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WEAK HOMOGENEITY OF LATTICE ORDERED GROUPS

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Abstract. In this paper we deal with weakly homogeneous direct factors of lattice ordered groups. The main result concerns the case when the lattice ordered groups under consideration are archimedean, projectable and conditionally orthogonally complete.

Keywords: lattice ordered group, weak homogeneity, direct product, cardinal property, f-homogeneity

MSC 2000: 06F15

1. Introduction

A lattice ordered group is *weakly homogeneous* if whenever $a_1, b_i \in G$ $(i = 1, 2)$ and $a_1 < a_2$, $b_1 < b_2$, then $\operatorname{card}[a_1, b_1] = \operatorname{card}[a_2, b_2]$.

The weak homogeneity of Boolean algebras or of MV-algebras is defined analogously.

Weakly homogeneous direct factors of a complete lattice ordered group were investigated in [4].

Earlier, weak homogeneity of direct factors of a complete Boolean algebra was dealt with by Sikorski [11], §25.

The notion of weak homogeneity of a Boolean algebra is a particular case of f-homogeneity which is defined by means of a cardinal property f (cf. Pierce [9], [10]).

The above mentioned result of Sikorski [11] and a result of Pierce [9] on complete Boolean algebras were generalized in [8] to MV-algebras which are archimedean, projectable and orthogonally complete.

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In this context the natural question arises whether similar conditions enable one to generalize the results of [4] concerning complete lattice ordered groups to a broader class of lattice ordered groups.

We denote by $\mathscr{C}_0$ the class of all lattice ordered groups which are archimedean, projectable and conditionally orthogonally complete.

Each complete lattice ordered group belongs to $\mathscr{C}_0$, but not conversely. E.g., the additive group Q of all rationals with the natural linear order belongs to $\mathscr{C}_0$ and it fails to be complete. The same holds for any direct product of lattice ordered groups which are isomorphic to Q.

Let $\mathbb{R}$ be the additive group of all reals with the natural linear order. We denote by $\mathscr{G}_1$ the class of all lattice ordered groups G_1 such that G_1 is isomorphic to some ℓ-subgroup of $\mathbb{R}$.

In this paper we prove the following result.

(W) Let G be a lattice ordered group belonging to $\mathscr{C}_0$. Then G can be represented as a complete subdirect product of lattice ordered groups G_i ($i \in I$) such that for each $i \in I$, either $G_i \in \mathscr{G}_1$ or G_i is weakly homogeneous. If, moreover, G is orthogonally complete, then the representation turns out to be a direct product decomposition of G.

This generalizes a result of [4].

If G has a strong unit, then the assertion of (W) can be deduced from a result of [8] concerning MV-algebras; cf. Section 3.

If G is not assumed to have a strong unit the result of [8] cannot be applied and the proof is longer. We show that, similarly as in the case of Boolean algebras or MV-algebras, (W) is a consequence of a stronger result concerning f-homogeneity, where f is an increasing cardinal property. We generalize the main results of Section 1 in [4] dealing with f-homogeneity of complete lattice ordered groups to lattice ordered groups belonging to $\mathscr{C}_0$.

2. Preliminaries

The group operation in a lattice ordered group will be written additively (cf. Birkhoff [1] and Conrad [3]).

Let G be a lattice ordered group. G is *complete* if each nonempty upper-bounded subset of G possesses the supremum in G. In that case, also the corresponding dual condition is satisfied. An indexed system $(x_i)_{i\in I}$ of elements of G^+ is *orthogonal* if $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

G is (*conditionally*) *orthogonally complete* if each (upper-bounded) orthogonal system of elements of G has the supremum in G.

G is archimedean if, whenever $g_1, g_2 \in G^+$ and $ng_1 \leqslant g_2$ for each $n \in \mathbb{N}$, then $g_1 = 0$.

The direct product $\prod_{i \in I} G_i$ of lattice ordered groups G_i is defined in the usual way. If $I = \{1, 2, \ldots, n\}$, then we apply the notation $G_1 \times \ldots \times G_n$.

Let H_1 and H_2 be convex ℓ-subgroups of G. Assume that for each $g \in G$ there exist uniquely defined elements $h_1 \in H_1$, $h_2 \in H_2$ with $g = h_1 + h_2$ such that, whenever for $g' \in G$ we have the analogous representation $g' = h_1' + h_2'$, then

$$g \circ g' = (h_1 \circ h_1') + (h_2 \circ h_2')$$

for each operation $\circ \in \{+, \wedge, \vee\}$. In that case, the mapping $\varphi\colon g \to (h_1, h_2)$ is an isomorphism of G onto the direct product $H_1 \times H_2$. We call H_1 and H_2 *internal direct factors* of G; the mapping φ is an *internal direct product decomposition* of G. For $g \in G$ and $i \in \{1, 2\}$, h_i is the *component* of g in H_i.

Assume that $(H_i)_{i \in I}$ is an indexed system of internal direct factors of G. For $g \in G$ and $i \in I$ let g_i be the component of g in H_i. Suppose that the mapping

$$\psi\colon G \to \prod_{i \in I} H_i$$

defined by $\psi(g) = (g_i)_{i \in I}$ is an isomorphism of G onto $\prod_{i \in I} H_i$. Then ψ is called an internal direct product decomposition of G. (In the case $I = \{1, 2, \}$, this definition obviously coincides with that given above.) We often express this situation by writing

$$G = \prod_{i \in I} H_i. \tag{1}$$

More generally, let $\{H_i\}_{i \in I}$ be an indexed system of internal direct factors of a lattice ordered group G_0. For $g \in G_0$ put $\psi(g) = (g_i)_{i \in I}$ and suppose that

(i) $\psi(G_0)$ is an ℓ-subgroup of $\prod_{i \in I} H_i$;

(ii) ψ is an isomorphism of G_0 onto $\psi(G_0)$.

Then G_0 is said to be a complete subdirect product of lattice ordered groups H_i $(i \in I)$. In this situation we write $G_0 = (s) \prod_{i \in I} H_i$.

In particular, if (1) is valid and G_0 is a convex ℓ-subgroup of G such that $H_i \subseteq G_0$ for each $i \in I$, then G_0 is a complete subdirect product of H_i $(i \in I)$. The notion of the complete subdirect product of lattice ordered groups goes back to Šik [12].

We denote by $F(G)$ the system of all internal direct factors of a lattice ordered group G. The system $F(G)$ is partially ordered by the set-theoretical inclusion. It is well-known that $F(G)$ is a Boolean algebra.

For $X \subseteq G$ we put

$$X^\delta = \{g \in G\colon\ |g| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

X^δ is a *polar* of G.

G is projectable if for each one-element subset X of G, X^δ is an internal direct factor of G.

3. Unital lattice ordered groups

A lattice ordered group G is called unital if it has a strong unit. We will deal with a fixed strong unit u of G.

For MV-algebras we apply the notation as in the monograph [8].

For the notion of projectability of an MV-algebra cf. [6]. The orthogonal completeness of an MV-algebra is defined analogously as in the case of lattice ordered groups.

We denote by $\mathscr{C}$ the class of all MV-algebras which are archimedean, orthogonally complete and projectable. This class is studied in [8].

In the present section we apply the results from [5] concerning weak homogeneity of MV-algebras for investigating the weak homogeneity of unital lattice ordered groups.

Let G and u be as above. Consider the MV-algebra $\mathscr{A} = \Gamma(G, u)$.

The underlying set of $\mathscr{A}$ (i.e., the interval $[0, u]$ of G) will be denoted by A. We have $A = \{0\}$ if and only if $G = \{0\}$. For our purposes, this case is trivial. Thus we will suppose that $A \neq \{0\}$; we say that $\mathscr{A}$ is a nonzero MV-algebra.

Lemma 3.0. *Assume that $\mathscr{A}$ is an internal direct product $\prod\limits_{i\in I} \mathscr{A}_i$. For $i \in I$ let G_i be the ℓ-subgroup of G generated by the set A_i. Then $G = (s) \prod\limits_{i\in I} G_i$.*

Proof. Let $i \in I$. According to [7], G_i is an internal direct factor of G and for each $a \in A$, the component of a in $\mathscr{A}_i$ coincides with the component of a in G_i.

For $g \in G$ let g_i be the component of g in G_i. Hence the mapping $\varphi\colon\ g \to (g_i)_{i\in I}$ is a homomorphism of G into $\prod\limits_{i\in I} G_i = G'$ and $\varphi(G)$ is an ℓ-subgroup of G'.

If φ fails to be an isomorphism of G onto G' then there exists $0 < g \in G$ with $\varphi(g) = 0$. Put $a = g \wedge u$. Then $0 < a \in A$ and $\varphi(a) = 0$. Hence $a_i = 0$ for each $i \in I$. This yields $a = 0$, which is a contradiction. Therefore $g = (s) \prod\limits_{i\in I} G_i$. $\square$

It is well-known that G is archimedean if and only if $\mathscr{A}$ is archimedean (i.e., semisimple). Further, it is easy to verify that the following conditions are equivalent:

(i) G is conditionally orthogonally complete;
(ii) $\mathscr{A}$ is orthogonally complete.

Lemma 3.1 (Cf. [6]). *$\mathscr{A}$ is projectable if and only if G is projectable.*

Summarizing, we obtain

Lemma 3.2. *The lattice ordered group G belongs to $\mathscr{C}_0$ if and only if $\mathscr{A} \in \mathscr{C}$.*

Consider the following condition for $\mathscr{A}$:

($*$) For each $0 < a \in A$, $\operatorname{card}[0, a]$ is infinite.

Lemma 3.3 (Cf. [8], Theorem (C)). *The following conditions are equivalent:*

(i) *$\mathscr{A}$ is weakly homogeneous and satisfies the condition ($*$);*
(ii) *G is weakly homogeneous.*

Theorem 3.4 (Cf. [8]). *Let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Then $\mathscr{A}$ can be represented as an internal direct product $\prod\limits_{i\in I} \mathscr{A}_i$ such that for each $i \in I$, some of the following conditions is valid:*

(i) *$\mathscr{A}_i$ is weakly homogeneous;*
(ii) *$\mathscr{A}_i$ is a finite chain.*

Theorem 3.5. *Let G be a unital lattice ordered group belonging to the class $\mathscr{C}_0$. Then G can be represented as a complete subdirect product $(s) \prod\limits_{i\in I} G_i$ such that for each $i \in I$, some of the following conditions is satisfied:*

(i) *G_i is weakly homogeneous;*
(ii) *$G_i \simeq Z$.*

Proof. In view of the assumption, there exists a strong unit u in G. Put $\mathscr{A} = \Gamma(G, u)$. Since $G \in \mathscr{C}_0$, in view of 3.2 we have $\mathscr{A} \in \mathscr{C}$. Thus the assertion of 3.4 holds for $\mathscr{A}$. Let $\mathscr{A}_i$ $(i \in I)$ be as in 3.4 and let A_i be the underlying set of $\mathscr{A}_i$. We denote by G_i the ℓ-subgroup of G generated by A_i. Then for each $i \in I$ we have

$$\mathscr{A}_i = \Gamma(G_i, u_i), \tag{1}$$

where u_i is the greatest element of A_i.

From 3.4 and from 3.0 we obtain

$$G = (s)\prod_{i\in I} G_i.$$

Without loss of generality we can suppose that $A_i \neq \{0\}$ for each $i \in I$. If $i \in I$ and if A_i is finite, then in view of (1) we conclude that $G_i \simeq Z$.

Let $i \in I$ and assume that A_i is infinite. Then according to 3.4, $\mathscr{A}_i$ is weakly homogeneous. It is clear that in this case the condition $(*)$ must be satisfied. Hence in view of 3.3, G_i is weakly homogeneous. This completes the proof. $\square$

We have verified that for unital lattice ordered groups the assertion of (W) (in fact, a slightly stronger result) is valid.

4. Increasing cardinal properties

Assume that $\mathscr{C}_1$ is a nonempty class of lattice ordered groups which is closed with respect to isomorphisms. We denote by $\operatorname{Int}\mathscr{C}_1$ the class of all lattices L having the property that there exist $G \in \mathscr{C}_1$ and an interval $[a_1, a_2]$ of G such that $L \simeq [a_1, a_2]$.

Let f be a rule that assigns to each $L \in \operatorname{Int}\mathscr{C}_1$ a cardinal fL such that, whenever $L' \in \operatorname{Int}\mathscr{C}_1$ and $L' \simeq L$, then $fL' = fL$. We say that f is a cardinal property on the class $\mathscr{C}_1$.

For other types of ordered algebraic structures we can apply analogous definitions.

The cardinal property f is *increasing* (*decreasing*) if, whenever $L \in \operatorname{Int}\mathscr{C}_1$ and L_1 is a subinterval of L, then $fL_1 \leqslant fL_2$ (or $fL_1 \geqslant fL$, respectively).

A lattice ordered group $G \in \mathscr{C}_1$ is *f-homogeneous* if, whenever $a_i, b_i \in G$ $(i = 1, 2)$ and $a_1 < a_2$, $b_1 < b_2$, then $f[a_1, a_2] = f[b_1, b_2]$.

A lattice $L \in \operatorname{Int}\mathscr{C}_1$ is said to be f-homogeneous if for each subinterval L_1 of L with $\operatorname{card} L_1 > 1$ we have $fL_1 = fL$. Increasing cardinal properties on the class of complete lattice ordered groups were investigated in the author's paper [4].

Earlier, Pierce [9] studied increasing cardinal properties on the class of complete Boolean algebras. For the case of MV-algebras, cf. the author's paper [8].

In Section 6 we generalize some results of [4] concerning increasing cardinal properties on the class of complete lattice ordered groups for the larger class $\mathscr{C}_0$.

Let f be an increasing cardinal property on $\mathscr{C}_1$. We will use the following conditions for f:

(c_1) If $G \in \mathscr{C}_1$, $t_i \in G$, $0 < t_i$ $(i = 1, 2)$, $f[0, t_1] = f[0, t_2]$ and if $[0, t_1], [0, t_2]$ are f-homogeneous, then $f[0, t_1 + t_2] = f[0, t_1]$.

(c_2) If $G \in \mathscr{C}_1$, $t_n \in G$ $(n = 1, 2, \ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $\bigvee_{n\in\mathbb{N}} t_n = t$ and if all the intervals $[0, t_n]$ are f-homogeneous, then $f[0, t] = f[0, t_1]$.

Lemma 4.1. *Under the above notation, let f satisfy the condition* (c_1). *Let G, t_1 and t_2 be as in* (c_1). *Then the interval $[0, t_1 + t_2]$ of G is f-homogeneous.*

Proof. Denote $f[0, t_1] = \alpha$. Let $x_1, x_2 \in [0, t_1 + t_2]$, $x_1 < x_2$. We put $x = x_2 - x_1$. Since $[0, x] \simeq [x_1, x_2]$, we have $f[x_1, x_2] = f[0, x]$. From $0 < x \leqslant t_1 + t_2$ we get $f[0, x] \leqslant f[0, t_1 + t_2] = \alpha$. There exist $t_1', t_2' \in G$ such that $0 \leqslant t_i' \leqslant t_i$ for $i = 1, 2$ and $x = t_1' + t_2'$. Hence either $t_1' > 0$ or $t_2' > 0$. Suppose, e.g., that $0 < t_1'$. In view of the f-homogeneity of $[0, t_1]$ we get $f[0, t_1'] = \alpha$. Then $f[0, x] \geqslant f[0, t_1]$. Therefore $f[0, x] = \alpha$ and hence $f[x_1, x_2] = \alpha$. □

Lemma 4.2. *Under the above notation, let f satisfy the condition* (c_2). *Let G, t_n $(n \in \mathbb{N})$ and t be as in* (c_2). *Then the interval $[0, t]$ of G is f-homogeneous.*

Proof. Put $f[0, t_1] = \alpha$. Similarly as in the proof of 4.1 it suffices to verify that for each $0 < x \leqslant t$ we have $f[0, x] = \alpha$. From $t = \bigvee_{n \in \mathbb{N}} t_n$ we obtain

$$x = x \wedge t = \bigvee_{n \in \mathbb{N}} (x \wedge t_n).$$

Hence there exists $n \in \mathbb{N}$ with $x \wedge t_n > 0$. Since $[0, t_n]$ is f-homogeneous, we get $f[0, x \wedge t_n] = \alpha$ and thus $f[0, x] \geqslant \alpha$. On the other hand, from $x \leqslant t$ and from (c_2) we get $f[0, x] \leqslant f[0, t] = \alpha$, whence $f[0, x] = \alpha$. □

5. Auxiliary results

An element e of a lattice ordered group G is a *weak unit* of G if $e \wedge g > 0$ for each $0 < g \in G$.

Lemma 5.1. *Let G be an archimedean lattice ordered group and let e be a weak unit of G. Then*

$$\bigvee_{n=1}^{\infty} (ne \wedge g) = g \tag{1}$$

for each $0 \leqslant g \in G$.

Proof. Let us denote by G_1 the Dedekind completion of G. In view of 1.19 in [4], the relation (1) is valid in the lattice ordered group G_1. Since the elements ne and g belong to G, we infer that (1) holds also in G. □

We denote by $a(G)$ the set of all elements $0 < a \in G$ such that the interval $[0, a]$ of G is a chain.

Lemma 5.2 (Cf. [5]). *Let G be an archimedean lattice ordered group.*

(i) *For each $a_1 \in a(G)$ there exists an element $G(a_1)$ of $F(G)$ such that $a_1 \in G(a_1)$ and $G(a_1)$ is linearly ordered.*

(ii) *Let $a_1, a_2 \in a(G)$. Then either $G(a_1) = G(a_2)$ or $G(a_1) \cap G(a_2) = \{0\}$.*

Lemma 5.3. *Let G be a lattice ordered group and $0 < e \in G$. Then e is a weak unit of the lattice ordered group $\{e\}^{\delta\delta}$.*

Proof. Let $0 < g \in \{e\}^{\delta\delta}$. If $a \wedge g = 0$, then $g \in \{a\}^{\delta}$. Since $\{e\}^{\delta} \cap \{e\}^{\delta\delta} = \{0\}$, we obtain $g = 0$, which is a contradiction. $\square$

Lemma 5.4. *Let G be an archimedean lattice ordered group and let $0 < e \in G$. Let f be an increasing cardinal property on the class of all archimedean lattice ordered groups satisfying the conditions* (c_1) *and* (c_2). *Assume that the interval $[0, e]$ is f-homogeneous. Then the lattice ordered group $\{e\}^{\delta\delta}$ is f-homogeneous.*

Proof. By applying 4.1 and induction we obtain that for each $n \in \mathbb{N}$, the interval $[0, ne]$ is f-homogeneous. Put $f[0, g] = \alpha$. It suffices to verify that for each $0 < g \in \{e\}^{\delta\delta}$ we have $f[0, g] = \alpha$.

In view of 5.3 and 5.1, the relation (1) is valid. Further, $0 < ne \wedge g$ for each $n \in \mathbb{N}$. Thus $f[ne \wedge g] = \alpha$ for each $n \in \mathbb{N}$. From 4.2 we infer that $f[0, g] = \alpha$. $\square$

An indexed system $(G_i)_{i \in I}$ of elements of $F(G)$ is *orthogonal* if $G_{i(1)} \cap G_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

Assume that the lattice ordered group G is conditionally orthogonally complete and that $(G_i)_{i \in I}$ is an orthogonal indexed system of elements of $F(G)$. Let H_0 be the set of all elements h of G^+ which can be expressed in the form

$$h = \bigvee_{i \in I} h^i, \tag{2}$$

where $h^i \in G_i^+$ for each $i \in I$.

Lemma 5.5. *H_0 is an ideal of the lattice G^+; further, H_0 is closed with respect to the operation $+$.*

Proof. a) Let h be as above. Analogously, let $h_1 = \bigvee_{i \in I} h_1^i$, $h_1^i \in G_i^+$. Then we have

$$h \vee h_1 = \bigvee_{i \in I} (h^i \vee h_1^i)$$

with $h^i \vee h_1^i \in G_i^+$. Thus $h \vee h_1 \in H_0$.

b) Let h be as above and $h_2 \in G^+$, $h_2 \leqslant h$. Then

$$h_2 = h_2 \wedge h = \bigvee_{i \in I} (h_2 \wedge h^i)$$

and $h_2 \wedge h^i \in G_i^+$ for each $i \in I$. Hence $h_2 \in H_0$. We have verified that H_0 is an ideal of G^+.

c) Let h and h_1 be as in a). Then

$$h + h_1 = \Big(\bigvee_{i \in I} h^i\Big) + \Big(\bigvee_{j \in I} h_1^j\Big) = \bigvee_{i \in I} \bigvee_{j \in I} (h^i + h_1^j).$$

If $i = j$, then $h^i + h_1^j \in G_i$. If $i \neq j$, then $h^i \wedge h_1^j = 0$, whence $h^i + h_1^j = h^i \vee h_1^j$. Therefore

$$h + h_1 = \bigvee_{i \in I} (h^i + h_1^i)$$

and thus $h + h_1 \in H_0$. □

We denote by H_1 the set of all $g \in G$ having the property that there exist $h, h_1 \in H_0$ with $-h \leqslant g \leqslant h_1$. By a simple calculation we obtain from 5.5

Lemma 5.6. *H_1 is a convex ℓ-subgroup of G.*

Lemma 5.7. *Let $0 < g \in G$. Then the set $\{h \in H_0 \colon h \leqslant g\}$ has the greatest element.*

Proof. For $i \in I$ let g_i be the component of g in G_i. Hence $g_i \in G_i^+$ for each $i \in I$ and the indexed system $(g_i)_{i \in I}$ is orthogonal. Also, $g_i \leqslant g$ for each $i \in I$. Thus there exists $g_0 = \bigvee_{i \in I} g_i$ in G. In view of the definition of H_0 we have $g_0 \in H_0$. Clearly, $g_0 \leqslant g$. We want to show that g_0 is the greatest element of the set $\{h \in H_0 \colon h \leqslant g\} = K$.

By way of contradiction, assume that g_0 fails to be the greatest element of the set K. Then there exists $k \in K$ with $g_0 < k$.

If (2) is valid and $i \in I$, then the component h_i of h in G_i is equal to h^i. Hence $(g_0)_i = g_i$ for each $i \in I$. Since $g_0 < k$, there exists $i(1) \in I$ such that $(g_0)_{i(1)} < k_{i(1)}$. Further, from $k \in K$ we get $k \leqslant g$, whence $k_{i(1)} \leqslant g_{i(1)}$. We obtain $g_{i(1)} < k_{i(1)} \leqslant g_{i(1)}$, which is a contradiction. □

From 5.6 and 5.7 we infer

Lemma 5.8. *H_1 is an internal direct factor of G.*

For each $i \in I$ we have $G_i \subseteq H_1$. If $H_2 \in F(G)$ is such that $G_i \subseteq H_2$ for each $i \in I$, then from the definition of H_0 we get $H_0 \subseteq H_2$; this yields that $H_1 \subseteq H_2$. Therefore the relation

$$H_1 = \bigvee_{i \in I} G_i \tag{3}$$

is valid in $F(G)$. Thus each orthogonal indexed system of $F(G)$ has the sumpremum in $F(G)$. This property will be called, similarly as for lattice ordered groups, the *orthogonal completeness* of $F(G)$.

It is well-known that each orthogonally complete Boolean algebra is complete. Thus we have

Theorem 5.9. *If G is a conditionally orthogonally complete lattice ordered group, then the Boolean algebra $F(G)$ is complete.*

Lemma 5.10. *Let $(G_i)_{i \in I}$ be as above and let H_1 be as in* 5.8. *Then $H_1 = (s) \prod_{i \in I} G_i$.*

Proof. This is a consequence of 5.6 and of the fact that $G_i \subseteq H_1$ for each $i \in I$. □

6. f-homogeneous direct factors

In this section we assume that $G \neq \{0\}$ is a lattice ordered group belonging to $\mathscr{C}_0$ and that f is an increasing cardinal property on the class $\mathscr{C}$ such that the conditions (c_1) and (c_2) are satisfied.

We recall the notation introduced in [4].

Let $\mathscr{A}$ be the set of all cardinals α such that $f[a, b] = \alpha$ for some non-trivial interval $[a, b]$ of G. For any $\alpha \in \mathscr{A}$ we put

$$X_\alpha = \{x \in G \colon\ x > 0, f[0, x] \leqslant \alpha\} \cup \{0\};$$
$$Y_\alpha = \{y \in G \colon\ y > 0, f[0, y] < \alpha\} \cup \{0\};$$
$$Z_\alpha = (Y_\alpha)^\delta, \quad A_\alpha = X_\alpha \cap Z_\alpha.$$

Further, we set

$$B_\alpha = \{g \in G \colon\ -t_1 \leqslant g \leqslant t_2 \quad \text{for some } t_1, t_2 \in A\},$$
$$\bar{A}_\alpha = \Big\{g \in G \colon\ g = \bigvee_{j \in J} t_j \quad \text{for some } \{t_j\}_{j \in J} \subseteq A_\alpha\Big\},$$
$$\overline{B}_\alpha = \{g \in G \colon\ -t_1 \leqslant g \leqslant t_2 \quad \text{for some } t_1, t_2 \in \bar{A}_\alpha\}.$$

Though the main results of Section 1 in [4] are formulated for complete lattice ordered groups several auxiliary results proved in that section remain valid without the assumption of completeness. We will freely use such results in the present paper.

Lemma 6.1 (Cf. [4], 1.4). *Let $\alpha \in \mathscr{A}$. Then B_α is an ℓ-ideal of G and $f[a,b] = \alpha$ for each non-trivial interval of B_α. If $\beta \in \mathscr{A}$, $\beta \neq \alpha$, then $B_\alpha \cap B_\beta = \{0\}$.*

Lemma 6.2 (Cf. [4], 1.7.1). *Let $\alpha \in \mathscr{A}$. Then $\overline{B}_\alpha$ is an ℓ-ideal of G. If $\beta \in \mathscr{A}$, $\beta \neq \alpha$, then $\overline{B}_\alpha \cap \overline{B}_\beta = \{0\}$.*

Let H be a lattice ordered group and let $\{h_i\}_{i \in I}$ be an orthogonal subset of H such that

(i) $0 < h_i$ for each $i \in I$;
(ii) if $h \in H$ and $h \wedge h_i = 0$ for each $i \in I$, then $h = 0$.

Under these conditions we say that $\{h_i\}_{i \in I}$ is a maximal orthogonal subset of H.

From the Axiom of Choice it follows that if $H \neq \{0\}$, then there exists a maximal orthogonal system in H.

In view of [4], p. 91 we have

Lemma 6.3. *Let $\{a_i\}_{i \in I}$ be a maximal orthogonal subset of B_α. Then $\{a_i\}_{i \in I}$ is a maximal orthogonal subset of $\overline{B}_\alpha$.*

Let $(a_i)_{i \in I}$ be as in 6.3. For each $i \in I$ we put $G_i = \{a_i\}^{\delta\delta}$. From 6.3 we obtain that the indexed system $(G_i)_{i \in I}$ is orthogonal.

Because G belongs to $\mathscr{C}_0$ it is projectable and hence $G_i \in F(G)$ for each $i \in I$. Let H_1 be as in Section 5. In view of 5.8, H_1 is an element of $F(G)$; moreover, by virtue of the relation (3) in Section 5, H_1 is the join of the system $(G_i)_{i \in I}$ in $F(G)$.

Consider the convex ℓ-subgroups H_1 and $\overline{B}_\alpha$ of G.

Lemma 6.4. $\overline{B}_\alpha = H_1$.

P r o o f. In view of the definitions of $\overline{B}_\alpha$ and H_1 we have $H_1 \subseteq \overline{B}_\alpha$.

By way of contradiction, assume that $H_1 \subset \overline{B}_\alpha$. Then there exists $0 < b \in \overline{B}_\alpha$ such that $b \notin H_1$. Since $H_1 \in F(G)$ there is H_1' in $F(G)$ such that $G = H_1 \times H_1'$. From the relation $b \notin H_1$ we obtain $b(H_1') > 0$. Clearly $b(H_1') \leqslant b$, hence $b(H_1') \in \overline{B}_\alpha$. Then $b(H_1') \wedge h_1 = 0$ for each $0 < h_1 \in H_1$. In particular, $b(H_1') \wedge a_i = 0$ for each $i \in I$. According to 6.3, we have arrived at a contradiction. □

From 6.4 and 5.10 we get

Corollary 6.5. *For each $\alpha \in \mathscr{A}$, $\overline{B}_\alpha \in F(G)$. Moreover, under the above notation, $\overline{B}_\alpha$ is a complete subdirect product of lattice ordered groups G_i ($i \in I$).*

Let $\alpha \in \mathscr{A}$ and $g \in G^+$. In view of 6.5, there exists the component $g(\overline{B}_\alpha) = g_\alpha$ of g in $\overline{B}_\alpha$. It is easy to verify that

$$g_\alpha = \sup\{0 < t \in \overline{B}_\alpha \colon\ t \leqslant g\}.$$

Since $(\overline{B}_\alpha)^+ = \bar{A}_\alpha$, we have also

$$g_\alpha = \sup\{0 < t \in \bar{A}_\alpha \colon\ t \leqslant g\}. \tag{1}$$

Now we apply to $G \in \mathscr{C}_0$ the argument from 1.8–1.15 in [4]. (The completeness of G was used only in 1.10; at that place it was applied for showing that G is abelian. In our present case, the commutativity of G is a consequence of the archimedean property.)

Hence, looking at 1.15 in [4] we obtain

Theorem 6.6. *Let $G \neq \{0\}$ be a lattice ordered group belonging to the class $\mathscr{C}_0$. Assume that f is a cardinal property on $\mathscr{C}_0$ satisfying the conditions* (c_1) *and* (c_2). *For $\alpha \in \mathscr{A}$ let $\overline{\mathscr{B}}_\alpha$ be as above. Then $G = (s) \prod\limits_{\alpha \in \mathscr{A}} \overline{B}_\alpha$. If, moreover, G is orthogonally complete, then G is an internal direct product of lattice ordered groups $\overline{\mathscr{B}}_\alpha$ ($\alpha \in \mathscr{A}$).*

This generalizes Theorem 1.15 of [4].

Let us now slightly modify the notation applied in 6.4 and in 5.10. In 6.4, we write now H_1^α instead of H_1. Analogously, in 5.10 we write G_i^α instead of G_i and $I(\alpha)$ instead of I; we get

Lemma 6.7. *Let $\alpha \in \mathscr{A}$. Then $\overline{B}_\alpha$ is a complete subdirect product of lattice ordered groups G_i^α ($i \in I(\alpha)$).*

In view of 6.6 and 6.7, for $G \in \mathscr{C}_0$ we obtain

$$G = (s) \prod_{\alpha \in \mathscr{A}} \prod_{i \in I(\alpha)} G_i^\alpha. \tag{2}$$

Lemma 6.8. *For each $\alpha \in \mathscr{A}$ and each $i \in I(\alpha)$, the lattice ordered group G_i^α is f-homogeneous.*

Proof. Let $\alpha \in \mathscr{A}$ and $i \in I$. In view of the definition of G_i^α, there exists $a_i^\alpha \in A_\alpha$ such that

$$G_i^\alpha = \{a_i^\alpha\}^{\delta\delta}.$$

According to the assertion 1.1 of [4], the interval $[0, a_i^\alpha]$ of G is f-homogeneous. Hence 5.4 yields that G_i^α is f-homogeneous. □

From 6.6, (2) and 6.8 we obtain

Theorem 6.9. *Let $G \neq \{0\}$ be a lattice ordered group belonging to the class $\mathscr{C}_0$. Assume that f is a cardinal property on $\mathscr{C}_0$ satisfying the conditions* (c_1) *and* (c_2). *Then G can be prepresented as a complete subdirect product* (2), *where all factors G_i^α are f-homogeneous. If, moreover, G is orthogonally complete, then* (2) *is an internal direct product decomposition of G.*

This generalizes theorem 1.21 of [4].

7. Weakly homogeneous direct factors

In this section we apply the results of Section 6 for dealing with weak homogeneity of lattice ordered groups which belong to $\mathscr{C}_0$ and are not assumed to be unital.

Again, assume that $G \neq \{0\}$ is a lattice ordered group belonging to $\mathscr{C}_0$. Let $\mathscr{G}_1$ be as in Section 1.

Lemma 7.1. *Let $G \in \mathscr{C}_0$. Then G can be expressed as an internal direct product $A \times B$ such that*

(i) *A is the complete subdirect product of linearly ordered groups belonging to $\mathscr{G}_1$;*
(ii) *if $0 < b \in B$, then the interval $[0, b]$ of B fails to be a chain.*

Proof. This is a consequence of 5.2, 5.8 and 5.10. □

From 7.1 we conclude that for proving the assertion (W) of Section 1 it suffices to deal with the lattice ordered group B. If $B = \{0\}$, then the assertion of (W) is valid for G; assume that $B \neq \{0\}$.

In view of 7.1 we have $\operatorname{card}[0, b] \geqslant \aleph_0$ for each $0 < b \in B$. Moreover, it is easy to verify that $[0, b]$ contains an infinite orthogonal subset.

In [4], Section 3 the cardinal function f_3 was considered on the class of all bounded lattices defined by

$$f_3[a,b] = \max\{\mathrm{card}[a,b], \aleph_0\}$$

for any non-trivial interval $[a,b]$; in the case $a = b$ we put $f_3[a,b] = 0$. We can apply f_3 to the class $\mathscr{C}_0$. If $[a,b]$ is a nontrivial interval of B, then we have

$$f_3[a,b] = \mathrm{card}[a,b]. \tag{1}$$

Lemma 7.2. *Let $0 < b \in B$ and assume that the interval $[0,b]$ is f_3-homogeneous, $f_3[0,b] = \alpha$. Then $\alpha^{\aleph_0} = \alpha$.*

Proof. Since the interval $[0,b]$ has an infinite orthogonal subset we can apply the argument used in the proof of 3.5 in [4]. □

We say that f_3 satisfies (c_2) with regard to B if, whenever all the elements considered in (c_2) belong to B, then the assertion of (c_2) is valid.

Lemma 7.3. *f_3 satisfies the condition (c_2) with regard to B.*

Proof. It suffices to apply 7.2 and the argument applied in the proof of 3.6 from [4]. □

According to 3.1 in [4], f_3 satisfies the condition (c_1). From this and from 7.3 we conclude that we can apply the assertion of 6.9 to the lattice ordered group G and to the cardinal property f_3. In view of (1), for intervals of B the f_3-homogeneity is the same as weak homogeneity. Further, if G is orthogonally complete, then B is orthogonally complete as well. Hence we have

Theorem 7.4. *Let $G \in \mathscr{C}_0$ and let B be as in 7.1. Then B can be represented as a complete subdirect product of weakly homogeneous lattice ordered groups. If, moreover, G is orthogonally complete, then the just mentioned complete subdirect product turns out to be an internal direct product.*

The assertion (W) of Section 1 is a consequence of 7.1 and 7.4.

We remark that if G is a complete lattice ordered group and if G_1 is some of linearly ordered groups mentioned in the assertion (i) of 7.1, then G_1 is complete, thus $G_1 \in \{\{0, \mathbb{Z}, \mathbb{R}\}$. In the cases $G_1 = \{0\}$ or $G_1 = \mathbb{R}$ we obtain that G_1 is weakly homogeneous. Hence 7.2 and 7.4 yield a generalization of Theorem 3.7 of [4].

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