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Marek Cezary Zdun; Wei Nian Zhang

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A GENERAL CLASS OF ITERATIVE EQUATIONS ON THE UNIT CIRCLE

Marek C. Zdun, Krakow, and Weinian Zhang, Sichuan

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Abstract. A class of functional equations with nonlinear iterates is discussed on the unit circle $\mathbb{T}^1$. By lifting maps on $\mathbb{T}^1$ and maps on the torus $\mathbb{T}^n$ to Euclidean spaces and extending their restrictions to a compact interval or cube, we prove existence, uniqueness and stability for their continuous solutions.

Keywords: iterative equation, circle, lift, orientation-preserving, continuation

MSC 2000: 39B22, 37E05

1. Introduction

Let X be a topological space and let us consider a map $f\colon X \to X$. The j-th iterate f^j of f is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = \mathrm{id}$, the identity map. Founded on the problem of iterative roots, the problem of invariant curves and some problems from dynamical systems (e.g. in [2], [8]), the iterative equation

$$\Phi(f(x), f^2(x), \ldots, f^n(x)) = F(x), \quad x \in X, \tag{$*$}$$

where F and Φ are given functions and f is unknown, was investigated actively ([2], [21]). When Φ is linear, i.e., $\Phi(y_1, \ldots, y_n) = \sum\limits_{j=1}^{n} \lambda_j y_j$, this equation assumes the form

$$\sum_{j=1}^{n} \lambda_j f^j(x) = F(x) \tag{$**$}$$

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and was discussed on $X=\mathbb{R}$. For linear F some results can be found e.g. in [6], [12], [13], [17] and [19]. For nonlinear F results are given mainly in a compact interval (see e.g. in [22], [23], [24]). Generalizations to $\mathbb{R}^N$ are given in [9] and [25]. The case of nonlinear Φ is considered in [11] and [15].

It is also interesting to study iteration on the unit circle $X=\mathbb{T}^1$ (or denoted by $\mathbb{S}^1$), i.e., the set $\{z\in\mathbb{C}\colon\ z=\mathrm{e}^{2\pi\mathrm{i}t}, t\in\mathbb{R}\}$. Many results have been given for iterative roots and iteration groups on $\mathbb{T}^1$, seen for example in [3], [7], [10], [16], [20], [26] and some references therein. In those works maps on $\mathbb{T}^1$ can be lifted to the whole real line $\mathbb{R}$ so that considered problems are reduced to problems of iteration on $\mathbb{R}$ even in some complicated cases, for example, where rotation numbers of considered maps are irrational. In contrast, because of the more complicated form of $(*)$, few published results are found for the more general form $(*)$ of iterative equations on $\mathbb{T}^1$.

In this paper we discuss solutions of the equation $(*)$ on $X=\mathbb{T}^1$, i.e., the equation

$$\Phi(f(z),f^2(z),\dots,f^n(z))=F(z),\quad z\in\mathbb{T}^1, \tag{1.1}$$

in the class of homeomorphisms

$$H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)=\{f\in C^0(\mathbb{T}^1,\mathbb{T}^1)\colon\ f(\mathbb{T}^1)=\mathbb{T}^1 \text{ homeomorphically and } f(\mathbf{1})=\mathbf{1}\},$$

where $C^0(\mathbb{T}^1,\mathbb{T}^1)$ consists of all continuous maps from $\mathbb{T}^1$ into itself and the notation $\mathbf{1}$ indicates the point $(1,0)$ in the complex plane $\mathbb{C}$ so as to distinguish it from $1\in\mathbb{R}$. We will lift F,f from the circle $\mathbb{T}^1$ to $\mathbb{R}$ and Φ from the n-dimensional torus $\mathbb{T}^n$ to $\mathbb{R}^n$. Moreover, we apply techniques of restricting and extending to those lifts so that the reduced problem can be discussed on the compact interval $I:=[0,1]$. We will prove existence, uniqueness and stability for solutions of equation (1.1) in the class $H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$.

2. Maps on $\mathbb{T}^1$ and induced maps

Let $h\colon\ t\in\mathbb{R}\mapsto\mathrm{e}^{2\pi\mathrm{i}t}\in\mathbb{T}^1$ and $h_*:=h|_{[0,1)}$. The map h_* is a continuous bijection. If $v,w,z\in\mathbb{T}^1$, then there exist unique $t_1,t_2\in[0,1)$ such that $wh_*(t_1)=z$ and $wh_*(t_2)=v$. As in [1], [3], [4] and [20], define the *cyclic order*, i.e.,

$$v\prec w\prec z\ \text{ if and only if }\ 0<t_1<t_2$$

and

$$v\preceq w\preceq z\ \text{ if and only if }\ t_1\leqslant t_2 \text{ or } t_2=0.$$

Obviously, the relations $v\prec w\prec z$, $w\prec z\prec v$ and $z\prec v\prec w$ are equivalent. More properties of $\prec$ and $\preceq$ can be found in [3]. Consider a nonempty set $A\subset\mathbb{T}^1$.

A map $F\colon A \to \mathbb{T}^1$ is said to be *increasing* (*strictly increasing*) if $F(v) \preceq F(w) \preceq F(z)$ ($F(v) \prec F(w) \prec F(z)$, respectively) for every $v, w, z \in A$ with $v \prec w \prec z$. Obviously, if $\operatorname{card} A \leqslant 2$ then every map is increasing.

If $v, z \in \mathbb{T}^1$ with $v \neq z$, there exist $t_v, t_z \in \mathbb{R}$ such that $t_v < t_z < t_v + 1$ and $v = h(t_v)$, $z = h(t_z)$. Define the *oriented arc*

$$\overrightarrow{(v, z)} := \{h(t)\colon\ t \in (t_v, t_z)\}.$$

This definition does not depend on the choice of t_v and t_z. Obviously, $v \prec w \prec z$ if and only if $w \in \overrightarrow{(v, z)}$. The map F is strictly increasing if $w \in \overrightarrow{(v, z)}$ yields $F(w) \in \overrightarrow{(F(v), F(z))}$.

As in [5], [14] and [18], the continuous map $\widetilde{F}\colon \mathbb{R} \to \mathbb{R}$ is referred to as a *lift* of $F \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ if

$$h \circ \widetilde{F} = F \circ h.$$

As shown in [5], [14] and [18], we know the following properties:

Lemma 2.1. (i) *Every $F \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ has a lift $\widetilde{F}$.* (ii) *There exists a constant $k \in \mathbb{Z}$ such that every lift $\widetilde{F}$ of F satisfies $\widetilde{F}(t+1) - \widetilde{F}(t) = k$ for all $t \in \mathbb{R}$.* (iii) *If $\widetilde{F}$ is a lift of F then for each $j \in \mathbb{Z}$ the map $\widetilde{F} + j$ is a lift of F and every lift of F can be expressed in this form.*

By Lemma 2.1, the integer k is determined uniquely and independently of the choice of lifts. It is called the *degree* of F and denoted by $\deg F$. One can show that $|\deg F| = 1$ if F is a homeomorphism, and a continuous map $F\colon \mathbb{T}^1 \to \mathbb{T}^1$ is strictly increasing if and only if $\deg F = 1$ and its lift $\widetilde{F}$ is strictly increasing in $\mathbb{R}$. A homeomorphism $F\colon \mathbb{T}^1 \to \mathbb{T}^1$ is said to be *orientation preserving* if it is strictly increasing.

A map $F \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is said to be *Lipschitzian* if its lift $\widetilde{F}$ satisfies

$$|\widetilde{F}(t_1) - \widetilde{F}(t_2)| \leqslant K|t_1 - t_2|, \quad \forall t_1, t_2 \in \mathbb{R}, \tag{2.1}$$

for a constant $K \geqslant 0$. By Lemma 2.1, the constant K is independent of the choice of lifts and is called a Lipschitz constant of F.

For $F \in H^0_1(\mathbb{T}^1, \mathbb{T}^1)$, define $\widetilde{F}_* = h_*^{-1} \circ F \circ h_*$, which is a self-map on $[0, 1)$. Clearly, F preserves orientation if and only if $\widetilde{F}_*$ is strictly increasing. In order to convert our problem from the circle $\mathbb{T}^1$ to the compact interval $I := [0, 1]$, we extend $\widetilde{F}_*$ to

$$G(t) := \begin{cases} \widetilde{F}_*(t), & t \in [0, 1), \\ 1, & t = 1. \end{cases} \tag{2.2}$$

For convenience we call G the *induced map* of F, which is a self-map on I.

Lemma 2.2. *The induced map G of an orientation-preserving $F \in H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ is continuous and strictly increasing on I and fixes 0 and 1. It can be extended to a lift of F, and there is a unique lift $\widetilde{F}$ which fixes 0 and 1 and maps I into itself.*

P r o o f. Obviously, G is continuous on the interval $[0, 1)$ and $G(0) = h_*^{-1} \circ F \circ h_*(0) = h_*^{-1} \circ F(\mathbf{1}) = h_*^{-1}(\mathbf{1}) = 0$. On the other hand, G is well defined on the closed interval $[0, 1]$ and $G(1) = 1$.

Concerning continuity at 1, we note that

$$\begin{aligned}\lim_{t\to 1^-} G(t) &= \lim_{t\to 1^-} h_*^{-1} \circ F \circ h_*(t) = \lim_{\varepsilon\to 0^+} h_*^{-1} \circ F \circ h_*(1-\varepsilon)\\ &= \lim_{\varepsilon\to 0^+} h_*^{-1} \circ F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}).\end{aligned}$$

By continuity of F at $\mathbf{1} \in \mathbb{T}^1$ we have $\lim\limits_{\varepsilon\to 0^+} F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) = F(\mathbf{1}) = \mathbf{1}$. More concretely, for every $0 < \varepsilon < 1$ there exists $0 < \delta < 1$ such that $F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) = \mathrm{e}^{2\pi\mathrm{i}(1-\delta)}$ since $F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) \in \mathbb{T}^1$. Let $\widetilde{F}$ be a lift of F such that $\widetilde{F}(1) = 1$. We have $F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) = \mathrm{e}^{2\pi\mathrm{i}\widetilde{F}(1-\varepsilon)}$, so $\widetilde{F}(1-\varepsilon) = 1-\delta$. Hence $\varepsilon \to 0^+$ implies that $\delta \to 0^+$. since $\widetilde{F}$ is increasing. Then $\lim\limits_{\varepsilon\to 0^+} F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) = \lim\limits_{\delta\to 0^+} \mathrm{e}^{2\pi\mathrm{i}(1-\delta)}$. Thus

$$\begin{aligned}\lim_{t\to 1^-} G(t) &= \lim_{\varepsilon\to 0^+} h_*^{-1} \circ F(\mathrm{e}^{2\pi\mathrm{i}(1-\varepsilon)}) = \lim_{\delta\to 0^+} h_*^{-1}(\mathrm{e}^{2\pi\mathrm{i}(1-\delta)})\\ &= \lim_{\delta\to 0^+} \frac{1}{2\pi\mathrm{i}} \ln(\mathrm{e}^{2\pi\mathrm{i}(1-\delta)}) = 1,\end{aligned}$$

implying the continuity of G at 1.

Note that $\widetilde{F}_*$ is strictly increasing on $[0, 1)$. For $t_1 \in (0, 1)$ and $t_2 = 1$ we have $0 < G(t_1) < 1 = G(t_2)$. Hence G is strictly increasing on $[0, 1]$.

Given $t \in \mathbb{R}$, let k be the integer such that $t \in [k, k+1)$. Define $\widetilde{F}(t) := G(t-k)+k$. One can check that $h \circ \widetilde{F} = F \circ h$, i.e., $\widetilde{F}$ is a lift of F. Assume that F has another lift $\hat{F} \in C^0(\mathbb{R}, \mathbb{R})$, mapping $[0, 1]$ into itself, such that $\hat{F}(0) = 0$ and $\hat{F}(1) = 1$. By Lemma 2.1, $\hat{F}(t) = \widetilde{F}(t) + j$ for some integer j. Clearly, $j = \hat{F}(0) - \widetilde{F}(0) = 0$. So $\hat{F}(t) \equiv \widetilde{F}(t)$ for all $t \in \mathbb{R}$. The proof is completed. □

What follows is a converse to Lemma 2.2.

Lemma 2.3. *Suppose that $G \in C^0(I, I)$ is strictly increasing and satisfies $G(0) = 0$ and $G(1) = 1$. Then the map $F := h_* \circ G \circ h_*^{-1}$ is in the class $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ and preserves orientation. Moreover, G can be extended to a lift of F.*

P r o o f. Clearly, $F(\mathbf{1}) = h_* \circ G \circ h_*^{-1}(\mathbf{1}) = h_* \circ G(0) = h_*(0) = \mathbf{1}$. One can verify that F preserves orientation. Then we only need to show the continuity of F at $\mathbf{1} \in \mathbb{T}^1$. Its continuity at $\mathbf{1}$ in “clockwise” direction, i.e., continuity of the function

$F|_{\overrightarrow{[1,i]}}$ at $\mathbf{1}$, is obvious. In "counter-clockwise" direction we shall verify continuity of $F|_{\overrightarrow{[-i,1]}}$ at $\mathbf{1}$. Actually, we have

$$\begin{aligned}\lim_{t\to 1^-} F(\mathrm{e}^{2\pi i t}) &= \lim_{t\to 1^-} h_*\circ G\circ h_*^{-1}(\mathrm{e}^{2\pi i t}) = \lim_{t\to 1^-} h_*\circ G(t)\\ &= \lim_{t\to 1^-} h\circ G(t) = h\circ G(1) = \mathbf{1}.\end{aligned}$$

This implies continuity of F at $\mathbf{1}$ in counter-clockwise direction. Hence F is continuous on $\mathbb{T}^1$.

Given $t\in\mathbb{R}$ let k be the integer such that $t\in[k,k+1)$. Define

$$\widetilde{F}(t) := G(t-k)+k. \tag{2.3}$$

It is easy to verify continuity of F on $\mathbb{R}$. Note that

$$F\circ h(t) = h\circ G(t), \qquad \forall t\in[0,1]. \tag{2.4}$$

In fact, $F\circ h(t) = F\circ h_*(t) = h_*\circ G(x) = h\circ G(t)$ for $t\in[0,1)$ and, moreover, $F\circ h(1) = F(\mathbf{1}) = \mathbf{1}$ and $h\circ G(1) = h(1) = \mathbf{1}$. It follows that

$$h\circ\widetilde{F}(t) = h(G(t-k)+k) = h(G(t-k)) = F\circ h(t-k) = F\circ h(t)$$

for all $t\in\mathbb{R}$. Therefore, $\widetilde{F}$ is a lift of F. □

3. Maps on $\mathbb{T}^n$ and induced maps

We also need a version similar to that of the last section for the multi-variate function Φ, but the generalization is much more complicated. For simplicity, let

$$\mathbb{T}^n := \overbrace{\mathbb{T}^1\times\ldots\times\mathbb{T}^1}^{n}, \qquad \mathbf{1}^n := (\overbrace{\mathbf{1},\ldots,\mathbf{1}}^{n}).$$

For $f\in H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$, let us introduce the notation

$$H_f(z) := (f(z),\ldots,f^n(z)). \tag{3.5}$$

Make the general assumption for the domain and range of Φ that $\operatorname{Dom}\Phi\subset\mathbb{T}^n$ and $\operatorname{Ran}\Phi\subset\mathbb{T}^1$. Then equation (1.1) can be written in the form

$$\Phi\circ H_f = F. \tag{3.6}$$

Before defining the lift of Φ and its induced map, we need to know more about $\operatorname{Dom}\Phi$ and $\operatorname{Ran}\Phi$.

Remark 1. *H_f maps $\mathbb{T}^1$ into $(\mathbb{T}^1\backslash\{\mathbf{1}\})^n\cup\{\mathbf{1}^n\}$.* In fact, if there exists an $x_0 \in \mathbb{T}^1$ such that $f^k(x_0) = \mathbf{1}$ for a certain $k \in \{1, \ldots, n\}$, then $f^{j+k}(x_0) = f^j(\mathbf{1}) = \mathbf{1}$ for all $j \in \mathbb{Z}$. In particular, for $j = 1 - k, \ldots, n - k$ we get $f(x_0) = \ldots = f^n(x_0) = \mathbf{1}$. So $H_f(x_0) = \mathbf{1}^n$.

Remark 2. *If equation* (3.6) *has a solution in $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ and $F \in H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$, then* $\operatorname{Ran}\Phi = \mathbb{T}^1$ *and* $\Phi(\mathbf{1}^n) = \mathbf{1}$. The former assertion is observed from the fact that $\Phi(H_f(\mathbb{T}^1)) = F(\mathbb{T}^1) = \mathbb{T}^1$. The latter comes from (3.6) and the fact that $H_f(\mathbf{1}) = \mathbf{1}^n$.

In contrast to Remark 2, we also want to know $\operatorname{Dom}\Phi$. For this purpose, we first discuss degree of Φ and give a result of nonexistence of solutions for (3.6) in Corollary 3.1. Then we answer to $\operatorname{Dom}\Phi$ after Corollary 3.1. As an immediate consequence of Lemma 2.1, we have its generalization in a multi-variate version:

Lemma 3.1. *If $\Phi\colon \mathbb{T}^n \to \mathbb{T}^1$ is continuous and $\Phi(\mathbf{1}^n) = \mathbf{1}$, then there exists a unique continuous function $\widetilde{\Phi}\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$\Phi(h(t_1), \ldots, h(t_n)) = \mathrm{e}^{2\pi i\widetilde{\Phi}(t_1,\ldots,t_n)}, \qquad \widetilde{\Phi}(0, \ldots, 0) = 0. \tag{3.7}$$

Moreover, for each $k \in \{1, \ldots, n\}$, there exists an $m_k \in \mathbb{Z}$ such that

$$\widetilde{\Phi}(t_1, \ldots, t_k + 1, \ldots, t_n) = \widetilde{\Phi}(t_1, \ldots, t_k, \ldots, t_n) + m_k, \qquad \forall t_1, \ldots, t_n \in \mathbb{R}. \tag{3.8}$$

Proof. Put

$$\Upsilon(t_1, \ldots, t_n) := \Phi(h(t_1), \ldots, h(t_n)). \tag{3.9}$$

Then $\Upsilon\colon \mathbb{R}^n \to \mathbb{T}^1$ is continuous and periodic and satisfies

$$\Upsilon(t_1, \ldots, t_k + 1, \ldots, t_n) = \Upsilon(t_1, \ldots, t_k, \ldots, t_n), \quad k = 1, \ldots, n, \tag{3.10}$$

and $\Upsilon(0, \ldots, 0) = \mathbf{1}$. By the continuity of Υ, for every $x \in I^n$ there exists an open neighborhood $S_x \subset \mathbb{R}^n$ of x such that $\Upsilon(S_x) \neq \mathbb{T}^1$. Actually, the image $\Upsilon(S_x)$ is an open arc in $\mathbb{T}^1$. Hence, for every $x \in I^n$ we can define on $\Upsilon(S_x)$ the branches of complex logarithm. Let

$$\varsigma_x(t_1, \ldots, t_n) := \frac{1}{2\pi\mathrm{i}} \ln \Upsilon(t_1, \ldots, t_n), \quad (t_1, \ldots, t_n) \in S_x, \tag{3.11}$$

where ln denotes one of the branches of logarithm. The function ς_x has the following property: If $S_x \cap S_y \neq \emptyset$ then there exists a constant $k \in \mathbb{Z}$ such that

$$\varsigma_x(t_1, \ldots, t_n) = \varsigma_y(t_1, \ldots, t_n) + k \quad \forall (t_1, \ldots, t_n) \in S_x \cap S_y. \tag{3.12}$$

In fact, for $(t_1,\dots,t_n)\in S_x\cap S_y$ we have

$$\mathrm{e}^{2\pi \mathrm{i}\varsigma_x(t_1,\dots,t_n)}=\mathrm{e}^{2\pi i\varsigma_y(t_1,\dots,t_n)}=\Psi(t_1,\dots,t_n),$$

that is, $\mathrm{e}^{2\pi i[\varsigma_x(t_1,\dots,t_n)-\varsigma_y(t_1,\dots,t_n)]}=1$. So

$$k(t_1,\dots,t_n):=\varsigma_x(t_1,\dots,t_n)-\varsigma_y(t_1,\dots,t_n)\in\mathbb{Z}. \tag{3.13}$$

On the other hand, being a difference of two continuous functions, $k(t_1,\dots,t_n)$ is also continuous, implying together with (3.13) that $k(t_1,\dots,t_n)$ is a constant $k\in\mathbb{Z}$, i.e., (3.12) is proved. The result (3.12) also implies that ς_x is determined uniquely up to an integer.

Obviously, $I^n\subset\bigcup\limits_{x\in I^n}S_x$. By the compactness of I^n,

$$I^n\subset\bigcup_{j=0}^{p}S_{x_j} \tag{3.14}$$

for some positive integer p. Without loss of generality, we can put $x_0=(0,\dots,0)$ and arrange the sequence (x_j) in (3.14) such that

$$S_{x_j}\cap S_{x_{j+1}}\neq\emptyset,\quad j=0,\dots,p-1.$$

Now, for each x_j, we exactly define ς_{x_j} by choosing an appropriate branch of logarithm in (3.11) inductively, so that

$$\varsigma_{x_j}(t_1,\dots,t_n)=\varsigma_{x_{j+1}}(t_1,\dots,t_n)\quad\forall(t_1,\dots,t_n)\in S_{x_j}\cap S_{x_{j+1}} \tag{3.15}$$

for $j=0,\dots,p-1$. First, for $x_0=(0,\dots,0)$ we choose such a branch that $\varsigma_{x_0}(0,\dots,0)=0$ because $\Upsilon(0,\dots,0)=\mathbf{1}$. Assume that functions ς_{x_j} $(j=0,\dots,\iota)$ are defined exactly such that (3.15) holds for $j=0,\dots,\iota-1$. Let $\widetilde{\varsigma}_{x_{\iota+1}}$ be defined as in (3.11) for an arbitrarily fixed branch of logarithm. By the property of (3.12), there exists an integer $k\in\mathbb{Z}$ such that

$$\varsigma_{x_\iota}(t_1,\dots,t_n)=\widetilde{\varsigma}_{x_{\iota+1}}(t_1,\dots,t_n)+k.$$

Then we define $\varsigma_{x_{\iota+1}}(t_1,\dots,t_n):=\widetilde{\varsigma}_{x_{\iota+1}}(t_1,\dots,t_n)+k$ and, therefore, the extended sequence of functions ς_{x_j} $(j=0,\dots,\iota+1)$ also satisfies (3.15). Thus, the full sequence $(\varsigma_{x_j}:\ j=0,\dots,p)$ that satisfies (3.15) is well defined inductively. By (3.14) and (3.15), it is reasonable to define

$$\varphi(t_1,\dots,t_n):=\varsigma_{x_j}(t_1,\dots,t_n)\quad\text{for }(t_1,\dots,t_n)\in S_{x_j},\quad j=0,\dots,p. \tag{3.16}$$

Obviously, φ is continuous on I^n and $e^{2\pi i\varphi(t_1,\dots,t_n)} = \Upsilon(t_1,\dots,t_n)$. It follows from (3.9) that

$$e^{2\pi i\varphi(t_1,\dots,t_n)} = \Phi(h(t_1),\dots,h(t_n)).$$

Let $v_l := (0,\dots,1,\dots,0)$, the vector in $\mathbb{R}^n$ whose components except for the l-th one being 1 are all equal to 0. Let $m_l := \varphi(v_l)$, $l = 1,\dots,n$. Since

$$\Phi(h(0),\dots,h(1),\dots,h(0)) = \Phi(\mathbf{1}^n) = \mathbf{1}$$

as assumed, where $h(1)$ appears at the l-th variable, we have $e^{2\pi i\varphi(v_l)} = \mathbf{1}$. it implies that $\varphi(v_l) \in \mathbb{Z}$, i.e., $m_l \in \mathbb{Z}$.

We further extend function φ on the whole $\mathbb{R}^n$. Consider $(t_1,\dots,t_n) \in \mathbb{R}^n$. Clearly,

$$(t_1,\dots,t_n) = (s_1,\dots,s_n) + (k_1,\dots,k_n)$$

for some $s_j \in [0,1)$ and $k_j \in \mathbb{Z}$, $j = 1,\dots,n$. Let

$$\widetilde{\Phi}(t_1,\dots,t_n) = \varphi(s_1,\dots,s_n) + k_1m_1 + \ldots + k_nm_n,$$

which is obviously a continuous map on $\mathbb{R}^n$. One can check (3.8) by (3.10). Moreover, we can also verify that

$$e^{2\pi i\widetilde{\Phi}(t_1,\dots,t_n)} = e^{2\pi i\varphi(s_1,\dots,s_n)} = \Phi(h(s_1),\dots,h(s_n)) = \Phi(h(t_1),\dots,h(t_n)),$$

i.e., (3.7) is proved.

Uniqueness of $\widetilde{\Phi}$ is obtained from the restriction $\widetilde{\Phi}(0,\dots,0) = 0$. □

By this lemma, it is reasonable to call $\widetilde{\Phi}$ the *lift* of Φ and define the *degree* of Φ by $\deg\Phi := (m_1,\dots,m_n)$.

Lemma 3.2. *Let $\widetilde{F}$ be the lift of F such that $\widetilde{F}(0) = 0$ and let $\widetilde{\Phi}$ be the lift of Φ such that $\widetilde{\Phi}(0,\dots,0) = 0$. Let $f \in H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$ be a solution of* (1.1) *and let $\tilde{f}$ be its lift such that $\tilde{f}(0) = 0$. Then equation* (1.1) *is equivalent to*

$$\widetilde{\Phi}(\tilde{f}(t),\dots,\tilde{f}^n(t)) = \widetilde{F}(t), \qquad t \in \mathbb{R}. \tag{3.17}$$

Proof. In fact, $f^j(h(t)) = h(\tilde{f}^j(t))$ for $t \in \mathbb{R}$. For $z = e^{2\pi it} \in \mathbb{T}^1$, equation (1.1) is equivalent to

$$\Phi(e^{2\pi i\tilde{f}(t)},\dots,e^{2\pi i\tilde{f}^n(t)}) = e^{2\pi i\widetilde{F}(t)}.$$

By Lemma 3.1, $\mathrm{e}^{2\pi\mathrm{i}\widetilde{\Phi}(\tilde{f}(t),\dots,\tilde{f}^n(t))} = \mathrm{e}^{2\pi\mathrm{i}\widetilde{F}(t)}$, that is, for each $t \in \mathbb{R}$ we have

$$\widetilde{\Phi}(\tilde{f}(t),\dots,\tilde{f}^n(t)) = \widetilde{F}(t) + k(t), \tag{3.18}$$

where $k(t) \in \mathbb{Z}$. Since $\widetilde{F}(0) = 0, \tilde{f}(0) = 0$ and $\widetilde{\Phi}(0,\dots,0) = 0$, from (3.18) we get $k(0) = 0$. By the continuity of $\widetilde{F}, \tilde{f}$ and $\widetilde{\Phi}$, the function $k(t)$ is continuous in $t \in \mathbb{R}$. This implies that $k(t) \equiv 0$ and the result of this lemma is proved. □

Theorem 3.1. *Suppose that $\Phi\colon \mathbb{T}^n \to \mathbb{T}^1$ is continuous, $\Phi(\mathbf{1}^n) = \mathbf{1}$, $F\colon \mathbb{T}^1 \to \mathbb{T}^1$ is continuous, $F(\mathbf{1}) = \mathbf{1}$ and equation* (3.6) *has a solution in $H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$. Let $\deg\Phi = (m_1,\dots,m_n)$. Then $\deg F = m_1 + \ldots + m_n$.*

P r o o f. Let $\widetilde{F}$ be the lift of F such that $\widetilde{F}(0) = 0$ and $\widetilde{\Phi}$ the lift of Φ such that $\widetilde{\Phi}(0,\dots,0) = 0$. Let $f \in H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$ be a solution of (3.6) and let $\tilde{f}$ be its lift such that $\tilde{f}(0) = 0$. By Lemma 3.2, equation (3.6) is equivalent to (3.17). Note that $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that

$$\tilde{f}(t+1) = \tilde{f}(t) + 1, \quad t \in \mathbb{R}.$$

As in (3.5), put $\widetilde{H}(t) := (\tilde{f}(t),\dots,\tilde{f}^n(t))$. Then equation (3.17) can be written as

$$\widetilde{\Phi} \circ \widetilde{H} = \widetilde{F}. \tag{3.19}$$

Since $\deg\Phi = (m_1,\dots,m_n)$, we have, by (3.19),

$$\widetilde{\Phi}(1,\dots,1) = \widetilde{\Phi}(0,\dots,0) + m_1 + \ldots + m_n = m_1 + \ldots + m_n.$$

Moreover, by (3.19),

$$\begin{aligned} \widetilde{F}(t+1) &= \widetilde{\Phi}(\widetilde{H}(t+1)) = \widetilde{\Phi}(\widetilde{H}(t) + (1,\dots,1)) \\ &= \widetilde{\Phi}(\widetilde{H}(t)) + m_1 + \ldots + m_n \\ &= \widetilde{F}(t) + m_1 + \ldots + m_n. \end{aligned} \tag{3.20}$$

This means that $\deg F = m_1 + \ldots + m_n$. □

Corollary 3.1. *Suppose that $\Phi\colon \mathbb{T}^n \to \mathbb{T}^1$ $(n \geqslant 2)$ such that $\Phi(\mathbf{1}^n) = \mathbf{1}$ and Φ is increasing with respect to each variable and nonconstant in at least two variables. If $F \in H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$, then equation* (3.6) *has no solution in $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$.*

Proof. Since F is a homeomorphism, we have $|\deg F| = 1$. Let $\deg \Phi = (m_1, \ldots, m_n)$. If Φ is increasing with respect to each variable then its lift is also increasing with respect to each variable. This follows by Theorem 1 in [4] and formula (3.7) where all $t_1, \ldots, t_n$ except a variable t_k are fixed. Hence, by (3.8), $m_1 \geqslant 0, \ldots, m_n \geqslant 0$ and $m_k = 0$ if and only if Φ is constant with respect to t_k. Thus, by (3.20), $\deg(\Phi \circ H_f) = m_1 + \ldots + m_n \geqslant 2$ since Φ is nonconstant in at least two variables. By (3.6) we have $\deg(\Phi \circ H_f) = \deg F$. This implies that $\deg F \geqslant 2$, a contradiction. $\square$

In view of Remark 1 and Corollary 3.1 it is natural to assume that $\operatorname{Dom}\Phi = (\mathbb{T}^1 \setminus \{\mathbf{1}\})^n \cup \{\mathbf{1}^n\}$ and it is not possible to extend it continuously on $\mathbb{T}^n$. Therefore, we make the following general assumptions:

(H1) $\Phi\colon (\mathbb{T}^1 \setminus \{\mathbf{1}\})^n \cup \{\mathbf{1}^n\} \to \mathbb{T}^1$ is continuous, $\Phi(\mathbf{1}^n) = \mathbf{1}$, $\Phi((\mathbb{T}^1 \setminus \{\mathbf{1}\})^n) = \mathbb{T}^1 \setminus \{\mathbf{1}\}$, and

(A) there exists a constant $\delta > 0$ such that if $0 < t_k < \delta$, $k = 1, \ldots, n$, then $\Phi(h_*(t_1), \ldots, h_*(t_n)) \in \overrightarrow{(1, i)}$ and for $1 - \delta < t_k < 1$, $k = 1, \ldots, n$, we have $\Phi(h_*(t_1), \ldots, h_*(t_n)) \in \overrightarrow{(-i, 1)}$.

Under assumptions (H1) and (A), we define

$$\Psi(t_1, \ldots, t_n) := h_*^{-1}(\Phi(h_*(t_1), \ldots, h_*(t_n))), \quad t_j \in (0,1),\ j = 1, \ldots, n, \tag{3.21}$$

and

$$\Psi(0, \ldots, 0) = 0, \qquad \Psi(1, \ldots, 1) = 1. \tag{3.22}$$

The function Ψ defined by (3.21) and (3.22) on $(0,1)^n \cup \{(0, \ldots, 0), (1, \ldots, 1)\}$ is called the *induced map* of Φ. Let us note that

$$\lim_{t_j \to 0,\ j=1,\ldots,n} \Psi(t_1, \ldots, t_n) = \lim_{t_j \to 0,\ j=1,\ldots,n} h_*^{-1}(\Phi(h_*(t_1), \ldots, h_*(t_n)) = 0,$$
$$\lim_{t_j \to 1,\ j=1,\ldots,n} \Psi(t_1, \ldots, t_n) = \lim_{t_j \to 1,\ j=1,\ldots,n} h_*^{-1}(\Phi(h_*(t_1), \ldots, h_*(t_n)) = 1.$$

Thus we get

Lemma 3.3. *Under assumptions* (H1) *and* (A), *the induced map of* Φ *is continuous.*

For further considerations it is sufficient that

$$\operatorname{Dom}\Psi = (0,1)^n \cup \{(0,\dots,0),(1,\dots,1)\}.$$

In particular, if Ψ is increasing with respect to each variable, then we can extend Ψ continuously on $[0,1]^n$.

Remark 3. It is obvious that if $\Psi\colon [0,1]^n \to [0,1]$ is continuous, strictly increasing with respect to each variable, $\Psi(0,\dots,0)=0$ and $\Psi(1,\dots,1)=1$ then the function Φ defined by

$$\Phi(z_1,\dots,z_n) := h(\Psi(h_*^{-1}(z_1),\dots,h_*^{-1}(z_n))), \quad z_i \in \mathbb{T}^1\setminus\{\mathbf{1}\},\ i=1,\dots,n,$$
$$\Phi(\mathbf{1},\dots,\mathbf{1}) := \mathbf{1}$$

satisfies assumptions (H1), (A) and Φ is increasing with respect to each variable.

Remark 4. It is also obvious that if Φ satisfies (H1) and is strictly increasing with respect to each variable then Φ satisfies (A).

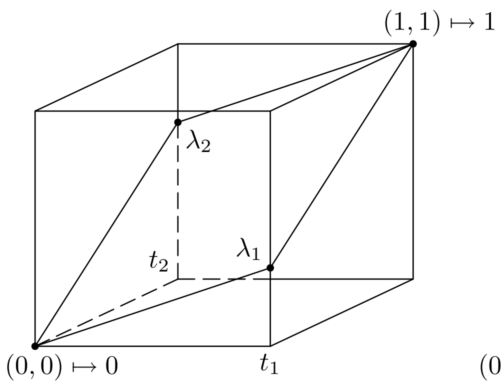

Fig. 1 Plane of $\Psi(t_1,t_2) = \lambda_1 t_1 + \lambda_2 t_2$ with $\lambda_j > 0$, $j=1,2$, $\lambda_1+\lambda_2=1$

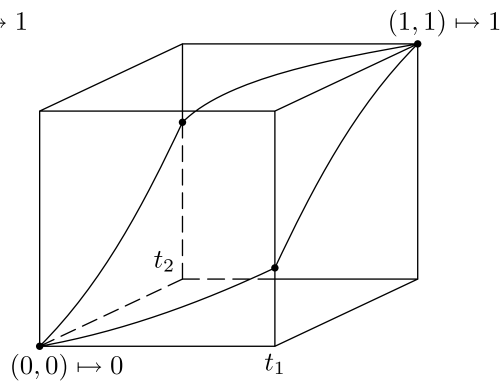

Fig. 2 Surface of nonlinear $\Psi(t_1,t_2)$ for understanding the limits at $\mathbf{1}\in\mathbb{T}^1$

One can understand the induced map Ψ at $\mathbf{1}$ in limit with the example of $\Phi(z_1,z_2) = z_1^{\lambda_1} z_2^{\lambda_2}$ for the special form

$$(f(z))^{\lambda_1}(f^2(z))^{\lambda_2}\dots(f^n(z))^{\lambda_n} = F(z), \quad z\in\mathbb{T}^1 \tag{3.23}$$

of equation (1.1), where $n=2$, $\lambda_1>0$, $\lambda_2>0$ and $\lambda_1+\lambda_2=1$. A comparison of a linear Ψ and a nonlinear Ψ is shown by Figures 1 and 2.

Besides hypothesis (H1), we need the Lipschitzian property of Φ. Similar to (2.1), such a property on the circle $\mathbb{T}^1$ can be defined directly for the induced map Ψ. Let us introduce the following hypotheses:

(H2) There are nonnegative real constants α_j, β_j $(j = 2, \dots, n)$ with $\beta_1 \geqslant \alpha_1 > 0$, $\beta_j \geqslant \alpha_j \geqslant 0$ such that

$$\sum_{j=1}^{n} \alpha_j (t_j - s_j) \leqslant \Psi(t_1, \dots, t_n) - \Psi(s_1, \dots, s_n) \leqslant \sum_{j=1}^{n} \beta_j (t_j - s_j)$$

for all $t_j \geqslant s_j$ in I $(j = 1, \dots, n)$.

(H3) For every $k \in \{1, \dots, n\}$ there exist $\alpha_k, \beta_k \geqslant 0$ with $\beta_1 \geqslant \alpha_1 > 0$ such that

$$\alpha_k (t_k - s_k) \leqslant \Psi(t_1, \dots, t_k, \dots, t_n) - \Psi(t_1, \dots, s_k, \dots, t_n) \leqslant \beta_k (t_k - s_k) \tag{3.24}$$

for all $s_j, t_j \in (0,1)$, $j = 1, \dots, n$ and $t_k \geqslant s_k$.

Remark 5. *Hypotheses* (H2) *and* (H3) *are equivalent.* In fact, (H2) implies (H3) obviously since putting $t_i = s_i$, $i \neq k$, in (H2) we get (H3). Conversely, having (H3), observe that

$$\begin{aligned}
&\Psi(t_1, t_2, \dots, t_n) - \Psi(s_1, s_2, \dots, s_n) \\
&\qquad = (\Psi(t_1, t_2, \dots, t_n) - \Psi(s_1, t_2, \dots, t_n)) \\
&\qquad\quad + (\Psi(s_1, t_2, t_3, \dots, t_n) - \Psi(s_1, s_2, t_3, \dots, t_n)) \\
&\qquad\quad + (\Psi(s_1, s_2, t_3, \dots, t_n) - \Psi(s_1, s_2, s_3, \dots, t_n)) + \dots \\
&\qquad\quad + (\Psi(s_1, s_2, \dots, s_{n-1}, t_n) - \Psi(s_1, s_2, \dots, , s_{n-1}, s_n).
\end{aligned}$$

In view of (3.24), we obtain

$$\sum_{k=0}^{n} \alpha_k (t_k - s_k) \leqslant \Psi(t_1, \dots, t_n) - \Psi(s_1, \dots, s_n) \leqslant \sum_{k=0}^{n} \beta_k (t_k - s_k).$$

Remark 6. It is clear that if Ψ satisfies (H2) then Ψ is increasing with respect to each variable and strictly increasing with respect to those variables t_j that α_j is positive. Moreover, if $\Psi(0, \dots, 0) = 0$ and $\Psi(1, \dots, 1) = 1$, then

$$\sum_{k=1}^{n} \alpha_k \leqslant 1 \leqslant \sum_{k=1}^{n} \beta_k.$$

Therefore, under (H1) and (H2) equation (1.1) includes (3.23) as a special case.

Lemma 3.4. *If $\Psi\colon (0,1)^n \to \mathbb{R}$ is differentiable with respect to each variable and for every k there exist α_k, β_k such that $\alpha_1 > 0$, $0 \leqslant \alpha_k \leqslant \partial\Psi/\partial t_k \leqslant \beta_k$, then Ψ satisfies* (H2).

P r o o f. Let us note that (3.24) is equivalent to the inequalities

$$\begin{aligned}\alpha_k t_k - \Psi(t_1,\dots,t_k,\dots,t_n) &\leqslant \alpha_k s_k - \Psi(t_1,\dots,s_k,\dots,t_n),\\ \beta_k t_k - \Psi(t_1,\dots,t_k,\dots,t_n) &\geqslant \beta_k s_k - \Psi(t_1,\dots,s_k,\dots,t_n)\end{aligned}$$

for $t_k \geqslant s_k$, $t_i \in (0,1)$, $i = 1,\dots,n$. This means that the maps

$$t_k \longmapsto \alpha_k t_k - \Psi(t_1,\dots,t_k,\dots,t_n)$$

are decreasing and

$$t_k \longmapsto \beta_k t_k - \Psi(t_1,\dots,t_k,\dots,t_n)$$

are increasing. This is equivalent to

$$\alpha_k \leqslant \frac{\partial\Psi(t_1,\dots,t_n)}{\partial t_k} \leqslant \beta_k, \quad t_1,\dots,t_n \in (0,1)$$

for $k = 1,\dots,n$. □

4. Existence of solutions

Theorem 4.1. *Assume that $F \in H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$ preserves orientation with a Lipschitz constant $M > 0$ and that* (H1) *and* (H2) *hold. Then equation* (1.1) *has a solution $f \in H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$ which preserves orientation with a Lipschitz constant M/α_1.*

P r o o f. Let G and Ψ be the induced maps of F and Φ, defined as in Sections 2 and 3, respectively. Since we want to find solutions f in $H^0_{\mathbf{1}}(\mathbb{T}^1,\mathbb{T}^1)$, let g be the induced map of f. Similarly to Lemma 3.2, the problem of (1.1) is reduced to that of the continuous and strictly increasing solutions g of the equation

$$\Psi(g(t), g^2(t), \dots, g^n(t)) = G(t), \qquad t \in I. \tag{4.1}$$

In the sequel, we use the method given in [22] and [23], applying Schauder's fixed point theorem, to show the existence of a solution g. Although such a procedure was given in [15], we still need the procedure with a simpler statement to show that the solution g found in a compact subset of $C^0(I)$, which cannot require strict monotonicity of g, is actually strictly increasing.

By Lemma 2.2, $G \in C^0(I,I)$ is strictly increasing and $G(0)=0$, $G(1)=1$. Lemma 2.2 also implies that G can be extended to a lift of F. Thus

$$|G(x_2)-G(x_1)| \leqslant M|x_2-x_1|, \qquad \forall x_1,x_2 \in I,$$

because F is Lipschitzian with a Lipschitz constant M. Concerning Ψ, besides (H2) we know that $\Psi\colon (0,1)^n \cup \{(0,\dots,0),(1,\dots,1)\} \to [0,1]$ is continuous. In what follows, Lemma 3.2 in [25] is useful and its proof can be found in [23]. For convenience, we state it as

Lemma 4.1. *Let $i=1,2$ and suppose that g_i is a self-homeomorphism of I such that $|g_i(x)-g_i(y)| \leqslant M|x-y|$ for all $x,y\in I$, where $M>0$ is a constant. Then*

(i) $\|g_1^n - g_2^n\| \leqslant \Bigl(\sum\limits_{i=0}^{n-1} M^i\Bigr)\|g_1-g_2\|$ *for all $n=1,2\dots$, and*

(ii) $\|g_1-g_2\| \leqslant M\|g_1^{-1}-g_2^{-1}\|$.

For $0\leqslant m\leqslant M$, let

$$\begin{aligned}\mathscr{F}(I;m,M) = \{g\in C^0(I)\colon\ & g(0)=0,\ g(1)=1,\\ & m(t-s)\leqslant g(t)-g(s)\leqslant M(t-s),\ \forall s\leqslant t\in I\}.\end{aligned} \tag{4.2}$$

As in [22] and [25], this subset is compact and convex in the Banach space $C^0(I)$, equipped with the supremum norm $\|g\| = \max\{|g(t)|\colon\ t\in I\}$. Define an operator $L\colon \mathscr{F}(I;0,\alpha_1^{-1}M)\to C^0(I)$ by $g\mapsto L_g$, where

$$L_g(t) := \Psi(t,g(t),\dots,g^{n-1}(t)), \quad t\in I, \tag{4.3}$$

where $g\in\mathscr{F}(I;0,\alpha_1^{-1}M)$. Let $M_0 := \sum\limits_{j=1}^{n}\beta_j(\alpha_1^{-1}M)^{j-1}$. Then $L_g\in\mathscr{F}(I;\alpha_1,M_0)$ because for any $t\geqslant s\in I$,

$$\begin{aligned}
L_g(t)-L_g(s) &= \Psi(t,g(t),\dots,g^{n-1}(t)) - \Psi(s,g(s),\dots,g^{n-1}(s))\\
&\geqslant \alpha_1(t-s) + \sum_{j=2}^{n}\alpha_j(g^{j-1}(t)-g^{j-1}(s))\\
&\geqslant \alpha_1(t-s),\\
L_g(t)-L_g(s) &= \Psi(t,g(t),\dots,g^{n-1}(t)) - \Psi(s,g(s),\dots,g^{n-1}(s))\\
&\leqslant \beta_1(t-s) + \sum_{j=2}^{n}\beta_j(g^{j-1}(t)-g^{j-1}(s))\\
&\leqslant \beta_1(t-s) + \sum_{j=2}^{n}\beta_j(\alpha_1^{-1}M)^{j-1}(t-s)\\
&= M_0(t-s),
\end{aligned}$$

where (H2) is applied. In particular, L_g is an orientation-preserving homeomorphism on I since $\alpha_1 > 0$. Thus $L_g^{-1} \in \mathscr{F}(I; M_0^{-1}, \alpha_1^{-1})$.

Define $\mathscr{T}\colon \mathscr{F}(I; 0, \alpha_1^{-1}M) \to C^0(I)$ by

$$\mathscr{T}g(t) = L_g^{-1} \circ G(t), \qquad t \in I. \tag{4.4}$$

Then $\mathscr{T}$ maps $\mathscr{F}(I; 0, \alpha_1^{-1}M)$ into itself because $\mathscr{T}g(0) = 0$, $\mathscr{T}g(1) = 1$ and

$$\begin{aligned} 0 \leqslant \mathscr{T}g(t) - \mathscr{T}g(s) &= L_g^{-1} \circ G(t) - L_g^{-1} \circ G(s) \\ &\leqslant \alpha_1^{-1}(G(t) - G(s)) \leqslant \alpha_1^{-1}M(t-s) \end{aligned} \tag{4.5}$$

for all $t, s \in I$ with $t \geqslant s$. Furthermore, for any $g_1, g_2 \in \mathscr{F}(I; 0, M)$,

$$\begin{aligned} \|\mathscr{T}g_1 - \mathscr{T}g_2\| &= \|L_{g_1}^{-1} \circ G - L_{g_2}^{-1} \circ G\| \\ &= \|L_{g_1}^{-1} - L_{g_2}^{-1}\| \leqslant \alpha_1^{-1}\|L_{g_1} - L_{g_2}\| \\ &\leqslant \alpha_1^{-1} \max_{t\in I} |\Psi(t, g_1(t), \ldots, g_1^{n-1}(t)) - \Psi(t, g_2(t), \ldots, g_2^{n-1}(t))| \\ &\leqslant \alpha_1^{-1} \sum_{j=2}^{n} \beta_j \|g_1^{j-1} - g_2^{j-1}\| \\ &\leqslant \alpha_1^{-1} \sum_{j=2}^{n} \beta_j \sum_{k=1}^{j-1} (\alpha_1^{-1}M)^{k-1} \|g_1 - g_2\|, \end{aligned} \tag{4.6}$$

where Lemma 4.1 and (H2) are applied. Hence $\mathscr{T}$ maps $\mathscr{F}(I; 0, \alpha_1^{-1}M)$ continuously into itself. By Schauder's fixed point theorem $\mathscr{T}$ has a fixed point g in $\mathscr{F}(I; 0, \alpha_1^{-1}M)$, that is, $L_g \circ g(t) = G(t)$. Therefore, g is a continuous solution of equation (4.1). In consequence the map f defined by $f(\mathrm{e}^{2\pi\mathrm{i}t}) = \mathrm{e}^{2\pi\mathrm{i}g(t)}$ on $\mathbb{T}^1$ belongs to $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ and is a solution of equation (1.1).

The definition of $\mathscr{F}(I; 0, \alpha_1^{-1}M)$ does not guarantee strict monotonicity of the obtained g, but g actually is strictly increasing. In fact, both G and L_g^{-1} are proved to be strictly increasing. So is the function $g(t) = L_g^{-1} \circ G(t)$ by (4.4). Thus, it follows from (4.5) that

$$0 < g(t) - g(s) \leqslant \frac{M}{\alpha_1}(t-s), \quad \forall t \geqslant s \in I. \tag{4.7}$$

Let

$$f(z) := h_* \circ g \circ h_*^{-1}(z), \qquad \forall z \in \mathbb{T}^1. \tag{4.8}$$

By Lemma 2.3 and (4.7), $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ preserves orientation and $f(\mathbf{1}) = \mathbf{1}$. Thus $\Phi(f(z), \ldots, f^n(z)) = F(z)$ for $z \in \mathbb{T}^1$, i.e., f is a solution of equation (1.1) in the class $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$.

Further, by Lemma 2.3, g can be extended to a lift $\tilde{f}$ of f. From Lemma 2.1 we have $\tilde{f}(t+1) = \tilde{f}(t)+1$ for all $t \in \mathbb{R}$. For any t, s in $\mathbb{R}$ with $t < s$ there exists an integer k and a nonnegative integer m such that $t \in [k, k+1)$ and $s \in [k+m, k+m+1)$. Note that $\tilde{f}(t) = g(t)$ for $t \in I$. It follows from (4.7) that

$$\begin{aligned}
(4.9)\quad &|\tilde{f}(t) - \tilde{f}(s)| \\
&\leqslant |\tilde{f}(t) - \tilde{f}(k+1)| + \sum_{j=1}^{m-1} |\tilde{f}(k+j) - \tilde{f}(k+1+j)| + |\tilde{f}(s) - \tilde{f}(k+m)| \\
&\leqslant |\tilde{f}(t-k) - \tilde{f}(1)| + (m-1)|\tilde{f}(0) - \tilde{f}(1)| + |\tilde{f}(s-k-m) - \tilde{f}(0)| \\
&\leqslant \frac{M}{\alpha_1}[1-(t-k)+m-1+s-k-m] = \frac{M}{\alpha_1}(s-t)
\end{aligned}$$

since $t-k, s-k-m \in [0,1)$ and $\tilde{f}(t) = \tilde{f}(t-k)+k$. This implies that $\tilde{f}(t)$ is Lipschitzian and thus f is Lipschitzian with the Lipschitz constant M/α_1. This completes the proof. $\square$

Remark 7. If we assume that $\Phi\colon \mathbb{T}^n \to \mathbb{T}^1$ is a continuous map satisfying (H2), F is continuous, a lift of F is Lipschitz strictly increasing and $\deg F = m_1+m_2+\ldots+m_n$, where $\deg \Phi = (m_1, m_2, \ldots, m_n)$, then we get the same result as in Theorem 4.1. The proof is almost the same except the assumption that $\Psi(1,\ldots,1) = 1$. However we have $\Psi(1,\ldots,1) = m_1 + \ldots + m_n$.

5. Uniqueness and stability

As in [14] (p. 75), let $F_1, F_2 \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ and $\widetilde{F}_1, \widetilde{F}_2$ be their lifts respectively. For a given small constant $\varepsilon > 0$, we say that F_1 is ε *C^0-close* to F_2 if

$$(5.10)\qquad \|\widetilde{F}_1 - \widetilde{F}_2\| = \sup_{t\in\mathbb{R}} |\widetilde{F}_1(t) - \widetilde{F}_2(t)| < \varepsilon.$$

As usual, we say equation (1.1) is *stable* if for arbitrarly $\varepsilon > 0$ there exists $\sigma > 0$ such that, provided $F \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ being σ C^0-close to $F_0 \in C^0(\mathbb{T}^1, \mathbb{T}^1)$, the corresponding solutions f, f_0 are ε C^0-close to each other.

Theorem 5.1. *Suppose that the conditions in Theorem* 4.1 *hold and*

$$(5.11)\qquad \sum_{j=2}^{n} \beta_j \sum_{k=1}^{j-1} \alpha_1^{-k} M^{k-1} < 1.$$

Then equation (1.1) *has a unique solution* $f \in H_{\mathbf{1}}^0(\mathbb{T}^1, \mathbb{T}^1)$ *which preserves orientation with the Lipschitz constant* M/α_1. *Moreover, equation* (1.1) *is stable.*

P r o o f. Since (1.1) satisfies the conditions of Theorem 4.1, the existence of solutions for equation (4.1) is given in the proof of Theorem 4.1. As in [22] and [23], condition (5.11) guarantees that Banach's Contraction Theorem is applicable. Hence, equation (4.1) has a unique solution g on I. This implies uniqueness of the solution f given in Theorem 4.1.

Suppose that $F_1, F_2 \in H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ both satisfy conditions in Theorem 4.1 and that f_j are the unique solutions of equation (1.1) corresponding to the given F_j and Φ_j, $j = 1, 2$, where Φ_j's satisfy conditions (H1) and (H2). Assume that $\widetilde{F}_j$, $\tilde{f}_j$ are lifts of F_j and f_j, respectively. Let $\widetilde{F}_{*j}$ and $\tilde{f}_{*j}$ be restrictions of $\widetilde{F}_j$ and $\tilde{f}_j$ on I, respectively. Correspondingly we introduce the restrictions $\widetilde{\Phi}_{*j}$ for the lifts of Φ_j. Under condition (5.11), by the uniqueness as proved above, the corresponding continuations G_j and g_j of $\widetilde{F}_{*j}$ and $\tilde{f}_{*j}$ as in (2.2) must satisfy

$$g_j(x) = L^{-1}_{g_j, \Psi_j} \circ G_j(x), \qquad j = 1,\ 2,$$

where Ψ_j is the continuation of $\widetilde{\Phi}_{*j}$ as in (3.21) and L_{g_j,Ψ_j} is defined as in (4.3) with an emphasis on the dependence on Ψ_j. In the sequel, let $\|\cdot\|$ denote the norm $\|\varphi\| = \max\limits_{t\in I} |\varphi(t)|$ for $\varphi \in C^0(I)$. Since

$$\begin{aligned} \|L^{-1}_{g_1,\Psi_1} - L^{-1}_{g_2,\Psi_2}\| &\leqslant \alpha_1^{-1}\|L_{g_1,\Psi_1} - L_{g_2,\Psi_2}\|, \\ \|L^{-1}_{g_2,\Psi_2} \circ G_1 - L^{-1}_{g_2,\Psi_2} \circ G_2\| &\leqslant \alpha_1^{-1}\|G_1 - G_2\|, \end{aligned}$$

similarly to (4.6) we obtain

$$\begin{aligned} \|g_1 - g_2\| &= \|L^{-1}_{g_1,\Psi_1} \circ G_1 - L^{-1}_{g_2,\Psi_2} \circ G_2\| \\ &\leqslant \|L^{-1}_{g_1,\Psi_1} \circ G_1 - L^{-1}_{g_2,\Psi_2} \circ G_1\| + \|L^{-1}_{g_2,\Psi_2} \circ G_1 - L^{-1}_{g_2,\Psi_2} \circ G_2\| \\ &\leqslant \alpha_1^{-1}(\|L_{g_1,\Psi_1} - L_{g_2,\Psi_2}\| + \|G_1 - G_2\|) \\ &\leqslant \alpha_1^{-1}\{\max\nolimits_{t\in I}|\Psi_1(t, g_1(t), \ldots, g_1^{n-1}(t)) - \Psi_2(t, g_2^2(t), \ldots, g_2^{n-1}(t))| \\ &\qquad + \|G_1 - G_2\|\} \\ &\leqslant \alpha_1^{-1}\{\max\nolimits_{t\in I}|\Psi_1(t, g_1(t), \ldots, g_1^{n-1}(t)) - \Psi_1(t, g_2(t), \ldots, g_2^{n-1}(t))| \\ &\quad + \max\nolimits_{t\in I}|\Psi_1(t, g_2(t), \ldots, g_2^{n-1}(t)) - \Psi_2(t, g_2(t), \ldots, g_2^{n-1}(t))| \\ &\qquad\qquad + \|G_1 - G_2\|\} \\ &\leqslant \alpha_1^{-1}\left\{\sum_{j=2}^{n} \beta_j \|g_1^{j-1} - g_2^{j-1}\| + \|\Psi_1 - \Psi_2\| + \|G_1 - G_2\|\right\} \\ &\leqslant \alpha_1^{-1}\left\{\sum_{j=2}^{n} \beta_j \sum_{k=1}^{j-1} (\alpha_1^{-1}M)^{k-1}\|g_1 - g_2\| + \|\Psi_1 - \Psi_2\| + \|G_1 - G_2\|\right\} \\ &\leqslant r\|g_1 - g_2\| + \alpha_1^{-1}(\|\Psi_1 - \Psi_2\| + \|G_1 - G_2\|), \end{aligned}$$

where $r := \alpha_1^{-1} \sum\limits_{j=2}^{n} \beta_j \sum\limits_{k=1}^{j-1} (\alpha_1^{-1} M)^{k-1} < 1$ by (5.11). Therefore,

$$\|g_1 - g_2\| \leqslant \frac{\|G_1 - G_2\| + \|\Psi_1 - \Psi_2\|}{\alpha_1 - \sum\limits_{j=2}^{n} \beta_j \sum\limits_{k=1}^{j-1} (\alpha_1^{-1} M)^{k-1}}, \tag{5.12}$$

which implies the continuous dependence of the solution g on functions G and Ψ.

Now we partially focus at the dependence on the function F in (1.1). Then (5.12) implies

$$\|\tilde{f}_{*1} - \tilde{f}_{*2}\| \leqslant \mu \|\widetilde{F}_{*1} - \widetilde{F}_{*2}\| \tag{5.13}$$

for some constant $\mu > 0$. For $t \in \mathbb{R}$ let k be an appropriate integer such that $t \in [k, k+1)$. As in (2.3),

$$|\tilde{f}_1(t) - \tilde{f}_2(t)| = |\tilde{f}_{*1}(t-k) + k - \tilde{f}_{*2}(t-k) - k| = |\tilde{f}_{*1}(t-k) - \tilde{f}_{*2}(t-k)|.$$

Thus $\|\tilde{f}_1 - \tilde{f}_2\| = \|\tilde{f}_{*1} - \tilde{f}_{*2}\|$. Similarly we also have $\|\widetilde{F}_1 - \widetilde{F}_2\| = \|\widetilde{F}_{*1} - \widetilde{F}_{*2}\|$. Hence by (5.13),

$$\|\tilde{f}_1 - \tilde{f}_2\| \leqslant \mu \|\widetilde{F}_1 - \widetilde{F}_2\|,$$

implying that f_1 is ε; C^0-close to f_2 if F_1 is ε/μ; C^0-close to F_2. This proves stability in the C^0 sense. □

The proof of Theorem 5.1 also implies continuous dependence on Φ.

6. Examples

Consider equation (3.23), where $F \in H_{\mathbf{1}}^0(\mathbb{T}^1, \mathbb{T}^1)$ preserves orientation with a Lipschitz constant $M > 0$ and $\sum\limits_{j=1}^{n} \lambda_j = 1$, where $\lambda_1 > 0$, $\lambda_j \geqslant 0$, $j = 2, 3, \ldots, n$. As stated at the end of Section 2, the map

$$\Phi(z_1, \ldots, z_n) = z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_n^{\lambda_n} \tag{6.1}$$

satisfies (H1) and has the induced map

$$\Psi(t_1, \ldots, t_n) = \lambda_1 t_1 + \lambda_2 t_2 + \ldots + \lambda_n t_n$$

on $I = [0, 1]$. Obviously Ψ satisfies (H2) with $\alpha_j = \beta_j = \lambda_j$, $j = 1, 2, \ldots, n$. By Theorem 4.1, equation (3.23) has a solution $f \in H_{\mathbf{1}}^0(\mathbb{T}^1, \mathbb{T}^1)$ which preserves orientation with the Lipschitz constant M/λ_1. Further, by Theorem 5.1, we can see the results on uniqueness and stability under the additional condition

$$\sum_{j=2}^{n} \lambda_j \sum_{k=1}^{j-1} \lambda_1^{-k} M^{k-1} < 1.$$

For another example of no expression in (6.1), consider the equation

$$(f(z))^{6/7}(f^2(z))^{(1/14\pi\mathrm{i}) \ln f^2(z)} = \exp\Big(\frac{2\pi\mathrm{i}(z^{1/2\pi\mathrm{i}} - 1)}{\mathrm{e} - 1}\Big), \qquad z \in \mathbb{T}^1. \tag{6.2}$$

Let $F(z) = \exp(2\pi\mathrm{i}(z^{1/2\pi\mathrm{i}} - 1)/(\mathrm{e} - 1))$ and $\Phi(z_1, z_2) = z_1^{6/7} z_2^{(1/14\pi\mathrm{i}) \ln z_2}$. Clearly, $F \in H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ and has a lift $\widetilde{F}(t) := (\mathrm{e}^t - 1)/(\mathrm{e} - 1)$. It obviously is strictly increasing on $[0, 1]$, so F preserves orientation. Moreover,

$$|\widetilde{F}(t) - \widetilde{F}(s)| = \Big|\frac{\mathrm{e}^t - 1}{\mathrm{e} - 1} - \frac{\mathrm{e}^s - 1}{\mathrm{e} - 1}\Big| = \Big|\frac{\mathrm{e}^\xi}{\mathrm{e} - 1}(t - s)\Big| \leqslant M|t - s|, \ \forall t, s \in [0, 1],$$

where $M := \mathrm{e}/(\mathrm{e} - 1) > 1$. Using the same arguments as in (4.9), we obtain

$$|\widetilde{F}(t) - \widetilde{F}(s)| \leqslant M|t - s| \qquad \forall t, s \in \mathbb{R}, \tag{6.3}$$

i.e., F is Lipschitzian on $\mathbb{T}^1$ with the Lipschitz constant M. On the other hand, concerning Φ we see that $\Phi(\mathbf{1}, \mathbf{1}) = \mathbf{1}$. Consider its induced map

$$\Psi(t_1, t_2) = h_*^{-1}(\Phi(h_*(t_1), h_*(t_2))) = \frac{6}{7} t_1 + \frac{1}{7} t_2^2, \quad 0 < t_j < 1, \ j = 1, 2.$$

It is easy to check (H2) with constants $\alpha_1 = \beta_1 = 6/7$, $\alpha_2 = 0$, $\beta_2 = 2/7$. Moreover, Ψ can be extended continuously to I^2 so that $\Psi(0, 0) = 0$, $\Psi(1, 1) = 1$. Therefore both (H1) and (H2) are satisfied. By Theorem 4.1, equation (6.2) has a continuous solution $f\colon \mathbb{T}^1 \to \mathbb{T}^1$ such that $f(\mathbf{1}) = \mathbf{1}$. Moreover, f has a Lipschitz constant $7\mathrm{e}/(6(\mathrm{e} - 1))$ and preserves orientation on $\mathbb{T}^1$.

Since $\alpha_1 > \beta_2$, condition (5.11) is also satisfied. By Theorem 5.1, the solution of equation (6.2) is unique in the class of orientation-preserving maps in $H^0_{\mathbf{1}}(\mathbb{T}^1, \mathbb{T}^1)$ with the Lipschitz constant $7\mathrm{e}/(6(\mathrm{e} - 1))$ and continuously dependent on the given F.

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Authors' addresses: Marek C. Zdun, Institute of Mathematics, Pedagogical University of Krakow, Podchorążych 2, 30-084 Krakow, Poland; Weinian Zhang, Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China, e-mail: `matzwn@126.com`, `matwnzhang@yahoo.com.cn`.