \Theta_2[y] \Leftrightarrow \Theta_2[xz] = \Theta_2[yz]$$

and

$$(5) \qquad \Theta_2[x] < \Theta_2[y] \Leftrightarrow r_x = r_y,\ a_x^{(2)} < a_y^{(2)} \Leftrightarrow r_x = r_y,\ a_x^{(2)} \neq a_y^{(2)},\ b_x < b_y.$$

Since $Q\uparrow = A_1 \circ B$, where A_1, B are normal subloops of Q, from (5) and 4.3 it follows that

$$\begin{aligned}\Theta_2[x] < \Theta_2[y] &\Leftrightarrow a^{(2)}_{xz} \neq a^{(2)}_{yz},\ r_{xz} = r_{yz},\ b_{xz} < b_{yz} \\ &\Leftrightarrow a^{(2)}_{xz} < a^{(2)}_{yz},\ r_{xz} = r_{yz} \Leftrightarrow \Theta_2[x]\Theta_2[z] < \Theta_2[y]\Theta_2[z],\end{aligned}$$

thus (ii) from 2.2 holds. Using similar methods as above we obtain (cf. also the proof of 4.10)

$$\overline{Q}_2\uparrow = H_2^{\uparrow} \text{ and } \overline{Q}_2\downarrow = H_2^{\downarrow},$$

which yields $\overline{Q}_2 = \overline{Q}_2\uparrow \cup \overline{Q}_2\downarrow$. Now, since $\overline{Q}_2\uparrow$ is a linearly ordered set, we can conclude that $\overline{Q}_2$ is an hl-loop. □

5.5. Lemma. *Let Q be an hl-loop. Let $Q\uparrow = A_1 \circ A_2 \circ \ldots \circ A_n$, $n \neq 1$, where A_i $(i = 1, 2, \ldots, n)$ are nontrivial normal subloops of Q. Then for each $i = 1, 2, \ldots, n$ $\overline{Q}_i$ is an hl-loop and $\overline{Q}_i\uparrow = H_i^{\uparrow}$, $\overline{Q}_i\downarrow = H_i^{\downarrow}$.*

Proof. In view of 4.10 and 5.4 it suffices to consider the case $n \geqslant 4$. Let $i \neq 1$ and $i \neq n$. We denote $B_1 = A_1 \circ A_2 \circ \ldots \circ A_{i-1}$, $B_2 = A_{i+1} \circ \ldots \circ A_n$. Then $Q\uparrow = B_1 \circ A_i \circ B_2$, thus, by 5.4, $\overline{Q}_i$ is an hl-loop. For $i = 1$ and $i = n$ the assertion of the lemma follows from 5.4, where we set $Q\uparrow = A_1 \circ B \circ A_n$, $B = A_2 \circ \ldots \circ A_{n-1}$. □

Let Q be an hl-loop. We assume that (1) holds, $n \neq 1$ and A_i, $i = 1, 2, \ldots, n$, are the nontrivial normal subloops of Q. For each $i = 1, 2, \ldots, n$ we set

(6) $$\psi_i\colon\ \overline{Q}_1/\ \overline{Q}_1\uparrow \to \overline{Q}_i/\ \overline{Q}_i\uparrow;\ T_{\Theta_1[x]} \to T_{\Theta_i[x]}.$$

Analogously as in Section 4 it can be shown that ψ_i is a loop isomorphism. Denote $\Psi = (\psi_i; i = 1, 2, \ldots, n)$ the system of isomorphisms from (6) (it is obvious that ψ_1 is the identity permutation of $\overline{Q}_1/\ \overline{Q}_1\uparrow$). Let

$$\alpha_1\colon\ Q \to (\Psi) \mathop{\Gamma}_{i=1}^{n} \overline{Q}_i;\ \alpha_1(x) = (\Theta_1[x], \Theta_2[x], \ldots, \Theta_n[x]).$$

Then α_1 is a Ψ-lexicographic product decomposition of Q (the proof is analogous with that of 4.12). Also, it is easy to see that the linearly ordered loops $\overline{Q}_i\uparrow$ and A_i are isomorphic. The decomposition α_1 will be called an extension of the decomposition (1).

Now, for each $i = 1, 2, \ldots, n$, $n \geqslant 2$, let G_i be an hl-loop and let

(7) $$\gamma\colon\ Q \to (\Phi) \mathop{\Gamma}_{i=1}^{n} G_i$$

be a Φ-lexicographic product decomposition of an hl-loop Q. The component of $\gamma(x)$ in G_i will be denoted by $\gamma(x)_i$. For $i = 1, 2, \ldots, n$ we consider the relation Θ_i^* defined on Q by

$$x\Theta_i^* y \Leftrightarrow \gamma(x)_i = \gamma(y)_i.$$

Clearly, Θ_i^* is a normal congruence relation on Q. Under the relation

$$\Theta_i^*[x] \leqslant \Theta_i^*[y] \Leftrightarrow \gamma(x)_i \leqslant \gamma(y)_i,$$

Q/Θ_i^* is an hl-loop, $Q/\Theta_i^*{\uparrow} = \{\Theta_i^*[x];\ x \in Q{\uparrow}\}$, $Q/\Theta_i^*{\downarrow} = \{\Theta_i^*[x];\ x \in Q{\downarrow}\}$. It is routine to verify that G_i, Q/Θ_i^* are isomorphic hl-loops. For each $i = 1, 2, \ldots n$ let

$$B_i = \{x \in Q{\uparrow}\colon\ \gamma(x)_j = 1 \ \text{ for each } j \neq i\}.$$

Obviously B_i are normal, nontrivial subloops of Q and

$$Q{\uparrow} = B_1 \circ B_2 \circ \ldots \circ B_n. \tag{8}$$

Denote

$$\gamma_1\colon\ Q \to (\Psi) \mathop{\Gamma}_{i=1}^{n} \overline{Q}_i;\ \gamma_1(x) = (\Theta_1[x], \Theta_2[x], \ldots, \Theta_n[x]) \tag{9}$$

an extension of (8).

5.6. Lemma. *γ and γ_1 are isomorphic decompositions.*

Proof. From (8) it follows that elements $x, y \in Q$ can be uniquely written in the form $x = (b_x^{(1)} b_x^{(2)} \ldots b_x^{(n)}) \cdot r_x$, $y = (b_y^{(1)} b_y^{(2)} \ldots b_y^{(n)}) \cdot r_y$, where $b_x^{(i)}, b_y^{(i)} \in B_i$, $r_x, r_y \in R$. If $\Theta_i^*[x] \leqslant \Theta_i^*[y]$, then $r_x = r_y$. Indeed, from $\Theta_i^*[x] \leqslant \Theta_i^*[y]$ it follows that $T_{\gamma(x)_i} = T_{\gamma(y)_i}$, thus, by 3.2 and 3.3, $T_{\gamma(x)} = T_{\gamma(y)}$ and since γ is an isomorphism with respect to the partial order, we have $T_x = T_y$, i.e., $r_x = r_y$. Thus we can write

$$\begin{aligned}
\Theta_i^*[x] \leqslant \Theta_i^*[y] &\Leftrightarrow \gamma(x)_i \leqslant \gamma(y)_i \\
&\Leftrightarrow \gamma(b_x^{(1)} b_x^{(2)} \ldots b_x^{(n)} \cdot r_x)_i \leqslant \gamma(b_y^{(1)} b_y^{(2)} \ldots b_y^{(n)} \cdot r_y)_i \\
&\Leftrightarrow \gamma(b_x^{(i)})_i \leqslant \gamma(b_y^{(i)})_i \Leftrightarrow \gamma(b_x^{(i)}) \leqslant \gamma(b_y^{(i)}) \\
&\Leftrightarrow b_x^{(i)} \leqslant b_y^{(i)} \Leftrightarrow \Theta_i[x] \leqslant \Theta_i[y].
\end{aligned}$$

At the same time

$$\Theta_i^*[x] = \Theta_i^*[y] \Leftrightarrow \Theta_i[x] = \Theta_i[y].$$

Hence $Q/\Theta_i^* = \overline{Q}_i$, and since the hl-loops Q/Θ_i^* and G_i are isomorphic, we can conclude that $\overline{Q}_i$ and G_i are isomorphic hl-loops. □

5.7. Lemma. *Let Q be an hl-loop. Let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two decompositions $Q\upharpoonleft = A \circ B$ and $Q\upharpoonleft = C \circ B$, where A, B, C are normal nontrivial subloops of Q, have isomorphic extensions.*

Proof. Each element $x \in Q$ can be uniquely written in the form $x = a_x b_x \cdot r_x$, where $a_x \in A$, $b_x \in B$, $r_x \in R$ and at the same time in the form $x = c_x d_x \cdot r_x$, where $c_x \in C$, $d_x \in B$. Let

$$\alpha_1 \colon Q \to (\Phi)(Q/\Theta_1 \circ Q/\Theta_2) \tag{10}$$

be an extension of the decomposition $Q\upharpoonleft = A \circ B$ and

$$\beta_1 \colon Q \to (\Phi')(Q/\eta_1 \circ Q/\eta_2) \tag{11}$$

be an extension of the decomposition $Q\upharpoonleft = C \circ B$. We are going to show that

$$\psi \colon Q/\Theta_1 \to Q/\eta_1;\ \psi(\Theta_1[x]) = \eta_1[x]$$

is an isomorphism of the hl-loop Q/Θ_1 onto Q/η_1. Let $x = a_x b_x \cdot r_x$, $y = a_y b_y \cdot r_y$. If $\Theta_1[x] = \Theta_1[y]$, then, by 5.3, $r_x = r_y, a_x = a_y$. There are unique $c \in C$, $d \in B$ such that $a_x = cd$. Hence $x = (cd \cdot b_x)r_x = (c \cdot db_x)r_x$ and $y = (cd \cdot b_y)r_x = (c \cdot db_y)r_x$. Thus $\eta_1[x] = \eta_1[y]$. Analogously, if $\eta_1[x] = \eta_1[y]$, then $\Theta_1[x] = \Theta_1[y]$. We see that ψ is an injective map. Obviously, ψ is a surjection which preserves the loop operation. Since ψ is an injection, we have that $r_x = r_y$ implies

$$a_x \neq a_y \Leftrightarrow c_x \neq c_y.$$

Thus, provided $r_x = r_y$ we obtain

$$\begin{aligned} \Theta_1[x] < \Theta_1[y] &\Leftrightarrow a_x < a_y \Leftrightarrow x < y,\ a_x \neq a_y \\ &\Leftrightarrow c_x < c_y \Leftrightarrow \eta_1[x] < \eta_1[y]. \end{aligned}$$

Hence ψ is an isomorphism of the hl-loop Q/Θ_1 onto Q/η_1.

Now, we are going to show that Q/Θ_2 and Q/η_2 are isomorphic hl-loops. Consider

$$\xi \colon Q/\Theta_2 \to Q/\eta_2;\ \xi(\Theta_2[x]) = \eta_2[b_x r_x].$$

It is routine to verify that ξ is a bijection which preserves the partial order. Since B is a normal subloop of Q and R is a subgroupoid of Q, for each $b, d \in B$ and $r, s \in R$ we obtain $br \cdot ds = b_0 r_0$, where $b_0 \in B$, $r_0 = rs \in R$. Using this fact we get that ξ preserves the operation. Thus Q/Θ_2 and Q/η_2 are isomorphic hl-loops. □

6. Isomorphic refinements

Let Q be an hl-loop and let

$$\alpha\colon\ Q \to (\Phi) \underset{i\in I}{\Gamma} G_i,\ I = \{1, 2, \dots n\}, \tag{1}$$

$$\beta\colon\ Q \to (\Psi) \underset{k\in K}{\Gamma} H_k,\ K = \{1, 2, \dots m\} \tag{2}$$

be two lexicographic product decompositions of Q.

6.1. Definition (Cf. [11]). The lexicographic product decomposition β is said to be a refinement of α if for each $i \in I$ there exists a subset $K(i)$ of K and a lexicographic product decomposition

$$\alpha_i\colon\ G_i \to (\Phi_i) \underset{k\in K(i)}{\Gamma} H_k$$

such that, whenever $x \in Q$, $i \in I$ and $k \in K(i)$, then

$$\beta(x)_k = \alpha_i(\alpha(x)_i)_k.$$

We obviously have

6.2. Lemma. *Let α and β be isomorphic lexicographic product decompositions of Q and let α' be a refinement of α. Then there exists a refinement β' of β such that α' and β' are isomorphic.*

Let

$$Q\!\uparrow = A_1 \circ A_2 \circ \dots \circ A_n, \tag{3}$$

where $A_1, A_2, \dots A_n$ are normal subloops of Q. Suppose that for each $i = 1, 2, \dots, n$ there exists a lexicographic product decomposition

$$A_i = A_{i1} \circ A_{i2} \circ \dots \circ A_{ik(i)},$$

where A_{ij} are normal subloops of Q. Then (cf. [3])

$$Q\!\uparrow = A_{11} \circ A_{12} \circ \dots \circ A_{ij} \circ \dots \circ A_{nk(n)}. \tag{4}$$

Now, let

$$\alpha_1\colon\ Q \to (\Phi)(\overline{Q}_1 \circ \overline{Q}_2 \circ \dots \circ \overline{Q}_n)$$

be an extension of (3) and

$$\beta_1\colon\ Q \to (\Psi)(\overline{Q}_{11} \circ \overline{Q}_{12} \circ \dots \circ \overline{Q}_{12} \dots \circ \overline{Q}_{nk(n)})$$

be an extension of (4). From the construction of the extensions α_1 and β_1 we obtain

6.3. Lemma. *β_1 is a refinement of α_1.*

6.4. Theorem. *Let Q be an hl-loop and let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two lexicographic product decompositions of Q have isomorphic refinements.*

Proof. If $Q{\downarrow} = \emptyset$, then the assertion is valid in view of [3]. Suppose that $Q{\downarrow} \neq \emptyset$. Let

$$\alpha\colon\ Q \to (\Phi) \mathop{\Gamma}_{i=1}^{n} G_i,$$
$$\beta\colon\ Q \to (\Psi) \mathop{\Gamma}_{k=1}^{m} H_k$$

be two lexicographic product decompositions of Q. We prove the theorem by induction on $n+m$, $n+m \geqslant 2$. It is clear for $n+m=2$. Let $n+m>2$. The case $m=1$ or $n=1$ is trivial. Assume that $m,n \neq 1$. In the same way as we have constructed the decomposition (8) in Section 5 for γ and the extension γ_1 of γ, we can construct

$$Q{\uparrow} = A_1 \circ A_2 \circ \ldots \circ A_n \quad \text{for } \alpha, \tag{5}$$
$$Q{\uparrow} = B_1 \circ B_2 \circ \ldots \circ B_m \quad \text{for } \beta \tag{6}$$

and the extensions α_1 of (5) and β_1 of (6)

$$\alpha_1\colon\ Q \to (\Phi_1)(\overline{Q}_1 \circ \overline{Q}_2 \circ \ldots \circ \overline{Q}_n),$$
$$\beta_1\colon\ Q \to (\Psi_1)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_m).$$

By 5.6, α, α_1 are isomorphic decompositions and also β, β_1 are isomorphic decompositions. According to [3; Lemma 4.7(i)] we can suppose without loss of generality that $A_n \subseteq B_m$. Hence, by (6) and [3; Lemma 4.7 (ii)], we have

$$Q{\uparrow} = B_1 \circ B_2 \circ \ldots \circ B_{m-1} \circ B_{m1} \circ B_{m2}, \tag{7}$$

where $B_{m1} = B_m \cap (A_1 \circ A_2 \circ \ldots \circ A_{n-1})$ and $B_{m2} = A_n$. From the construction of (5) and (6) it follows that the subloops A_i and B_j are normal in Q for each $i = 1,2,\ldots n$, $j = 1,2,\ldots m$. Since, according to 5.2, B_{m1} is an intersection of normal subloops of Q, B_{m1} is normal in Q. Thus there exists an extension β_2 of (7)

$$\beta_2\colon\ Q \to (\Psi_2)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_{m-1} \circ \overline{G}_{m1} \circ \overline{G}_{m2}).$$

In view of 6.3 β_2 is a refinement of β_1. Denote

$$A = A_1 \circ A_2 \circ \ldots \circ A_{n-1}$$

and

$$B = B_1 \circ B_2 \circ \ldots \circ B_{m-1} \circ B_{m1}$$

(by 5.2, A, B are normal subloops of Q). Then

$$Q\!\uparrow = A \circ A_n \tag{8}$$

and at the same time

$$Q\!\uparrow = B \circ A_n. \tag{9}$$

Let $Q \to (\Phi')(\overline{Q}_A \circ \overline{Q}_{A_n})$ be an extension of (8) and $Q \to (\Psi')(\overline{G}_B \circ \overline{G}_{A_n})$ be an extension of (9). According to 5.7, $\overline{Q}_A$, $\overline{G}_B$ and also $\overline{Q}_{A_n}$, $\overline{G}_{A_n}$ are isomorphic hl-loops. Moreover, it can be verified that

$$\overline{Q}_{A_n} = \overline{Q}_n \text{ and } \overline{G}_{A_n} = \overline{G}_{m2}. \tag{10}$$

We denote by $\varphi_1, \varphi_2, \ldots, \varphi_n$ the isomorphisms from the system Φ_1 and by $\psi_1, \psi_2, \ldots, \psi_{m2}$ the isomorphisms from Ψ_2. Put $\Phi_1^* = \Phi_1 - \{\varphi_n\}$ and $\Psi_2^* = \Psi_2 - \{\psi_{m2}\}$. There exist decompositions

(I) $\overline{Q}_A \to (\Phi_1^*)(\overline{Q}_1 \circ \overline{Q}_2 \circ \ldots \circ \overline{Q}_{n-1})$;
(II) $\overline{G}_B \to (\Psi_2^*)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_{m-1} \circ \overline{G}_{m1})$.

By the induction hypothesis there exist lexicographic product decompositions α_1', β_1' such that

- α_1' is a refinement of (I), β_1' is a refinement of (II)
- α_1', β_1' are isomorphic decompositions.

Hence according to (10), α_1 and β_2 have isomorphic refinements. Therefore, by 6.2, the lexicographic product decompositions α and β have isomorphic refinements. □

References

[1] *V. D. Belousov*: Foundations of the theory of quasigroups and loops. Nauka Moscow, 1967. (In Russian.) zbl
[2] *Š. Černák*: Lexicographic products of cyclically ordered groups. Math. Slovaca *45* (1995), 29–38. zbl
[3] *M. Demko*: Lexicographic product decompositions of partially ordered quasigroups. Math. Slovaca *51* (2001), 13–24. zbl
[4] *M. Demko*: On half linearly ordered quasigroups. Acta Facultatis Prešov *39* (2002), 39–45.
[5] *L. Fuchs*: Partially ordered algebraic systems. Pergamon Press, Oxford-London-New York-Paris, 1963. zbl
[6] *M. Giraudet and F. Lucas*: Groupes à moitié ordonnés. Fund. Math. *139* (1991), 75–89. zbl

[7] *J. Jakubík*: Lexicographic products of partially ordered groupoids. Czech. Math. J. *14* (1964), 281–305. (In Russian.) zbl

[8] *J. Jakubík*: Lexicographic product decompositions of cyclically ordered groups. Czech. Math. J. *48* (1998), 229–241. zbl

[9] *J. Jakubík*: Lexicographic products of half linearly ordered groups. Czech. Math. J. *51* (2001), 127–138.

[10] *A. I. Maltsev*: On ordered group. Izv. Akad. Nauk SSSR, Ser. Matem. *13* (1949), 473–482. (In Russian.) zbl

Author's address: Milan Demko, KM FHPV PU, 17. novembra 1, 081 16 Prešov, Slovakia, e-mail: `demko@unipo.sk`.