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Subdirectly irreducible sectionally pseudocomplemented semilattices

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 2, 725–735

Persistent URL: <http://dml.cz/dmlcz/128201>

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SUBDIRECTLY IRREDUCIBLE SECTIONALLY
PSEUDOCOMPLEMENTED SEMILATTICES

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(Received April 19, 2005)

Abstract. Sectionally pseudocomplemented semilattices are an extension of relatively pseudocomplemented semilattices—they are meet-semilattices with a greatest element such that every section, i.e., every principal filter, is a pseudocomplemented semilattice. In the paper, we give a simple equational characterization of sectionally pseudocomplemented semilattices and then investigate mainly their congruence kernels which leads to a characterization of subdirectly irreducible sectionally pseudocomplemented semilattices.

Keywords: sectionally pseudocomplemented semilattice, weakly standard element

MSC 2000: 06A12, 06D15

A *pseudocomplemented semilattice* is an algebra $(S, \wedge, *, 0)$ of type $(2, 1, 0)$ such that $(S, \wedge, 0)$ is a meet-semilattice with a least element and for all $x, y \in S$,

$$(1) \quad y \leq x^* \quad \text{iff} \quad y \wedge x = 0.$$

A *relatively pseudocomplemented semilattice* is an algebra $(S, \wedge, *, 1)$ of type $(2, 2, 0)$, where $(S, \wedge, 1)$ is a meet-semilattice with a greatest element and for all $x, y, z \in S$,

$$(2) \quad z \leq x * y \quad \text{iff} \quad z \wedge x \leq y.$$

Relatively pseudocomplemented semilattices appear in the literature also under the name *Brouwerian semilattices* or *implicative semilattices*, respectively (see [7], [8]).

For every $a \in S$, we call the principal filter $[a] = \{x \in S : x \geq a\}$ a *section* of S . It is easy to see that if $(S, \wedge, *, 1)$ is a relatively pseudocomplemented semilattice then for any $a \in S$ and $x \in [a]$, $x^a := x * a$ is the pseudocomplement of x in

the section $[a]$, i.e., $y \leq x^a$ iff $y \wedge x = a$ for every $y \in [a]$. Thus $([a], \wedge, \cdot^a, a)$ is a pseudocomplemented semilattice, and consequently, for every interval $[a, b]$ of S , $([a, b], \wedge, \cdot^{ab}, a)$ is a pseudocomplemented semilattice with $x^{ab} := x^a \wedge b = (x * a) \wedge b$.

We know that a lattice is relatively complemented if every interval is a complemented lattice. From this point of view, the concept of a relatively pseudocomplemented (semi)lattice may seem to be a bit misleading since a (semi)lattice whose intervals are pseudocomplemented (semi)lattices in general need not be relatively pseudocomplemented. For instance, the pentagon N_5 (see Figure 1) is such a (semi)lattice. This observation leads to the extension of relative pseudocomplementation we deal with in this paper.

1. SECTIONALLY PSEUDOCOMPLEMENTED SEMILATTICES

Definition 1 [4]. A meet-semilattice $(S, \wedge, 1)$ with a greatest element is said to be *sectionally pseudocomplemented* if for every $a \in S$, the structure $([a], \wedge, \cdot^a, a)$ is a pseudocomplemented semilattice, i.e., every $x \in [a]$ has the pseudocomplement x^a in the section $[a]$.

Remark. The concept of a sectionally pseudocomplemented lattice was invented by I. Chajda in [2]; similar ideas are contained also in J. C. Varlet’s paper [9].

The difficulty arises concerning the number of partial unary operations $\cdot^a: [a] \rightarrow [a]$ which we overcome by defining a new total binary operation “ \circ ” as follows:

$$(3) \quad x \circ y := x^{x \wedge y}.$$

Thus $x \circ y$ is the pseudocomplement of x in the section $[x \wedge y]$.

It can be easily seen that if the relative pseudocomplement $x * y$ of x with respect to y exists then $x \circ y = x * y$. Indeed, we have $(x * y) \wedge x = x \wedge y$, so that $x * y \leq x \circ y$, and conversely, $(x \circ y) \wedge x = x \wedge y \leq y$ entails $x \circ y \leq x * y$.

On the other hand, $x \circ y$ need not be the relative pseudocomplement of x with respect to y . For instance, a non-distributive lattice (that is not relatively pseudocomplemented since relatively pseudocomplemented lattices are distributive) may be sectionally pseudocomplemented.

Example 2. Consider the (meet-semi)lattice that is shown in Figure 1. We obviously have $c \circ a = c^a = a$, while $c * a$ does not exist since the set of all x with $x \wedge c \leq a$ has no top element. The operation “ \circ ” is given by Table 1.

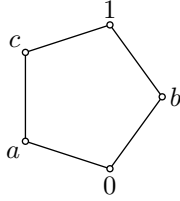


Figure 1

\circ	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	c	c	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Table 1

Remark. In the case of sectionally pseudocomplemented lattices, one can define another binary operation “ \bullet ” by

$$x \bullet y := (x \vee y)^y.$$

Obviously, $x \circ y = x \bullet (x \wedge y)$ and $x \bullet y = (x \vee y) \circ y$.

Theorem 3. A meet-semilattice $(S, \wedge, 1)$ is sectionally pseudocomplemented if and only if there exists a binary operation “ \circ ” on S such that, for all $x, y, z \in S$,

- (4) $x \circ x = 1,$
- (5) $x \wedge (x \circ y) = x \wedge y,$
- (6) $x \wedge ((x \wedge y) \circ z) = x \wedge (y \circ (x \wedge z)).$

Proof. Let $(S, \wedge, 1)$ be a sectionally pseudocomplemented semilattice and let a binary operation “ \circ ” be defined by (3). Then clearly $x \circ x = x^x = 1$ and $x \wedge (x \circ y) = x \wedge x^{x \wedge y} = x \wedge y$ since $x^{x \wedge y}$ is the pseudocomplement of x in the section $[x \wedge y]$. Thus (4) and (5) hold.

It is known and straightforward to show that any pseudocomplemented semilattice satisfies the identity

$$x \wedge (x \wedge y)^* = x \wedge y^*.$$

Hence for the section $[x \wedge y \wedge z]$ we have

$$x \wedge ((x \wedge y) \circ z) = x \wedge (x \wedge y)^{x \wedge y \wedge z} = x \wedge y^{x \wedge y \wedge z} = x \wedge (y \circ (x \wedge z))$$

which is (6).

Conversely, let “ \circ ” be a binary operation on S that fulfils the identities (4), (5) and (6). Let $a \in S$. We have to show that for any $x \in [a]$, $x \circ a$ is the pseudocomplement of x in the section. By (5) we see that $x \wedge (x \circ a) = x \wedge a = a$, and so $y \wedge x = a$ for every $y \in [a]$ with $y \leq x \circ a$. On the other hand, if $y \wedge x = a$ then (6) and (4) yield $y \wedge (x \circ a) = y \wedge (x \circ (y \wedge a)) = y \wedge ((x \wedge y) \circ a) = y \wedge (a \circ a) = y \wedge 1 = y$, so $y \leq x \circ a$. Thus $x^a = x \circ a$. □

Remark. In the light of the previous theorem, sectionally pseudocomplemented semilattices can be treated as algebras $(S, \wedge, \circ, 1)$ of type $(2, 2, 0)$. Of course, they form a variety that is axiomatized, relatively to the variety of meet-semilattices with 1, by the above identities (4), (5) and (6).

Let us recall that a meet-semilattice is said to be *distributive* if $a \geq b \wedge c$ implies the existence of $b_1 \geq b$ and $c_1 \geq c$ with $a = b_1 \wedge c_1$ (see [6]). It is worth noticing that a semilattice is distributive if and only if its filters form a distributive lattice.

Theorem 4. *Every distributive sectionally pseudocomplemented semilattice is relatively pseudocomplemented.*

Proof. Let $(S, \wedge, \circ, 1)$ be a distributive sectionally pseudocomplemented semilattice. We prove that $z \leq x \circ y$ is equivalent to $z \wedge x \leq y$. Obviously, if $z \leq x \circ y$ then $z \wedge x \leq (x \circ y) \wedge x = x \wedge y \leq y$. Conversely, if $z \wedge x \leq y$ then $y = x_1 \wedge z_1$, where $x_1 \geq x$ and $z_1 \geq z$, whence we obtain $x \wedge y = x \wedge x_1 \wedge z_1 = x \wedge z_1$ which yields $z \leq z_1 \leq x^{x \wedge y} = x \circ y$. Therefore $x \circ y$ is the relative pseudocomplement of x with respect to y . \square

2. CONGRUENCE KERNELS

First, we recall several well-known concepts from universal algebra (see e.g. [1, [3]).

Let A be an algebra with a constant 1. By the *kernel* of a congruence $\Theta \in \text{Con}(A)$ we mean the equivalence class $[1]_\Theta = \{a \in A : (a, 1) \in \Theta\}$. An algebra A is called *weakly regular* if $\Theta = \Phi$ whenever $[1]_\Theta = [1]_\Phi$ for any $\Theta, \Phi \in \text{Con}(A)$. A variety \mathcal{V} with a constant 1 is *weakly regular* if every $A \in \mathcal{V}$ is weakly regular. It is known that \mathcal{V} is weakly regular if and only if there exist binary terms t_1, \dots, t_n ($n \in \mathbb{N}$) such that

$$t_1(x, y) = \dots = t_n(x, y) = 1 \quad \text{iff} \quad x = y.$$

An algebra A with a constant 1 is *arithmetical at 1* if for all $\Theta, \Phi, \Psi \in \text{Con}(A)$,

$$[1]_{\Theta \circ \Phi} = [1]_{\Phi \circ \Theta} \quad \text{and} \quad [1]_{\Theta \cap (\Phi \vee \Psi)} = [1]_{(\Theta \cap \Phi) \vee (\Theta \cap \Psi)}.$$

A variety \mathcal{V} is *arithmetical at 1* if so is each $A \in \mathcal{V}$. Arithmeticity at 1 can be captured by a simple Maltsev type condition: a variety \mathcal{V} is arithmetical at 1 if and only if there exists a binary term t with

$$t(x, x) = t(1, x) = 1 \quad \text{and} \quad t(x, 1) = x.$$

Finally, a variety \mathcal{V} is called *congruence distributive* if the congruence lattice $\text{Con}(A)$ of every $A \in \mathcal{V}$ is distributive.

Theorem 5. *The variety of all sectionally pseudocomplemented semilattices is weakly regular and arithmetical at 1.*

Proof. Consider the terms $t_1(x, y) = x \circ y$ and $t_2(x, y) = y \circ x$. Clearly, $t_1(x, x) = t_2(x, x) = 1$. Conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $x = x \wedge (x \circ y) = x \wedge y = y \wedge (y \circ x) = y$ by the identity (5) of Theorem 3.

For arithmeticity at 1, consider the term $t(x, y) = y \circ x$. Then certainly $t(x, x) = 1$, $t(x, 1) = 1$ and $t(1, x) = x$. \square

Corollary 6. *The variety of all sectionally pseudocomplemented semilattices is congruence distributive.*

As known (e.g. [8], [7]), filters of relatively pseudocomplemented semilattices are in a one-to-one correspondence with their congruence relations. More precisely, given a filter F of $(S, \wedge, *, 1)$, the relation Θ_F defined via

$$(x, y) \in \Theta_F \quad \text{iff} \quad (x * y) \wedge (y * x) \in F,$$

or equivalently,

$$(x, y) \in \Theta_F \quad \text{iff} \quad x \wedge a = y \wedge a \text{ for some } a \in F,$$

is a congruence on $(S, \wedge, *, 1)$ such that $[1]_{\Theta_F} = F$. This is in contrast to the situation for sectionally pseudocomplemented semilattices: there exist filters that are not congruence kernels (see Example 8). However, any congruence is determined by its kernel in the following manner:

Lemma 7. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice and let F be a filter of a semilattice (S, \wedge) . Define a binary relation Φ_F by*

$$(7) \quad (x, y) \in \Phi_F \quad \text{iff} \quad x \wedge a = y \wedge a \text{ for some } a \in F.$$

Then F is the kernel of a congruence $\Theta \in \text{Con}(S)$ if and only if $\Theta = \Phi_F$.

In particular, a principal filter $[a]$ is the kernel of $\Theta \in \text{Con}(S)$ if and only if $\Theta = \Phi_a$, where

$$(8) \quad (x, y) \in \Phi_a \quad \text{iff} \quad x \wedge a = y \wedge a.$$

Proof. Let Θ be a congruence such that $F = [1]_{\Theta}$. If $(x, y) \in \Theta$ then $(x \circ y, 1), (y \circ x, 1) \in \Theta$, i.e., $x \circ y, y \circ x \in F$ whence it follows that $(x \circ y) \wedge (y \circ x) \in F$. It is obvious that $x \wedge (x \circ y) \wedge (y \circ x) = x \wedge y = y \wedge (x \circ y) \wedge (y \circ x)$, so we may take $a = (x \circ y) \wedge (y \circ x)$ which yields $(x, y) \in \Phi_F$. If $(x, y) \in \Phi_F$ then $x \wedge a = y \wedge a$ for some $a \in F = [1]_{\Theta}$. Since $(a, 1) \in \Theta$ implies $(x, x \wedge a) \in \Theta$ and $(y, y \wedge a) \in \Theta$, we have $(x, y) \in \Theta$. Thus $\Theta = \Phi_F$.

Conversely, one readily sees that $[1]_{\Phi_F} = F$, so if $\Theta = \Phi_F \in \text{Con}(S)$ then F is the kernel of Θ . \square

Example 8. Let us return to Example 2. Then Φ_b is an equivalence with the partition $\{b, 1\}, \{0, a, c\}$, but it is not a congruence since $(a, c) \in \Phi_b$ while $(c \circ a, c \circ c) = (a, 1) \notin \Phi_b$.

Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice, $a \in S$ and $[a] = \{x \in S : x \leq a\}$. Then upon defining

$$x \circ_a y := (x \circ y) \wedge a,$$

the structure $([a], \wedge, \circ_a, a)$ is a sectionally pseudocomplemented semilattice, too. Hence

Corollary 9. *A principal filter $[a]$ is a congruence kernel if and only if the mapping $f : x \mapsto x \wedge a$ is a homomorphism of $(S, \wedge, \circ, 1)$ onto $([a], \wedge, \circ_a, a)$.*

Proof. Assume first that $[a] = [1]_{\Theta}$, where $\Theta = \Phi_a \in \text{Con}(S)$. From $(a, 1) \in \Theta$ it follows that $(x, x \wedge a), (y, y \wedge a) \in \Theta$ and hence $(x \circ y, (x \wedge a) \circ (y \wedge a)) \in \Theta$. Seeing that $\Theta = \Phi_a$, we obtain $(x \circ y) \wedge a = ((x \wedge a) \circ (y \wedge a)) \wedge a$, i.e., $f(x \circ y) = f(x) \circ_a f(y)$. On the other hand, if $f : x \mapsto x \wedge a$ is a homomorphism then Φ_a is equal to Θ_f , the congruence induced by f , thus $[a] = [1]_{\Theta_f}$. \square

As an immediate consequence we obtain:

Corollary 10. *A principal filter $[a]$ is a congruence kernel if and only if*

$$(x \circ y) \wedge a = ((x \wedge a) \circ (y \wedge a)) \wedge a$$

for all $x, y \in S$.

It turns out that relatively pseudocomplemented semilattices are those sectionally pseudocomplemented semilattices in which every filter is a congruence kernel:

Theorem 11. Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice. Then the mapping $\Theta \mapsto [1]_{\Theta}$ is a one-to-one correspondence between congruences and filters, the inverse of which is given by $F \mapsto \Phi_F$, if and only if $(S, \wedge, \circ, 1)$ is a relatively pseudocomplemented semilattice.

Proof. If every filter F is a congruence kernel then the lattice of all filters is isomorphic to the congruence lattice $\text{Con}(S)$. Thus the lattice of filters is distributive which implies that S is a distributive semilattice, and hence by Theorem 4, $(S, \wedge, \circ, 1)$ is a relatively pseudocomplemented semilattice. \square

Given a lattice (L, \vee, \wedge) , an element $a \in L$ is called *standard* (see [6]) if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$$

for all $x, y \in L$.

Let $(S, \wedge, 1)$ be a meet-semilattice with a greatest element. A filter F is called *standard* if it is a standard element of the lattice of all filters of $(S, \wedge, 1)$; this is equivalent to

$$[x] \cap (F \vee [y]) = ([x] \cap F) \vee ([x] \cap [y])$$

for all $x, y \in S$.

It was proved in [5] that the congruence kernels of sectionally pseudocomplemented lattices are precisely the standard filters, but this is not the case of sectionally pseudocomplemented semilattices:

Example 12. In the pentagon (cf. Example 2, Figure 1), $F = \{c, 1\}$ is the kernel of the congruence given by the partition $\{c, 1\}, \{a\}, \{0, b\}$, but F is not a standard filter since $[a] \cap (F \vee [b]) = [a]$ while $([a] \cap F) \vee ([a] \cap [b]) = [c]$.

In order to capture the congruence kernels of sectionally pseudocomplemented semilattice, we extend the concept of standardness as follows:

Let (L, \vee, \wedge) be a lattice. We say that $a \in L$ is *weakly standard* if for all $x, y \in L$,

$$x \leq y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y.$$

Theorem 13. Let (L, \vee, \wedge) be a lattice. An element $a \in L$ is weakly standard if and only if there exist no $x_1, y_1 \in L$ such that $a \wedge x_1 = a \wedge y_1$, x_1, a, y_1 and $a \vee x_1 = a \vee y_1$ form a sublattice isomorphic to the pentagon N_5 (see Figure 2).

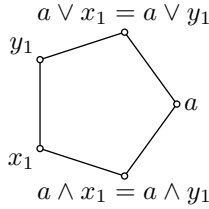


Figure 2

Proof. It is clear that if there exist such $x_1, y_1 \in L$ then a is not weakly standard since $x_1 \vee (a \wedge y_1) = x_1$ while $(x_1 \vee a) \wedge y_1 = y_1$.

Conversely, assume that $a \in L$ is not weakly standard, i.e., there are $x, y \in L$ with $x \leq y$, but $x \vee (a \wedge y) < (x \vee a) \wedge y$. We put $x_1 = x \vee (a \wedge y)$ and $y_1 = (x \vee a) \wedge y$. Then $a \vee x_1 = a \vee x \vee (a \wedge y) = a \vee x$ and $a \vee y_1 = a \vee ((x \vee a) \wedge y) \leq a \vee (x \vee a) = a \vee x \leq a \vee ((x \vee a) \wedge y) = a \vee y_1$ as $x = (x \vee a) \wedge x \leq (x \vee a) \wedge y$, so $a \vee x_1 = a \vee y_1$.

Similarly, $a \wedge y_1 = a \wedge (x \vee a) \wedge y = a \wedge y$ and $a \wedge x_1 = a \wedge (x \vee (a \wedge y)) \geq a \wedge y \geq a \wedge (x \vee (a \wedge y)) = a \wedge x_1$ since $y = y \vee (a \wedge y) \geq x \vee (a \wedge y)$, thus $a \wedge x_1 = a \wedge y_1$.

Therefore, one readily sees that $a \wedge x_1 = a \wedge y_1, x_1, a, y_1, a \vee x_1 = a \vee y_1$ form a sublattice of L that is isomorphic to N_5 (cf. Figure 2). \square

In a semilattice $(S, \wedge, 1)$, a filter F is called *weakly standard* if F is a weakly standard element of the lattice of all filters of S . It is straightforward to prove that F is weakly standard if and only if for all $x, y \in S$,

$$x \leq y \text{ implies } [x] \cap (F \vee [y]) = ([x] \cap F) \vee [y].$$

Lemma 14. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice, F a weakly standard filter and $x, y \in S$ with $x \leq y$. Then $(x, y) \in \Phi_F$ if and only if there exists $b \in F$ such that $x = y \wedge b$.*

Proof. Let $(x, y) \in \Phi_F$, i.e. $x \wedge a = y \wedge a$ for some $a \in F$. Clearly, $x \wedge a = y \wedge a \in F \vee [y]$ and so $x \in [x] \cap (F \vee [y]) = ([x] \cap F) \vee [y]$ since F is a weakly standard filter. Then $x \geq b \wedge y$ for some $b \in [x] \cap F$, i.e. $b \geq x$, whence $b \wedge y \geq x \geq b \wedge y$, so $x = b \wedge y$.

Conversely, if there is $b \in F$ with $x = y \wedge b$, then $x \wedge b = y \wedge b$, so that $(x, y) \in \Phi_F$. \square

For sectionally pseudocomplemented semilattices we have the following analogue of [5]:

Theorem 15. *Let $(S, \wedge, \circ, 1)$ a sectionally pseudocomplemented semilattice and F a filter of a semilattice $(S, \wedge, 1)$. Then F is a congruence kernel if and only if F is a weakly standard filter.*

Proof. Let $F = [1]_{\Theta}$ for some $\Theta \in \text{Con}(S)$. We have to show that $[x] \cap (F \vee [y]) \subseteq ([x] \cap F) \vee [y]$ provided $x \leq y$. For, let $z \in [x] \cap (F \vee [y])$. Then $z \geq x$ and $z \geq a \wedge y$ for some $a \in F$, so that $a \wedge y \wedge z = a \wedge y$ which means $(y, y \wedge z) \in \Phi_F = \Theta$. Hence $1 = y \circ y \Theta y \circ (y \wedge z) = y \circ z$, thus $y \circ z \in F = [1]_{\Theta}$. But we also have $y \circ z \in [x]$ since $x \leq y \wedge z \leq y \circ z$, so $y \circ z \in F \cap [x]$. Finally, $y \wedge (y \circ z) = y \wedge z \leq z$ yields $z \in (F \cap [x]) \vee [y]$.

Conversely, suppose that a filter F is weakly standard. First, we note that the relation Φ_F defined by (8) is compatible with “ \wedge ”.

Now, we put $\Theta = \Phi_F$. It is obvious that $F = [1]_{\Theta}$, so we have to prove that Θ is compatible with the operation “ \circ ”. For that purpose, it suffices to show that the quotient semilattice $(S/\Theta, \wedge, [1]_{\Theta})$ is a sectionally pseudocomplemented semilattice in which $[x]_{\Theta} \circ [y]_{\Theta} = [x \circ y]_{\Theta}$.

Let $[a]_{\Theta} \in S/\Theta$ and $[x]_{\Theta} \geq [a]_{\Theta}$. Without loss of generality we may assume that $x \geq a$. We show that $[x \circ a]_{\Theta}$ is the pseudocomplement of $[x]_{\Theta}$ in the section $[[a]_{\Theta}]$ of the quotient semilattice. One readily sees that $[x]_{\Theta} \wedge [x \circ a]_{\Theta} = [x \wedge (x \circ a)]_{\Theta} = [x \wedge a]_{\Theta} = [a]_{\Theta}$. Let now $[y]_{\Theta} \wedge [x]_{\Theta} = [a]_{\Theta}$; again, we assume that $y \geq a$. Then by Lemma 14, $[x \wedge y]_{\Theta} = [a]_{\Theta}$ along with $a \leq x \wedge y$ yields the existence of $b \in F$ with $a = x \wedge y \wedge b$ whence $y \wedge b \leq x^a = x \circ a$. This implies that $[y]_{\Theta} = [y \wedge b]_{\Theta} \leq [x \circ a]_{\Theta}$.

Therefore, $(S/\Theta, \wedge, [1]_{\Theta})$ is a sectionally pseudocomplemented semilattice with $[x]_{\Theta} \circ [y]_{\Theta} = [x]_{\Theta}^{[x]_{\Theta} \wedge [y]_{\Theta}} = [x]_{\Theta}^{[x \wedge y]_{\Theta}} = [x \circ (x \wedge y)]_{\Theta} = [x \circ y]_{\Theta}$. \square

Corollary 16. *A sectionally pseudocomplemented semilattice $(S, \wedge, \circ, 1)$ is subdirectly irreducible if and only if it has a smallest non-trivial weakly standard filter.*

Since each standard filter is weakly standard, we obtain

Corollary 17. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice. Then every standard filter of $(S, \wedge, 1)$ is the kernel of some congruence $\Theta \in \text{Con}(S)$.*

It is well-known (e.g. [7], [8]) that a relatively pseudocomplemented semilattice $(S, \wedge, *, 1)$ is subdirectly irreducible if and only if it has a smallest non-trivial filter; in other words, the set $S \setminus \{1\}$ has a greatest element. This easily follows from the fact that filters agree with congruence kernels. Sectionally pseudocomplemented semilattices generalize relative pseudocomplemented ones, however, the situation is rather different.

Lemma 18. For any sectionally pseudocomplemented semilattice $(S, \wedge, \circ, 1)$, if $S \setminus \{1\}$ has a greatest element then $(S, \wedge, \circ, 1)$ is subdirectly irreducible.

Proof. Let $\Theta \in \text{Con}(S) \setminus \{\omega\}$ and let u be a greatest element of $S \setminus \{1\}$. Then $(a, b) \in \Theta$ for some $a \neq b$; of course, we may assume that $a < b$. If $b = 1$ then clearly $(u, 1) \in \Theta$. If $b \leq u$ then $(a, b) \in \Theta$ yields $(b \circ a, 1) \in \Theta$. But $b \circ a \leq u$ since $b \circ a$ is the pseudocomplement of b in the section $[a]$, and hence $(b \circ a, 1) \in \Theta$ implies $(u, 1) \in \Theta$. Thus $\Theta(u, 1) \subseteq \Theta$ proving that $\Theta(u, 1)$ is the monolith of the congruence lattice $\text{Con}(S)$. \square

Unfortunately, the converse statement fails to be true:

Example 19. Consider the sectionally pseudocomplemented (semi)lattice S as shown in Figure 3; the operation “ \circ ” is given by Table 2. By Theorem 13 it is easy to see that S and $[1]$ are the only weakly standard filters of S and so S is simple by Theorem 15.

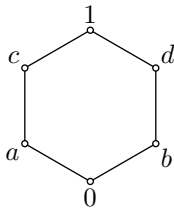


Figure 3

\circ	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	c	1	1
c	d	a	d	1	d	1
d	c	c	b	c	1	1
1	0	a	b	c	d	1

Table 2

Example 20. Another example of a subdirectly irreducible sectionally pseudocomplemented semilattice such that the set of all $x \neq 1$ has no greatest element is that from Example 2. There are two proper weakly standard filters, namely, $[a]$ and $[c]$. Thus the congruence lattice is a four-element chain $\omega \subset \Phi_c \subset \Phi_a \subset \iota$, and consequently, N_5 (as a sectionally pseudocomplemented semilattice) is subdirectly irreducible.

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