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Erhan Çalışkan

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IDEALS OF HOMOGENEOUS POLYNOMIALS AND WEAKLY COMPACT APPROXIMATION PROPERTY IN BANACH SPACES

Erhan Çalışkan, İstanbul

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Abstract. We show that a Banach space E has the weakly compact approximation property if and only if each continuous Banach-valued polynomial on E can be uniformly approximated on compact sets by homogeneous polynomials which are members of the ideal of homogeneous polynomials generated by weakly compact linear operators. An analogous result is established also for the compact approximation property.

Keywords: compact approximation property, weakly compact approximation property, ideals of homogeneous polynomials

MSC 2000: 46G20, 46B28, 46G25, 47B10

1. Introduction

A Banach space E is said to have the *approximation property* (abbreviated AP) if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a finite rank operator $T \in L(E;E)$ such that $\sup\limits_{x\in K} \|Tx - x\| \leqslant \varepsilon$, and a Banach space E is said to have the *compact approximation property* (abbreviated CAP) if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a compact operator $T \in L(E;E)$ such that $\sup\limits_{x\in K} \|Tx - x\| \leqslant \varepsilon$. The AP implies the CAP, but Willis [30] has shown that the reverse implication is not true.

We say that E has the *weakly compact approximation property* (abbreviated WCAP) if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a weakly compact operator $T \in L(E;E)$ such that $\sup\limits_{x\in K} \|Tx - x\| \leqslant \varepsilon$. It is easily seen that every reflexive Banach space has the WCAP. Clearly the CAP implies the WCAP, but the converse of this implication fails since there are separable reflexive Banach spaces without the CAP (see [13]). On the other hand, since every weakly compact operator on a Schur space is a compact operator, the CAP coincides with the WCAP on Schur spaces. This in particular shows the existence of Banach spaces without the WCAP, since there are subspaces of l1 which fail the CAP (see [9, Proposition 2.12]). We note that this notion was considered also by other authors (see [17] and [19]) and investigated in different directions. Also, a similar (but strictly stronger) notion was introduced by Astala and Tylli [3], where the identity operator is approximated on weakly compact sets by weakly compact linear operators with a uniform bound.

The purpose of this paper is to approach the (weakly) compact approximation problem in connection with the ideals of homogeneous polynomials generated by (weakly) compact linear operators. In Section 2 we give some necessary definitions and a brief information on ideals of homogeneous polynomials. Section 3 is devoted to the study of the WCAP in Banach spaces for the linear case.

In [24] Mujica asserts that every Banach-valued homogeneous polynomial on E can be uniformly approximated on compact sets by homogeneous polynomials which are weakly continuous on bounded sets, whenever E has the CAP. In [11] the author shows that the converse of this implication holds also true, that is, a Banach space E has the CAP if and only if each continuous Banach-valued polynomial on E can be uniformly approximated on compact sets by compact polynomials, or equivalently, by polynomials which are weakly continuous on bounded sets. In this paper, in Section 4 we first show that a Banach space E has the WCAP if and only if each continuous Banach-valued homogeneous polynomial on E can be uniformly approximated on compact sets by weakly compact (homogeneous) polynomials. Then using this result we characterize the WCAP in terms of density of the ideal of homogeneous polynomials generated by weakly compact linear operator ideals. In our main result we prove that a Banach space E has respectively the WCAP or CAP if and only if the ideal of m-homogeneous polynomials generated by the ideal of weakly compact or compact linear operators is τ_c-dense in the space of m-homogeneous polynomials if and only if the predual of the space of m-homogeneous polynomials $P(^mE)$, $Q(^mE)$ has WCAP or CAP for every $m \in \mathbb{N}$, or equivalently, for some $m \in \mathbb{N}$. Some of the results of this paper improve and extend earlier results obtained by the author in [11].

Our terminology is rather standard. We refer to [15] or [22] for the properties of polynomials and holomorphic mappings in infinite dimensional spaces, to [21] for the theory of Banach spaces, and to [27] and [28] for details on ideals of operators and ideals of multilinear mappings.

2. Notation and preliminaries

The symbol $\mathbb{C}$ represents the field of all complex numbers, $\mathbb{N}$ represents the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol τ_c will denote the compact-open topology. The letters E and F will always represent complex Banach spaces. The symbol B_E represents the closed unit ball of E. Given a subset M of E, the symbol $T|_M$ will denote the restriction of a mapping $T\colon E \longrightarrow F$ to the subset M, the symbol I_M will denote the identity mapping on M, and $\overline{M}^{\tau_c}$ will denote the τ_c-closure of M in E for the compact-open topology τ_c on E.

Let $L(E;F)$ be the Banach space of all continuous linear operators, and let $(L(E;F),\tau_c)$ denote the vector space $L(E;F)$ endowed with the topology of uniform convergence on all compact subsets of E. When $F = \mathbb{C}$ we write E' and E'_c instead of $L(E;\mathbb{C})$ and $(L(E;F),\tau_c))$, respectively.

An operator T in $L(E;F)$ is said to have a finite rank if $T(E)$ is finite dimensional, and an operator T in $L(E;F)$ is called a compact or weakly compact operator if T takes bounded subsets of E respectively to relatively compact or weakly compact subsets of F. Let $L_{\mathrm{k}}(E;F)$ and $L_{\mathrm{wk}}(E;F))$ denote respectively the subspace of all compact or weakly compact operators of $L(E;F)$.

For each integer $n \in \mathbb{N}$, let $L(E_1,\ldots,E_n;F)$ be the Banach space of all continuous n-linear mappings $A\colon E_1 \times \ldots \times E_n \longmapsto F$ endowed with the sup norm $\|A\| = \sup\{\|A(x_1,\ldots,x_n)\|\colon \|x_i\| \leqslant 1, i = 1,\ldots,n\}$. When $E_1 = \ldots = E_n = E$ we write $L(^nE;F)$ instead of $L(\underbrace{E,\ldots,E}_{n\text{ times}};F)$.

Given a continuous n-linear mapping $A \in L(^nE;F)$, the map $P\colon E \longrightarrow F$, defined by $P(x) = A(\underbrace{x,\ldots,x}_{n\text{ times}})$ for every $x \in E$, is said to be a continuous n-homogeneous polynomial. $\mathscr{P}(^nE;F)$ will denote the vector space of all continuous n-homogeneous polynomials from E into F, which is a Banach space under the norm $\|P\| = \sup\{\|P(x)\|\colon \|x\| \leqslant 1\}$. When $F = \mathbb{C}$ we will write $\mathscr{P}(^mE)$ instead of $\mathscr{P}(^mE;\mathbb{C})$.

If A is a multilinear mapping and P is the polynomial generated by A, we write $P = \widehat{A}$. Conversely, given a polynomial $P \in \mathscr{P}(^nE;F)$, there is a unique symmetric continuous n-linear mapping $\overset{\vee}{P} \in L(^nE;F)$ such that $P(x) = \overset{\vee}{P}(\underbrace{x,\ldots,x}_{n\text{ times}})$. It is well known that the correspondence $A \longleftrightarrow \widehat{A}$ is a topological isomorphism between $L^s(^nE;F)$, the space of all symmetric continuous n-linear mappings from E to F, and $\mathscr{P}(^nE;F)$ (see, for example, [22, Theorem 2.2]).

A mapping $P\colon E \longrightarrow F$ is said to be a continuous polynomial if it can be represented as a sum $P = P^0 + P^1 + \ldots + P^n$, with $P^j \in \mathscr{P}(^jE;F)$ for $j = 0,1,\ldots,n$.

By $\mathscr{P}(E;F)$ we will denote the vector space of all continuous polynomials from E into F.

Let us recall that a polynomial $P \in \mathscr{P}(^mE;F)$ is said to be of finite type if it can be represented as a sum $P(x) = \sum_{j=1}^{p} \varphi_j^m(x) y_j$ with $\varphi_j \in E'$ and $y_j \in F$. We say that a polynomial $P \in \mathscr{P}(E;F)$ is compact or weakly compact if P takes bounded subsets of E respectively to relatively or relatively weakly compact subsets of F. Let $\mathscr{P}_{\mathrm{k}}(E;F)$ and $\mathscr{P}_{\mathrm{wk}}(E;F)$ denote respectively the subspace of all compact and weakly compact members of $\mathscr{P}(E;F)$, and let $\mathscr{P}_{\mathrm{k}}(^mE;F)$ and $\mathscr{P}_{\mathrm{wk}}(^mE;F)$ denote respectively the subspace of all compact and weakly compact members of $\mathscr{P}(^mE;F)$, for every $m \in \mathbb{N}$.

Operator ideals and ideals of homogeneous polynomials generated by operator ideals

For each $i = 1, \ldots, n$, let $\Psi_i^{(n)}\colon L(E_1, \ldots, E_n; F) \longrightarrow L(E_i; L(E_1, \overset{[i]}{\ldots}, E_n; F))$ denote the canonical isometric isomorphism defined by

$$\Psi_i^{(n)}(T)(x_i)(x_1, \overset{[i]}{\ldots}, x_n) = T(x_1, \ldots, x_n), \quad T \in L(E_1, \ldots, E_n; F)$$

where the notation $\overset{[i]}{\ldots}$ means the ith coordinate is not involved. When $E_1 = \ldots = E_n = E$, we will write $\Psi^{(n)}$ instead of $\Psi_i^{(n)}$.

An operator ideal $\mathscr{I}$ is a subclass of the class of all continuous linear operators between Banach spaces such that for all Banach spaces E and F, its components $\mathscr{I}(E;F) = L(E;F) \cap \mathscr{I}$ satisfy:

(a) $\mathscr{I}(E;F)$ is a linear subspace of $L(E;F)$ which contains finite rank operators.

(b) If $T \in L(E;F)$, $R \in \mathscr{I}(F;G)$ and $S \in L(G;H)$, then the composition $S \circ R \circ T$ is in $\mathscr{I}(E;H)$.

Some of the well-known examples of ideals of linear operators are the family of finite rank operators, compact operators, weakly compact operators, approximable operators, absolutely summing operators, nuclear operators and integral operators.

Similarly to the linear case, an ideal of homogeneous polynomials $\mathfrak{P}$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and Banach spaces E and F, the components $\mathfrak{P}(^nE;F) = \mathscr{P}(^nE;F) \cap \mathfrak{P}$ satisfy:

(a) $\mathfrak{P}(^nE;F)$ is a linear subspace of $\mathscr{P}(^nE;F)$ which contains n-homogeneous polynomials of finite type.

(b) If $T \in L(G;E)$, $P \in \mathfrak{P}(^nE;F)$ and $S \in L(F;H)$ then the composition $S \circ P \circ T$ is in $\mathfrak{P}(^nG;H)$.

The ideal of all compact or weakly compact homogeneous polynomials are important examples.

There are different ways how to construct an ideal of multilinear mappings and an ideal of polynomials from a given operator ideal $\mathscr{I}$. Two methods, called factorization and linearization, outlined by Pietsch [28] are the following:

Definition 1. Let $\mathscr{I}$ be an operator ideal.

I. Factorization method: An n-linear mapping $T \in L(E_1, \ldots, E_n; F)$ is said to be of type $L[\mathscr{I}]$ ($T \in L[\mathscr{I}](E_1, \ldots, E_n; F)$) if there are Banach spaces $G_1, \ldots, G_n$, linear operators $R_j \in \mathscr{I}(E_j; G_j)$, $j = 1, \ldots, n$, and a continuous n-linear mapping $S \in L(G_1, \ldots, G_n; F)$ such that $T = S \circ (R_1, \ldots, R_n)$. A continuous n-homogeneous polynomial $P \in \mathscr{P}(^nE; F)$ is said to be of type $L[\mathscr{I}]$ ($P \in P_{L[\mathscr{I}]}(^nE; F)$) if there are a Banach space G, a linear operator $R \in \mathscr{I}(E; G)$ and a polynomial $Q \in \mathscr{P}(^nG; F)$ such that $P = Q \circ R$.

II. Linearization method: An n-linear mapping $T \in L(E_1, \ldots, E_n; F)$ is said to be of type $[\mathscr{I}]$ ($T \in [\mathscr{I}](E_1, \ldots, E_n; F)$) if $\Psi_j^{(n)}(T) \in \mathscr{I}(E_j; L(E_1, \overset{[j]}{\ldots}, E_n; F))$ for every $j = 1, \ldots, n$. A continuous n-homogeneous polynomial $P \in \mathscr{P}(^nE; F)$ is said to be of type $[\mathscr{I}]$ ($P \in P_{[\mathscr{I}]}(^nE; F)$) if $\overset{\vee}{P}$ is of type $[\mathscr{I}]$.

The classes $L[\mathscr{I}]$ and $[\mathscr{I}]$ are ideals of multilinear mappings and the classes $\mathscr{P}_{L[\mathscr{I}]}$ and $\mathscr{P}_{[\mathscr{I}]}$ are ideals of homogeneous polynomials (see Botelho [5, Proposition 4.6]). If $\mathscr{I}$ is a closed injective operator ideal (see [8, Definition 1.2]), then $L[\mathscr{I}] = [\mathscr{I}]$ (see [8, Theorem 3.4]), and $\mathscr{P}_{L[\mathscr{I}]} = \mathscr{P}_{[\mathscr{I}]}$ (see [5, Proposition 4.6]).

In this paper we shall study the ideal of homogeneous polynomials generated by compact linear operators and by weakly compact linear operators. Let $\mathscr{K}$ and $\mathscr{W}$ denote respectively the ideal of compact and weakly compact linear operators between Banach spaces. It is well known that $\mathscr{K}$ and $\mathscr{W}$ are closed injective operator ideals (see, for example, [18, Theorem 2.3]). Hence we have that $P_{L[\mathscr{K}]} = P_{[\mathscr{K}]}$ and $P_{L[\mathscr{W}]} = P_{[\mathscr{W}]}$. In the linear case the equality $L[\mathscr{K}] = [\mathscr{K}]$ was proved in [2] and the equality $L[\mathscr{W}] = [\mathscr{W}]$ was proved in [1]. It also follows from [16, Corollary 3.3] that we always have $L[\mathscr{K}](E; F) = \mathscr{K}(E; F) = L_{\mathrm{k}}(E; F)$, and from [14, Corollary 1] that we have $L[\mathscr{W}](E; F) = \mathscr{W}(E; F) = L_{\mathrm{wk}}(E; F)$ for any Banach spaces E and F.

On the other hand, it is clear that $\mathscr{P}_{L[\mathscr{K}]} \subset \mathscr{P}_{L[\mathscr{W}]}$. But it follows from the following simple example, shown to us by Geraldo Botelho, that these ideals do not coincide.

Example 2. Let E be an infinite dimensional reflexive Banach space. Let $\mathscr{P}_{\mathscr{K}}$ denote the ideal of compact homogeneous polynomials. Since $\mathscr{P}_{L[\mathscr{K}]} \subset \mathscr{P}_{\mathscr{K}}$, a noncompact polynomial does not belong to $\mathscr{P}_{L[\mathscr{K}]}$. Given $n \in \mathbb{N}$, fix $\varphi \in E'$, $\varphi \neq 0$, and consider

$$Q_n\colon E \longrightarrow E; \quad Q_n(x) = \varphi(x)^{n-1} \cdot x; \quad Q_n \in \mathscr{P}(^nE; E).$$

Since E is reflexive, $Q_n \in \mathscr{P}_{L[\mathscr{W}]}({}^nE;E)$ (see [5, Proposition 5.5 (a)]). But Q_n is not compact (see [4, Proposition (b)]), so $Q_n \notin \mathscr{P}_{L[\mathscr{K}]}({}^nE;E)$.

We note that $\mathscr{P}_{\mathscr{K}}$ is not contained in $\mathscr{P}_{L[\mathscr{K}]}$. Also, $\mathscr{P}_{\mathscr{W}}$, the ideal of weakly compact homogeneous polynomials, is not contained in $\mathscr{P}_{L[\mathscr{W}]}$ and, contrary to the compact case, $\mathscr{P}_{L[\mathscr{W}]}$ is not contained in $\mathscr{P}_{\mathscr{W}}$ (see [5, pp. 16–17]).

From now on we will use the notation $\mathscr{P}_{[\mathscr{K}]}$ or $\mathscr{P}_{[\mathscr{W}]}$ to represent respectively both $\mathscr{P}_{[\mathscr{K}]}$ and $\mathscr{P}_{L[\mathscr{K}]}$ or $\mathscr{P}_{[\mathscr{W}]}$ and $\mathscr{P}_{L[\mathscr{W}]}$, and in the linear case the symbol $\mathscr{W}(E;F)$ will be used in place of $[\mathscr{W}](E;F)$ (or $L[\mathscr{W}](E;F)$) because of the equalities given above.

3. The weakly compact approximation property

In this section we establish basic properties of the WCAP. We begin with the following characterization of the WCAP which is parallel to a result given in [11, Proposition 1] (for a related result see also [19, Theorem 4.1 and Corollary 4.2]).

Proposition 3. *For a Banach space E the following statements are equivalent.*

(a) *E has the WCAP.*

(b) *$L(E;E) = \overline{\mathscr{W}(E;E)}^{\tau_c}$.*

(c) *For every Banach space F, $L(F;E) = \overline{\mathscr{W}(F;E)}^{\tau_c}$.*

(d) *For every Banach space F, $L(E;F) = \overline{\mathscr{W}(E;F)}^{\tau_c}$.*

(e) *For every choice of $(x_n)_{n=1}^{\infty} \subset E$ and $(x'_n)_{n=1}^{\infty} \subset E'$ such that $\sum\limits_{n=1}^{\infty} \|x_n\|\|x'_n\| < \infty$ and satisfying $\sum\limits_{n=1}^{\infty} x'_n(Tx_n) = 0$ for every $T \in \mathscr{W}(E;E)$, we have $\sum\limits_{n=1}^{\infty} x'_n(x_n) = 0$.*

Proposition 3 can be proved easily using a traditional argument. It is also easily checked that every complemented subspace of a Banach space E has the WCAP whenever E has the WCAP.

In the next definition, which is a common generalization of the notions of a separable and a reflexive Banach space, a nice classification of Banach spaces is given.

Definition 4. A Banach space E is said to be generated by a weakly compact set (WCG, in short) if there is a weakly compact subset K of E such that $E = \overline{\left(\bigcup\limits_{n=1}^{\infty} nK\right)}$.

For examples of WCG Banach spaces other than reflexive and separable Banach spaces, and properties they enjoy, we refer to [20]. Using Proposition 3 and [20, Theorem 2.1] we get the following characterization of the WCAP for WCG Banach spaces.

Proposition 5. *For a WCG Banach space E, the following statements are equivalent:*

(a) *E has the WCAP.*

(b) *Every complemented subspace of E has the WCAP.*

(c) *Every complemented and separable subspace of E has the WCAP.*

Parallelling to the weakly compact case, using [11, Proposition 1] and [20, Theorem 2.1] we obtain the following characterization of the CAP for WCG Banach spaces, which is stated in [11] for reflexive Banach spaces.

Proposition 6 ([10])**.** *For a WCG Banach space E, the following statements are equivalent:*

(a) *E has the CAP.*

(b) *Every complemented subspace of E has the CAP.*

(c) *Every complemented and separable subspace of E has the CAP.*

Proposition 6 can be seen as a generalization of [11, Proposition 3].

The motivation for the next proposition, which appeared in [10], is a question whether or not E has the CAP whenever its dual E' has this property, which we include here since we did not find it in literature.

Proposition 7 ([10])**.** *Let E be a Banach space. If E' has the CAP then for every reflexive Banach space F we have $L(E;F) = \overline{\mathscr{K}(E;F)}^{\tau_c}$.*

P r o o f. Let F be a reflexive Banach space and let $T \in L(E;F)$. Let $\varphi \in (L(E;F),\tau_c)'$ be such that $\varphi(S) = 0$ for every $S \in \mathscr{K}(E;F)$. Hence by [21, Proposition 1.e.3] there are sequences $(x_i)_{i=1}^{\infty} \subset E$ and $(y_i')_{i=1}^{\infty} \subset F'$ such that $\sum\limits_{i=1}^{\infty} \|x_i\| \cdot \|y_i'\| < \infty$ and $\sum\limits_{i=1}^{\infty} y_i'(Sx_i) = \varphi(S) = 0$ for every $S \in \mathscr{K}(E;F)$. Since F is reflexive, given $R \in \mathscr{K}(F';E')$, by the theorem of Schauder, there always exists a unique $S \in \mathscr{K}(E;F)$ such that $R = S'$. Thus it follows that $\sum\limits_{i=1}^{\infty} (Ry_i')(x_i) = 0$ for every $R \in \mathscr{K}(F';E')$, and hence $\sum\limits_{i=1}^{\infty} (Ry_i')(x_i) = 0$ for every $R \in L(F';E')$, which shows that $\varphi(T) = \sum\limits_{i=1}^{\infty} (T'y_i')(x_i) = 0$. Now, it follows from the locally convex space version of the Hahn-Banach Theorem that T belongs to $\overline{\mathscr{K}(E;F)}^{\tau_c}$. $\square$

From the preceding proposition, in particular, we conclude that a reflexive Banach space E has the CAP if and only if E' has the CAP, a result that was already stated in [11]. Hence, Proposition 7 sharpens [11, Corollary 2]. However, contrary to the case of the AP, in general we do not know if the CAP passes from the dual space down to the space (see Casazza [9, Problem 8.5]). We also note that, as in the case of the AP, there are (non-reflexive) Banach spaces with the CAP whose dual space fails to have the CAP (see [9, pp. 17–18]). As we have mentioned in the introduction every reflexive Banach space has the WCAP (actually, in this case there is nothing to do due to the fact that a Banach space E is reflexive if and only if every bounded subset of E is weakly compact), and for this reason, the WCAP does not imply the CAP since there are closed subspaces of sequence spaces l_p, $2 \leqslant p < \infty$, without the CAP (see [13]). We end this section with the following natural question.

Question. Let E be a non-reflexive Banach space. If E' has the WCAP must E have the WCAP?

4. The weakly compact approximation property and ideals of homogeneous polynomials generated by weakly compact linear operators

To prove our results for the WCAP and CAP in connection with the ideals of weakly compact homogeneous polynomials and of compact homogeneous polynomials, respectively, we need some preparation.

An important tool of this work is the following version of a theorem of Ryan [29], which appeared in [24] in the following form (see also [23, Theorems 2.4 and 4.1, and Proposition 3.4]).

Theorem 8 ([29]). *Let E be a Banach space and let $m \in \mathbb{N}$. Then there are a Banach space $Q(^mE)$ and a polynomial $\delta_m \in \mathscr{P}(^mE; Q(^mE))$ with the following universal property: For each Banach space F and each polynomial $P \in \mathscr{P}(^mE; F)$, there is a unique operator $T_P \in L(Q(^mE); F)$ such that $T_P \circ \delta_m = P$. The correspondence*

$$P \in \mathscr{P}(^mE; F) \longrightarrow T_P \in L(Q(^mE); F)$$

is an isometric isomorphism. These properties characterize $Q(^mE)$ uniquely up to an isometric isomorphism. This correspondence is also a topological isomorphism when both spaces are endowed with the compact-open topology τ_c. Moreover, $P \in \mathscr{P}_{\mathrm{k}}(^mE; F)$ or $P \in \mathscr{P}_{\mathrm{wk}}(^mE; F))$ if and only if $T_P \in L_{\mathrm{k}}(Q(^mE); F)$ or $T_P \in L_{\mathrm{wk}}(Q(^mE); F))$, respectively.

The space $Q(^mE)$ given in the above theorem is defined as the closed subspace of all linear continuous functionals $v \in \mathscr{P}(^mE)'$ such that $v|_{B_{\mathscr{P}(^mE)}}$ is τ_c-continuous, and the evaluation mapping $\delta_m\colon\ x \in E \longrightarrow \delta_x \in Q(^mE)$ is defined by $\delta_x(P) = P(x)$, for every $x \in E$ and $P \in \mathscr{P}(^mE)$. The space $Q(^mE)$ is a Banach space with the norm induced by $\mathscr{P}(^mE)'$, which is called the predual of $\mathscr{P}(^mE)$, $m \in \mathbb{N}$.

Using an idea given in [6] we obtain the following useful lemma.

Lemma 9. *Let E and F be Banach spaces.*

(a) *If $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$ for some $m \in \mathbb{N}$ then $L(E;F) = \overline{\mathscr{W}(E;F)}^{\tau_c}$.*

(b) *If $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{\mathscr{K}}(^mE;F)}^{\tau_c}$ for some $m \in \mathbb{N}$ then $L(E;F) = \overline{\mathscr{K}(E;F)}^{\tau_c}$.*

Proof. We will only prove (a) since the same proof works for (b). Since the case $m = 1$ is our hypothesis let $m > 1$ and suppose that $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$. Let $T \in L(E;F)$, let K be a compact subset of E and let $\varepsilon > 0$. Let $a \in K$ with $a \neq 0$ and choose $\varphi \in E'$, $\varphi \neq 0$, such that $\varphi(a) = 1$. Consider the set $K_1 := \varepsilon_1 K + \varepsilon_2 K + \ldots + \varepsilon_m K$ with $\varepsilon_i = \mp 1$, $i = 1, \ldots, m$. Since K_1 is compact in E, the set $K_1' := \bigcup_{\substack{\varepsilon_i = \mp 1 \\ i=1,\ldots,m}} (\varepsilon_1 K + \ldots + \varepsilon_m K)$ is also compact in E. Since $T \circ \varphi^{m-1} \in \mathscr{P}(^mE;F)$, where $T \circ \varphi^{m-1}(x) := T(x)\varphi^{m-1}(x)$ for every $x \in E$, by hypothesis there is a $P^m \in \mathscr{P}_{\mathscr{W}}(^mE;F)$ such that

$$(*) \qquad \|P^m(x) - T \circ \varphi^{m-1}(x)\| < \frac{m!}{m}\varepsilon \quad \text{for every } x \in K_1'.$$

Thus, for every $(x_1, \ldots, x_m) \in \underbrace{K \times \ldots \times K}_{m \text{ times}}$, we have

$$\begin{aligned} &\|\overset{\vee}{P^m}(x_1, \ldots, x_m) - (T \overset{\vee}{\circ} \varphi^{m-1})(x_1, \ldots, x_m)\| \\ &\qquad = \Big\| \frac{1}{m!2^m} \sum_{\varepsilon_i = \mp 1} \varepsilon_1 \ldots \varepsilon_m \Big[P^m\Big(\sum_{i=1}^m \varepsilon_i x_i\Big) - T \circ \varphi^{m-1}\Big(\sum_{i=1}^m \varepsilon_i x_i\Big)\Big]\Big\| \\ &\qquad < \frac{1}{m!2^m} \sum_{\varepsilon_i = \mp 1} \frac{m!}{m}\varepsilon = \frac{\varepsilon}{m}. \end{aligned}$$

Then, in particular, we obtain

$$\|\overset{\vee}{P^m}(x, a, \ldots, a) - (T \circ \overset{\vee}{\varphi^{m-1}})(x, a \ldots, a)\| < \frac{\varepsilon}{m} \quad \text{for every } x \in K.$$

Hence using that $\varphi(a) = 1$ we get

$$\begin{aligned} &\Big\|\overset{\vee}{P^m}(x, a \ldots, a) - \Big(\frac{1}{m}T(x) + \frac{m-1}{m}\varphi(x)T(a)\Big)\Big\| \\ &= \Big\|\overset{\vee}{P^m}(x, a \ldots, a) - \Big(\frac{m-1}{m}\Big)\varphi(x)T(a) - \frac{1}{m}T(x)\Big\| < \frac{\varepsilon}{m} \quad \text{for every } x \in K, \end{aligned}$$

or equivalently,

$$\|m\overset{\vee}{P}^m(x,a\ldots,a)-(m-1)T(a)\varphi(x)-T(x)\|<\varepsilon \quad \text{for every } x\in K.$$

Therefore, if we define a linear operator $T_{\rm wk}$ by

$$T_{\rm wk}(x):=m\overset{\vee}{P}^m(x,a\ldots,a)-(m-1)T(a)\varphi(x) \quad \text{for every } x\in E,$$

since $T_{\rm wk}\in\mathscr{W}(E;F)$, the last inequality completes the proof. $\square$

Remark. The proof of Lemma 9, in fact, gives us more: If $\mathscr{P}(^mE;F)=\overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$ for some $m\in\mathbb{N}$ then, by using the same argument one can prove that $\mathscr{P}(^nE;F)=\overline{\mathscr{P}_{\mathscr{W}}(^nE;F)}^{\tau_c}$ for every $n\leqslant m$. The same is true for the case $\mathscr{P}_{\mathscr{K}}$.

The next result gives a polynomial characterization of the WCAP.

Proposition 10. *For a Banach space E, the following statements are equivalent:*

(a) *E has the WCAP.*

(b) *$\mathscr{P}(E;F)=\overline{\mathscr{P}_{\rm wk}(E;F)}^{\tau_c}$ for every Banach space F.*

(c) *$\mathscr{P}(^mE;F)=\overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$ for every Banach space F and for every $m\in\mathbb{N}$.*

(d) *$\mathscr{P}(^mE;F)=\overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$ for every Banach space F and for some $m\in\mathbb{N}$.*

(e) *$Q(^mE)$ has the WCAP for every $m\in\mathbb{N}$.*

(f) *$Q(^mE)$ has the WCAP for some $m\in\mathbb{N}$.*

Proof. (a) $\Longrightarrow$ (b): Let $P\in\mathscr{P}(E;F)$, let K be a compact subset of E and let $\varepsilon>0$. Then there is a $\delta>0$ such that $\|P(x)-P(y)\|\leqslant\varepsilon$ whenever $x\in K$ and $\|y-x\|<\delta$. Since by Proposition 3 there is an operator $T\in L_{\rm wk}(E;E)$ with $\sup\limits_{x\in K}\|Tx-x\|<\delta$, we have that $\|P(Tx)-P(x)\|\leqslant\varepsilon$ for every $x\in K$, which shows (b) since $P\circ T\in\mathscr{P}_{\rm wk}(E;F)$.

(b) $\Longrightarrow$ (c): Let $m\in\mathbb{N}$, let $P\in\mathscr{P}(^mE;F)$, let K be a compact subset of E and let $\varepsilon>0$. By (b) there exists a polynomial $P_{\rm wk}\in\mathscr{P}_{\rm wk}(E;F)$ such that $\|P(x)-P_{\rm wk}(x)\|<\varepsilon$ for every $x\in K$. One can show that it is always possible to write $P_{\rm wk}=P^0_{\rm wk}+P^1_{\rm wk}+\ldots+P^n_{\rm wk}$, with $n\geqslant m$, where $P^j_{\rm wk}\in\mathscr{P}_{\mathscr{W}}(^jE;F)$ for each $j=0,1,\ldots,n$ (see, for example, the proof of [22, Proposition 2.9]). Now, by the Cauchy integral formula (see [22, Corollary 7.5]), for every $x\in K$ we obtain that

$$\|P(x)-P^m_{\rm wk}(x)\|=\left\|\frac{1}{2\pi{\rm i}}\int_{|\xi|=1}\frac{P(\xi x)-P_{\rm wk}(\xi x)}{\xi^{m+1}}\,{\rm d}\xi\right\|\leqslant\varepsilon,$$

which proves (c).

(c) $\Longrightarrow$ (e): If (c) holds then, by Theorem 8 we have that $L(Q(^mE);F) = \overline{\mathscr{W}(Q(^mE);F)}^{\tau_c}$ for every Banach space F and every $m \in \mathbb{N}$, which shows that $Q(^mE)$ has the WCAP for every $m \in \mathbb{N}$.

(f) $\Longrightarrow$ (d): By (f) we have that $L(Q(^mE);F) = \overline{\mathscr{W}(Q(^mE);F)}^{\tau_c}$ for every Banach space F. Hence it follows from Theorem 8 that $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{\mathscr{W}}(^mE;F)}^{\tau_c}$ for every Banach space F.

Since the implication (d) $\Longrightarrow$ (a) follows immediately from Lemma 9 and Proposition 3 and the implications (c) $\Longrightarrow$ (d) and (e) $\Longrightarrow$ (f) are obvious, the proof is complete. $\square$

If E is a Banach space (necessarily non-reflexive) without the WCAP then, by Proposition 10, $Q(^mE)$ fails to have the WCAP for every $m \in \mathbb{N}$. We remark that there are reflexive Banach spaces E for which $Q(^mE)$, $m > 1$, is not reflexive (for the results in this direction see [7]). In this case we conclude from Proposition 10 that the space $Q(^mE)$ (not necessarily reflexive) has always the WCAP whenever E is reflexive. Moreover, if $Q(^mE)$ is reflexive then, by Theorem 8, the space $\mathscr{P}(^mE)$ also has the WCAP for every $m \in \mathbb{N}$.

Now we give the characterization of the WCAP in connection with the ideal of homogeneous polynomials generated by weakly compact linear operators.

Theorem 11. *For a Banach space E, the following statements are equivalent:*

(a) *E has the WCAP.*

(b) $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{[\mathscr{W}]}(^mE;F)}^{\tau_c}$ *for every Banach space F and for every $m \in \mathbb{N}$.*

(c) $\mathscr{P}(^mE;F) = \overline{\mathscr{P}_{[\mathscr{W}]}(^mE;F)}^{\tau_c}$ *for every Banach space F and for some $m \in \mathbb{N}$.*

Proof. (a) $\Longrightarrow$ (b) : Let $m \in \mathbb{N}$, let $P \in \mathscr{P}(^mE;F)$, let K be a compact subset of E and let $\varepsilon > 0$. By Proposition 10 there is a polynomial $P^m \in \mathscr{P}_{\mathscr{W}}(^mE;F)$ such that

$$\|P(x) - P^m(x)\| \leqslant \frac{\varepsilon}{2} \quad \text{for every } x \in K.$$

By continuity there is a $\delta > 0$ such that $\|P^m(x) - P^m(y)\| \leqslant \frac{1}{2}\varepsilon$ whenever $x \in K$ and $\|x - y\| < \delta$. Thus, since by Proposition 3 there is an operator $T \in \mathscr{W}(E;F)$ with $\sup\limits_{x \in K} \|Tx - x\| < \delta$, we have

$$\|P^m(x) - P^m(Tx)\| \leqslant \frac{\varepsilon}{2} \quad \text{for every } x \in K.$$

If we let $Q := P^m \circ T$ then clearly $Q \in \mathscr{P}_{[\mathscr{W}]}(^mE;F)$, and it follows that

$$\|P(x) - Q(x)\| \leqslant \|P(x) - P^m(x)\| + \|P^m(x) - P^m(Tx)\| \leqslant \varepsilon \quad \text{for every } x \in K,$$

proving (b).

(c) $\Longrightarrow$ (a): Let $m \in \mathbb{N}$. We will show that $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{\mathscr{W}}(^m E; F)}^{\tau_c}$ for every Banach space F. Let $P \in \mathscr{P}(^m E; F)$, let K be a compact subset of E and let $\varepsilon > 0$. By (c) there is a polynomial $P^m \in \mathscr{P}_{[\mathscr{W}]}(^m E; F)$ such that $\|P(x) - P^m(x)\| \leqslant \varepsilon$ for every $x \in K$. Since $P^m \in \mathscr{P}_{[\mathscr{W}]}(^m E; F) = \mathscr{P}_{L[\mathscr{W}]}(^m E; F)$, there are a Banach space G, an operator $R \in \mathscr{W}(E; G)$ and a polynomial $Q \in \mathscr{P}(^m G; F)$ such that $P^m = Q \circ R$, showing that $P^m \in \mathscr{P}_{\mathscr{W}}(^m E; F)$ as we desire. Now, by Lemma 9 we have that $L(E; F) = \overline{\mathscr{W}(E; F)}^{\tau_c}$ for every Banach space F.

Since the implication (b) $\Longrightarrow$ (c) is obvious we have the proof. □

The next proposition was proved in [11] as a consequence of [11, Theorem 5]. But using the proof of Proposition 10 with [11, Proposition 1] in the case of the CAP we obtain an alternative direct proof of [11, Corollary 7], which complements this result as follows:

Proposition 12 ([11]). *For a Banach space E, the following statements are equivalent:*

(a) *E has the CAP.*

(b) $\mathscr{P}(E; F) = \overline{\mathscr{P}_{\mathrm{k}}(E; F)}^{\tau_c}$ *for every Banach space F.*

(c) $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{\mathscr{K}}(^m E; F)}^{\tau_c}$ *for every Banach space F and for every $m \in \mathbb{N}$.*

(d) $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{\mathscr{K}}(^m E; F)}^{\tau_c}$ *for every Banach space F and for some $m \in \mathbb{N}$.*

(e) *$Q(^m E)$ has the CAP for every $m \in \mathbb{N}$.*

(f) *$Q(^m E)$ has the CAP for some $m \in \mathbb{N}$.*

Considering Proposition 10 and Proposition 12 it is interesting to observe that if $Q(^m E)$ has the WCAP or CAP for some $m \in \mathbb{N}$, then actually it has the WCAP or CAP, respectively, for every $m \in \mathbb{N}$. Hence, Proposition 13 improves a result due to the author [11, Corollary 7].

In the proof of Theorem 11 using [11, Proposition 1] we extend Proposition 12 to the ideal case as follows:

Proposition 13. *For a Banach space E, the following statements are equivalent:*

(a) *E has the CAP.*

(b) $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{[\mathscr{K}]}(^m E; F)}^{\tau_c}$ *for every Banach space F and for every $m \in \mathbb{N}$.*

(c) $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{[\mathscr{K}]}(^m E; F)}^{\tau_c}$ *for every Banach space F and for some $m \in \mathbb{N}$.*

Remark. We can give another proof of the implication (c) $\Longrightarrow$ (a) of Proposition 13 in the following way in which it is not necessary to use Lemma 9: After showing that $\mathscr{P}(^m E; F) = \overline{\mathscr{P}_{\mathrm{k}}(^m E; F)}^{\tau_c}$ for every Banach space F (by using the proof of (c) $\Longrightarrow$ (a) of Theorem 11), from the proof of (h) $\Longrightarrow$ (c) of [12, Proposition 6.6] we conclude that $(\mathscr{P}(^m E), \tau_c)$ has the CAP. Since for every $n \leqslant m$, $(\mathscr{P}(^n E), \tau_c)$ is a complemented subspace of $(\mathscr{P}(^m E), \tau_c)$ (see [15, Exercise 1.78]), $(\mathscr{P}(^n E), \tau_c)$

has the CAP for every $n \leqslant m$. In particular, $(\mathscr{P}(^1E), \tau_c) = E'_c$ has the CAP. Since $E = (E'_c)'_c$ then by [12, Corollary 4.6] E has the CAP. $\square$

If $\mathscr{P}_w(^mE; F)$ denotes the vector space of all m-homogeneous polynomials from E into F which are weakly continuous on bounded subsets of E, it follows from [2, Theorem 2.9] that $\mathscr{P}_{[\mathscr{K}]}(^mE; F) = \mathscr{P}_w(^mE; F)$ for all $m \in \mathbb{N}$, E and F. Therefore we have that $\mathscr{P}_{[\mathscr{K}]} = \mathscr{P}_w$. Hence, if we replace $\mathscr{P}_{[\mathscr{K}]}$ by $\mathscr{P}_w$ in Proposition 13 this result remains true also for this class of m-homogeneous polynomials, which is a polynomial ideal.

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Author's address: Erhan Çalişkan, Yıldız Technical University, Faculty of Sciences and Arts, Department of Mathematics, Davutpaşa, 34210 Esenler, İstanbul, Turkey, e-mail: caliskan@yildiz.edu.tr.