Ján Jakubík
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WEAK HOMOGENEITY OF LATTICE ORDERED GROUPS

Ján Jakubík, Košice

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Abstract. In this paper we deal with weakly homogeneous direct factors of lattice ordered groups. The main result concerns the case when the lattice ordered groups under consideration are archimedean, projectable and conditionally orthogonally complete.

Keywords: lattice ordered group, weak homogeneity, direct product, cardinal property, $f$-homogeneity

MSC 2000: 06F15

1. Introduction

A lattice ordered group is weakly homogeneous if whenever $a_1, b_1 \in G$ ($i = 1, 2$) and $a_1 < a_2$, $b_1 < b_2$, then $\text{card}[a_1, b_1] = \text{card}[a_2, b_2]$.

The weak homogeneity of Boolean algebras or of $MV$-algebras is defined analogously.

Weakly homogeneous direct factors of a complete lattice ordered group were investigated in [4].

Earlier, weak homogeneity of direct factors of a complete Boolean algebra was dealt with by Sikorski [11], §25.

The notion of weak homogeneity of a Boolean algebra is a particular case of $f$-homogeneity which is defined by means of a cardinal property $f$ (cf. Pierce [9], [10]).

The above mentioned result of Sikorski [11] and a result of Pierce [9] on complete Boolean algebras were generalized in [8] to $MV$-algebras which are archimedean, projectable and orthogonally complete.

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In this context the natural question arises whether similar conditions enable one to generalize the results of [4] concerning complete lattice ordered groups to a broader class of lattice ordered groups.

We denote by $C_0$ the class of all lattice ordered groups which are archimedean, projectable and conditionally orthogonally complete.

Each complete lattice ordered group belongs to $C_0$, but not conversely. E.g., the additive group $Q$ of all rationals with the natural linear order belongs to $C_0$ and it fails to be complete. The same holds for any direct product of lattice ordered groups which are isomorphic to $Q$.

Let $\mathbb{R}$ be the additive group of all reals with the natural linear order. We denote by $\mathcal{G}_1$ the class of all lattice ordered groups $G_1$ such that $G_1$ is isomorphic to some $\ell$-subgroup of $\mathbb{R}$.

In this paper we prove the following result.

(W) Let $G$ be a lattice ordered group belonging to $C_0$. Then $G$ can be represented as a complete subdirect product of lattice ordered groups $G_i$ $(i \in I)$ such that for each $i \in I$, either $G_i \in \mathcal{G}_1$ or $G_i$ is weakly homogeneous. If, moreover, $G$ is orthogonally complete, then the representation turns out to be a direct product decomposition of $G$.

This generalizes a result of [4].

If $G$ has a strong unit, then the assertion of (W) can be deduced from a result of [8] concerning $MV$-algebras; cf. Section 3.

If $G$ is not assumed to have a strong unit the result of [8] cannot be applied and the proof is longer. We show that, similarly as in the case of Boolean algebras or $MV$-algebras, (W) is a consequence of a stronger result concerning $f$-homogeneity, where $f$ is an increasing cardinal property. We generalize the main results of Section 1 in [4] dealing with $f$-homogeneity of complete lattice ordered groups to lattice ordered groups belonging to $C_0$.

2. Preliminaries

The group operation in a lattice ordered group will be written additively (cf. Birkhoff [1] and Conrad [3]).

Let $G$ be a lattice ordered group. $G$ is complete if each nonempty upper-bounded subset of $G$ possesses the supremum in $G$. In that case, also the corresponding dual condition is satisfied. An indexed system $(x_i)_{i \in I}$ of elements of $G^+$ is orthogonal if $x_{i(1)} \land x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of $I$.

$G$ is (conditionally) orthogonally complete if each (upper-bounded) orthogonal system of elements of $G$ has the supremum in $G$. 850
$G$ is archimedean if, whenever $g_1, g_2 \in G^+$ and $ng_1 \leq g_2$ for each $n \in \mathbb{N}$, then $g_1 = 0$.

The direct product $\prod_{i \in I} G_i$ of lattice ordered groups $G_i$ is defined in the usual way. If $I = \{1, 2, \ldots, n\}$, then we apply the notation $G_1 \times \cdots \times G_n$.

Let $H_1$ and $H_2$ be convex $\ell$-subgroups of $G$. Assume that for each $g \in G$ there exist uniquely defined elements $h_1 \in H_1$, $h_2 \in H_2$ with $g = h_1 + h_2$ such that, whenever for $g' \in G$ we have the analogous representation $g' = h_1' + h_2'$, then

$$g \circ g' = (h_1 \circ h_1' + (h_2 \circ h_2')$$

for each operation $\circ \in \{+, \wedge, \vee\}$. In that case, the mapping $\varphi: g \rightarrow (h_1, h_2)$ is an isomorphism of $G$ onto the direct product $H_1 \times H_2$. We call $H_1$ and $H_2$ internal direct factors of $G$; the mapping $\varphi$ is an internal direct product decomposition of $G$. For $g \in G$ and $i \in \{1, 2\}$, $h_i$ is the component of $g$ in $H_i$.

Assume that $(H_i)_{i \in I}$ is an indexed system of internal direct factors of $G$. For $g \in G$ and $i \in I$ let $g_i$ be the component of $g$ in $H_i$. Suppose that the mapping

$$\psi: G \rightarrow \prod_{i \in I} H_i$$

defined by $\psi(g) = (g_i)_{i \in I}$ is an isomorphism of $G$ onto $\prod_{i \in I} H_i$. Then $\psi$ is called an internal direct product decomposition of $G$. (In the case $I = \{1, 2\}$, this definition obviously coincides with that given above.) We often express this situation by writing

$$G = \prod_{i \in I} H_i.$$  \hspace{1cm} (1)

More generally, let $\{H_i\}_{i \in I}$ be an indexed system of internal direct factors of a lattice ordered group $G_0$. For $g \in G_0$ put $\psi(g) = (g_i)_{i \in I}$ and suppose that

(i) $\psi(G_0)$ is an $\ell$-subgroup of $\prod_{i \in I} H_i$;

(ii) $\psi$ is an isomorphism of $G_0$ onto $\psi(G_0)$.

Then $G_0$ is said to be a complete subdirect product of lattice ordered groups $H_i$ ($i \in I$). In this situation we write $G_0 = (s) \prod_{i \in I} H_i$.

In particular, if (1) is valid and $G_0$ is a convex $\ell$-subgroup of $G$ such that $H_i \subseteq G_0$ for each $i \in I$, then $G_0$ is a complete subdirect product of $H_i$ ($i \in I$). The notion of the complete subdirect product of lattice ordered groups goes back to Šik [12].

We denote by $F(G)$ the system of all internal direct factors of a lattice ordered group $G$. The system $F(G)$ is partially ordered by the set-theoretical inclusion. It is well-known that $F(G)$ is a Boolean algebra.
For \( X \subseteq G \) we put
\[
X^\delta = \{ g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X \}.
\]

\( X^\delta \) is a polar of \( G \).

\( G \) is projectable if for each one-element subset \( X \) of \( G \), \( X^\delta \) is an internal direct factor of \( G \).

### 3. Unital Lattice Ordered Groups

A lattice ordered group \( G \) is called unital if it has a strong unit. We will deal with a fixed strong unit \( u \) of \( G \).

For \( MV \)-algebras we apply the notation as in the monograph [8].

For the notion of projectability of an \( MV \)-algebra cf. [6]. The orthogonal completeness of an \( MV \)-algebra is defined analogously as in the case of lattice ordered groups.

We denote by \( \mathcal{C} \) the class of all \( MV \)-algebras which are archimedean, orthogonally complete and projectable. This class is studied in [8].

In the present section we apply the results from [5] concerning weak homogeneity of \( MV \)-algebras for investigating the weak homogeneity of unital lattice ordered groups.

Let \( G \) and \( u \) be as above. Consider the \( MV \)-algebra \( \mathcal{A} = \Gamma(G, u) \).

The underlying set of \( \mathcal{A} \) (i.e., the interval \([0, u]\) of \( G \)) will be denoted by \( A \). We have \( A = \{0\} \) if and only if \( G = \{0\} \). For our purposes, this case is trivial. Thus we will suppose that \( A \neq \{0\} \); we say that \( \mathcal{A} \) is a nonzero \( MV \)-algebra.

**Lemma 3.0.** Assume that \( \mathcal{A} \) is an internal direct product \( \prod_{i \in I} \mathcal{A}_i \). For \( i \in I \) let \( G_i \) be the \( \ell \)-subgroup of \( G \) generated by the set \( A_i \). Then \( G = (s) \prod_{i \in I} G_i \).

**Proof.** Let \( i \in I \). According to [7], \( G_i \) is an internal direct factor of \( G \) and for each \( a \in A \), the component of \( a \) in \( \mathcal{A}_i \) coincides with the component of \( a \) in \( G_i \).

For \( g \in G \) let \( g_i \) be the component of \( g \) in \( G_i \). Hence the mapping \( \varphi : g \mapsto (g_i)_{i \in I} \) is a homomorphism of \( G \) into \( \prod_{i \in I} G_i = G' \) and \( \varphi(G) \) is an \( \ell \)-subgroup of \( G' \).

If \( \varphi \) fails to be an isomorphism of \( G \) onto \( G' \) then there exists \( 0 < g \in G \) with \( \varphi(g) = 0 \). Put \( a = g \wedge u \). Then \( 0 < a \in A \) and \( \varphi(a) = 0 \). Hence \( a_i = 0 \) for each \( i \in I \). This yields \( a = 0 \), which is a contradiction. Therefore \( g = (s) \prod_{i \in I} G_i \). \( \square \)
It is well-known that $G$ is archimedean if and only if $A$ is archimedean (i.e., semisimple). Further, it is easy to verify that the following conditions are equivalent:

(i) $G$ is conditionally orthogonally complete;
(ii) $A$ is orthogonally complete.

**Lemma 3.1** (Cf. [6]). $A$ is projectable if and only if $G$ is projectable.

Summarizing, we obtain

**Lemma 3.2.** The lattice ordered group $G$ belongs to $C_0$ if and only if $A \in C$.

Consider the following condition for $A$:

(*) For each $0 < a \in A$, card$[0, a]$ is infinite.

**Lemma 3.3** (Cf. [8], Theorem (C)). The following conditions are equivalent:

(i) $A$ is weakly homogeneous and satisfies the condition ($*$);
(ii) $G$ is weakly homogeneous.

**Theorem 3.4** (Cf. [8]). Let $A$ be an MV-algebra belonging to the class $C$. Then $A$ can be represented as an internal direct product $\prod_{i \in I} A_i$ such that for each $i \in I$, some of the following conditions is valid:

(i) $A_i$ is weakly homogeneous;
(ii) $A_i$ is a finite chain.

**Theorem 3.5.** Let $G$ be a unital lattice ordered group belonging to the class $C_0$. Then $G$ can be represented as a complete subdirect product $(s) \prod_{i \in I} G_i$ such that for each $i \in I$, some of the following conditions is satisfied:

(i) $G_i$ is weakly homogeneous;
(ii) $G_i \simeq Z$.

**Proof.** In view of the assumption, there exists a strong unit $u$ in $G$. Put $A = \Gamma(G, u)$. Since $G \in C_0$, in view of 3.2 we have $A \in C$. Thus the assertion of 3.4 holds for $A$. Let $A_i$ ($i \in I$) be as in 3.4 and let $A_i$ be the underlying set of $A_i$. We denote by $G_i$ the $\ell$-subgroup of $G$ generated by $A_i$. Then for each $i \in I$ we have

(1) $A_i = \Gamma(G_i, u_i),$

where $u_i$ is the greatest element of $A_i$.  

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From 3.4 and from 3.0 we obtain

\[ G = (s) \prod_{i \in I} G_i. \]

Without loss of generality we can suppose that \( A_i \neq \{0\} \) for each \( i \in I \). If \( i \in I \) and if \( A_i \) is finite, then in view of (1) we conclude that \( G_i \simeq Z \).

Let \( i \in I \) and assume that \( A_i \) is infinite. Then according to 3.4, \( A_i \) is weakly homogeneous. It is clear that in this case the condition \( (\ast) \) must be satisfied. Hence in view of 3.3, \( G_i \) is weakly homogeneous. This completes the proof. \( \square \)

We have verified that for unital lattice ordered groups the assertion of (W) (in fact, a slightly stronger result) is valid.

4. Increasing cardinal properties

Assume that \( \mathcal{C}_1 \) is a nonempty class of lattice ordered groups which is closed with respect to isomorphisms. We denote by \( \text{Int} \mathcal{C}_1 \) the class of all lattices \( L \) having the property that there exist \( G \in \mathcal{C}_1 \) and an interval \([a_1, a_2]\) of \( G \) such that \( L \simeq [a_1, a_2] \).

Let \( f \) be a rule that assigns to each \( L \in \text{Int} \mathcal{C}_1 \) a cardinal \( fL \) such that, whenever \( L' \in \text{Int} \mathcal{C}_1 \) and \( L' \simeq L \), then \( fL' = fL \). We say that \( f \) is a cardinal property on the class \( \mathcal{C}_1 \).

For other types of ordered algebraic structures we can apply analogous definitions.

The cardinal property \( f \) is increasing (decreasing) if, whenever \( L \in \text{Int} \mathcal{C}_1 \) and \( L_1 \) is a subinterval of \( L \), then \( fL_1 \leq fL_2 \) (or \( fL_1 \geq fL \), respectively).

A lattice ordered group \( G \in \mathcal{C}_1 \) is \( f \)-homogeneous if, whenever \( a_i, b_i \in G \) \((i = 1, 2)\) and \( a_1 < a_2, b_1 < b_2 \), then \( f[a_1, a_2] = f[b_1, b_2] \).

A lattice \( L \in \text{Int} \mathcal{C}_1 \) is said to be \( f \)-homogeneous if for each subinterval \( L_1 \) of \( L \) with \( \text{card} L_1 > 1 \) we have \( fL_1 = fL \). Increasing cardinal properties on the class of complete lattice ordered groups were investigated in the author’s paper [4].

Earlier, Pierce [9] studied increasing cardinal properties on the class of complete Boolean algebras. For the case of \( MV \)-algebras, cf. the author’s paper [8].

In Section 6 we generalize some results of [4] concerning increasing cardinal properties on the class of complete lattice ordered groups for the larger class \( \mathcal{C}_0 \).

Let \( f \) be an increasing cardinal property on \( \mathcal{C}_1 \). We will use the following conditions for \( f \):

\((c_1)\) If \( G \in \mathcal{C}_1, t_i \in G, 0 < t_i \ (i = 1, 2), f[0, t_1] = f[0, t_2] \) and if \([0, t_1], [0, t_2] \) are \( f \)-homogeneous, then \( f[0, t_1 + t_2] = f[0, t_1] \).

\((c_2)\) If \( G \in \mathcal{C}_1, t_n \in G \ (n = 1, 2, \ldots), 0 < t_1 \leq t_2 \leq \ldots, \bigvee_{n \in \mathbb{N}} t_n = t \) and if all the intervals \([0, t_n] \) are \( f \)-homogeneous, then \( f[0, t] = f[0, t_1] \).
Lemma 4.1. Under the above notation, let $f$ satisfy the condition (c1). Let $G, t_1$ and $t_2$ be as in (c1). Then the interval $[0, t_1 + t_2]$ of $G$ is $f$-homogeneous.

Proof. Denote $f[0, t_1] = \alpha$. Let $x_1, x_2 \in [0, t_1 + t_2]$, $x_1 < x_2$. We put $x = x_2 - x_1$. Since $[0, x] \simeq [x_1, x_2]$, we have $f[x_1, x_2] = f[0, x]$. From $0 < x \leq t_1 + t_2$ we get $f[0, x] \leq f[0, t_1 + t_2] = \alpha$. There exist $t_1', t_2' \in G$ such that $0 \leq t_i' \leq t_i$ for $i = 1, 2$ and $x = t_1' + t_2'$. Hence either $t_1' > 0$ or $t_2' > 0$. Suppose, e.g., that $0 < t_1'$. In view of the $f$-homogeneity of $[0, t_1]$ we get $f[0, t_1'] = \alpha$. Then $f[0, x] \geq f[0, t_1]$. Therefore $f[0, x] = \alpha$ and hence $f[x_1, x_2] = \alpha$. □

Lemma 4.2. Under the above notation, let $f$ satisfy the condition (c2). Let $G, t_n \ (n \in \mathbb{N})$ and $t$ be as in (c2). Then the interval $[0, t]$ of $G$ is $f$-homogeneous.

Proof. Put $f[0, t_1] = \alpha$. Similarly as in the proof of 4.1 it suffices to verify that for each $0 < x \leq t$ we have $f[0, x] = \alpha$. From $t = \bigvee_{n \in \mathbb{N}} t_n$ we obtain

$$x = x \land t = \bigvee_{n \in \mathbb{N}} (x \land t_n).$$

Hence there exists $n \in \mathbb{N}$ with $x \land t_n > 0$. Since $[0, t_n]$ is $f$-homogeneous, we get $f[0, x \land t_n] = \alpha$ and thus $f[0, x] \geq \alpha$. On the other hand, from $x \leq t$ and from (c2) we get $f[0, x] \leq f[0, t] = \alpha$, whence $f[0, x] = \alpha$. □

5. Auxiliary results

An element $e$ of a lattice ordered group $G$ is a weak unit of $G$ if $e \land g > 0$ for each $0 < g \in G$.

Lemma 5.1. Let $G$ be an archimedean lattice ordered group and let $e$ be a weak unit of $G$. Then

$$\bigvee_{n=1}^{\infty} (ne \land g) = g$$

for each $0 \leq g \in G$.

Proof. Let us denote by $G_1$ the Dedekind completion of $G$. In view of 1.19 in [4], the relation (1) is valid in the lattice ordered group $G_1$. Since the elements $ne$ and $g$ belong to $G$, we infer that (1) holds also in $G$. □

We denote by $a(G)$ the set of all elements $0 < a \in G$ such that the interval $[0, a]$ of $G$ is a chain.
Lemma 5.2 (Cf. [5]). Let $G$ be an archimedean lattice ordered group.

(i) For each $a_1 \in a(G)$ there exists an element $G(a_1)$ of $F(G)$ such that $a_1 \in G(a_1)$ and $G(a_1)$ is linearly ordered.

(ii) Let $a_1, a_2 \in a(G)$. Then either $G(a_1) = G(a_2)$ or $G(a_1) \cap G(a_2) = \{0\}$.

Lemma 5.3. Let $G$ be a lattice ordered group and $0 < e \in G$. Then $e$ is a weak unit of the lattice ordered group $\{e\}^{\delta\delta}$.

Proof. Let $0 < g \in \{e\}^{\delta\delta}$. If $a \land g = 0$, then $g \in \{a\}^\delta$. Since $\{e\}^{\delta} \cap \{e\}^{\delta\delta} = \{0\}$, we obtain $g = 0$, which is a contradiction. □

Lemma 5.4. Let $G$ be an archimedean lattice ordered group and let $0 < e \in G$. Let $f$ be an increasing cardinal property on the class of all archimedean lattice ordered groups satisfying the conditions $(c_1)$ and $(c_2)$. Assume that the interval $[0, e]$ is $f$-homogeneous. Then the lattice ordered group $\{e\}^{\delta\delta}$ is $f$-homogeneous.

Proof. By applying 4.1 and induction we obtain that for each $n \in \mathbb{N}$, the interval $[0, ne]$ is $f$-homogeneous. Put $f[0, g] = \alpha$. It suffices to verify that for each $0 < g \in \{e\}^{\delta\delta}$ we have $f[0, g] = \alpha$.

In view of 5.3 and 5.1, the relation (1) is valid. Further, $0 < ne \land g$ for each $n \in \mathbb{N}$. Thus $f[ne \land g] = \alpha$ for each $n \in \mathbb{N}$. From 4.2 we infer that $f[0, g] = \alpha$. □

An indexed system $(G_i)_{i \in I}$ of elements of $F(G)$ is orthogonal if $G_{i(1)} \cap G_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of $I$.

Assume that the lattice ordered group $G$ is conditionally orthogonally complete and that $(G_i)_{i \in I}$ is an orthogonal indexed system of elements of $F(G)$. Let $H_0$ be the set of all elements $h$ of $G^+$ which can be expressed in the form

\[ h = \bigvee_{i \in I} h_i, \]

where $h_i \in G_i^+$ for each $i \in I$.

Lemma 5.5. $H_0$ is an ideal of the lattice $G^+; \text{ further, } H_0$ is closed with respect to the operation $+$.

Proof. a) Let $h$ be as above. Analogously, let $h_1 = \bigvee_{i \in I} h_i^1$, $h_i^1 \in G_i^+$. Then we have

\[ h \lor h_1 = \bigvee_{i \in I} (h_i \lor h_i^1) \]

with $h_i \lor h_i^1 \in G_i^+$. Thus $h \lor h_1 \in H_0$.
b) Let $h$ be as above and $h^2 \in G^+$, $h^2 \leq h$. Then
\[ h^2 = h^2 \land h = \bigvee_{i \in I} (h^2 \land h^i) \]
and $h^2 \land h^i \in G^+_i$ for each $i \in I$. Hence $h^2 \in H_0$. We have verified that $H_0$ is an ideal of $G^+$.

c) Let $h$ and $h_1$ be as in a). Then
\[
h + h_1 = \left( \bigvee_{i \in I} h^i \right) + \left( \bigvee_{j \in I} h_1^j \right) = \bigvee_{i \in I} \bigvee_{j \in I} (h^i + h_1^j).
\]
If $i = j$, then $h^i + h_1^i \in G_i$. If $i \neq j$, then $h^i \land h_1^j = 0$, whence $h^i + h_1^j = h^i \lor h_1^j$.

Therefore
\[
h + h_1 = \bigvee_{i \in I} (h^i + h_1^i)
\]
and thus $h + h_1 \in H_0$. ☐

We denote by $H_1$ the set of all $g \in G$ having the property that there exist $h, h_1 \in H_0$ with $-h \leq g \leq h_1$. By a simple calculation we obtain from 5.5

**Lemma 5.6.** $H_1$ is a convex ℓ-subgroup of $G$.

**Lemma 5.7.** Let $0 < g \in G$. Then the set $\{h \in H_0: h \leq g\}$ has the greatest element.

**Proof.** For $i \in I$ let $g_i$ be the component of $g$ in $G_i$. Hence $g_i \in G^+_i$ for each $i \in I$ and the indexed system $(g_i)_{i \in I}$ is orthogonal. Also, $g_i \leq g$ for each $i \in I$. Thus there exists $g_0 = \bigvee_{i \in I} g_i$ in $G$. In view of the definition of $H_0$ we have $g_0 \in H_0$. Clearly, $g_0 \leq g$. We want to show that $g_0$ is the greatest element of the set $\{h \in H_0: h \leq g\} = K$.

By way of contradiction, assume that $g_0$ fails to be the greatest element of the set $K$. Then there exists $k \in K$ with $g_0 < k$.

If (2) is valid and $i \in I$, then the component $h_i$ of $h$ in $G_i$ is equal to $h^i$. Hence $(g_0)_i = g_i$ for each $i \in I$. Since $g_0 < k$, there exists $i(1) \in I$ such that $(g_0)_{i(1)} < k_{i(1)}$. Further, from $k \in K$ we get $k \leq g$, whence $k_{i(1)} \leq g_{i(1)}$. We obtain $g_{i(1)} < k_{i(1)} \leq g_{i(1)}$, which is a contradiction. ☐

From 5.6 and 5.7 we infer

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Lemma 5.8. \(H_1\) is an internal direct factor of \(G\).

For each \(i \in I\) we have \(G_i \subseteq H_1\). If \(H_2 \in F(G)\) is such that \(G_i \subseteq H_2\) for each \(i \in I\), then from the definition of \(H_0\) we get \(H_0 \subseteq H_2\); this yields that \(H_1 \subseteq H_2\). Therefore the relation

\[
H_1 = \bigvee_{i \in I} G_i
\]

is valid in \(F(G)\). Thus each orthogonal indexed system of \(F(G)\) has the supremum in \(F(G)\). This property will be called, similarly as for lattice ordered groups, the orthogonal completeness of \(F(G)\).

It is well-known that each orthogonally complete Boolean algebra is complete. Thus we have

Theorem 5.9. If \(G\) is a conditionally orthogonally complete lattice ordered group, then the Boolean algebra \(F(G)\) is complete.

Lemma 5.10. Let \((G_i)_{i \in I}\) be as above and let \(H_1\) be as in 5.8. Then \(H_1 = (s) \prod_{i \in I} G_i\).

Proof. This is a consequence of 5.6 and of the fact that \(G_i \subseteq H_1\) for each \(i \in I\). \qed

6. \(f\)-Homogeneous Direct Factors

In this section we assume that \(G \neq \{0\}\) is a lattice ordered group belonging to \(C_0\) and that \(f\) is an increasing cardinal property on the class \(C\) such that the conditions \((c_1)\) and \((c_2)\) are satisfied.

We recall the notation introduced in [4].

Let \(A\) be the set of all cardinals \(\alpha\) such that \(f[a,b] = \alpha\) for some non-trivial interval \([a,b]\) of \(G\). For any \(\alpha \in A\) we put

\[
X_\alpha = \{x \in G: x > 0, f[0,x] \leq \alpha\} \cup \{0\};
\]
\[
Y_\alpha = \{y \in G: y > 0, f[0,y] < \alpha\} \cup \{0\};
\]
\[
Z_\alpha = (Y_\alpha)^\delta, \quad A_\alpha = X_\alpha \cap Z_\alpha.
\]

Further, we set

\[
B_\alpha = \{g \in G: -t_1 \leq g \leq t_2 \quad \text{for some } t_1, t_2 \in A\},
\]
\[
\bar{A}_\alpha = \left\{ g \in G: g = \bigvee_{j \in J} t_j \quad \text{for some } \{t_j\}_{j \in J} \subseteq A_\alpha \right\},
\]
\[
\overline{B}_\alpha = \{g \in G: -t_1 \leq g \leq t_2 \quad \text{for some } t_1, t_2 \in \bar{A}_\alpha\}.
\]
Though the main results of Section 1 in [4] are formulated for complete lattice
ordered groups several auxiliary results proved in that section remain valid without
the assumption of completeness. We will freely use such results in the present paper.

**Lemma 6.1** (Cf. [4], 1.4). Let $\alpha \in \mathcal{A}$. Then $B_{\alpha}$ is an $\ell$-ideal of $G$ and $f[a,b] = \alpha$
for each non-trivial interval of $B_{\alpha}$. If $\beta \notin \mathcal{A}$, $\beta \neq \alpha$, then $B_{\alpha} \cap B_{\beta} = \{0\}$.

**Lemma 6.2** (Cf. [4], 1.7.1). Let $\alpha \in \mathcal{A}$. Then $B_{\alpha}$ is an $\ell$-ideal of $G$. If $\beta \notin \mathcal{A}$,
$\beta \neq \alpha$, then $B_{\alpha} \cap B_{\beta} = \{0\}$.

Let $H$ be a lattice ordered group and let $\{h_i\}_{i \in I}$ be an orthogonal subset of $H$
such that

(i) $0 < h_i$ for each $i \in I$;
(ii) if $h \in H$ and $h \wedge h_i = 0$ for each $i \in I$, then $h = 0$.

Under these conditions we say that $\{h_i\}_{i \in I}$ is a maximal orthogonal subset of $H$.

From the Axiom of Choice it follows that if $H \neq \{0\}$, then there exists a maximal
orthogonal system in $H$.

In view of [4], p. 91 we have

**Lemma 6.3.** Let $\{a_i\}_{i \in I}$ be a maximal orthogonal subset of $B_{\alpha}$. Then $\{a_i\}_{i \in I}$ is a maximal orthogonal subset of $\overline{B}_{\alpha}$.

Let $(a_i)_{i \in I}$ be as in 6.3. For each $i \in I$ we put $G_i = \{a_i\}^{\delta \delta}$. From 6.3 we obtain
that the indexed system $(G_i)_{i \in I}$ is orthogonal.

Because $G$ belongs to $\mathcal{C}_0$ it is projectable and hence $G_i \in F(G)$ for each $i \in I$.
Let $H_1$ be as in Section 5. In view of 5.8, $H_1$ is an element of $F(G)$; moreover, by
virtue of the relation (3) in Section 5, $H_1$ is the join of the system $(G_i)_{i \in I}$ in $F(G)$.

Consider the convex $\ell$-subgroups $H_1$ and $\overline{B}_{\alpha}$ of $G$.

**Lemma 6.4.** $\overline{B}_{\alpha} = H_1$.

**Proof.** In view of the definitions of $\overline{B}_{\alpha}$ and $H_1$ we have $H_1 \subseteq \overline{B}_{\alpha}$.

By way of contradiction, assume that $H_1 \subset \overline{B}_{\alpha}$. Then there exists $0 < b \in \overline{B}_{\alpha}$
such that $b \notin H_1$. Since $H_1 \in F(G)$ there is $H'_1 \in F(G)$ such that $G = H_1 \times H'_1$.
From the relation $b \notin H_1$ we obtain $b(H'_1) > 0$. Clearly $b(H'_1) \leq b$, hence $b(H'_1) \in \overline{B}_{\alpha}$.
Then $b(H'_1) \wedge h_1 = 0$ for each $0 < h_1 \in H_1$. In particular, $b(H'_1) \wedge a_i = 0$ for each
$i \in I$. According to 6.3, we have arrived at a contradiction. □

From 6.4 and 5.10 we get
Corollary 6.5. For each \( \alpha \in \mathcal{A} \), \( B_\alpha \in F(G) \). Moreover, under the above notation, \( B_\alpha \) is a complete subdirect product of lattice ordered groups \( G_i \) (\( i \in I \)).

Let \( \alpha \in \mathcal{A} \) and \( g \in G^+ \). In view of 6.5, there exists the component \( g(B_\alpha) = g_\alpha \) of \( g \) in \( B_\alpha \). It is easy to verify that

\[
g_\alpha = \sup\{0 < t \in B_\alpha : t \leq g\}.
\]

Since \( (B_\alpha)^+ = \bar{A}_\alpha \), we have also

\[
(1) \quad g_\alpha = \sup\{0 < t \in \bar{A}_\alpha : t \leq g\}.
\]

Now we apply to \( G \in \mathcal{C}_0 \) the argument from 1.8–1.15 in [4]. (The completeness of \( G \) was used only in 1.10; at that place it was applied for showing that \( G \) is abelian. In our present case, the commutativity of \( G \) is a consequence of the archimedean property.)

Hence, looking at 1.15 in [4] we obtain

**Theorem 6.6.** Let \( G \neq \{0\} \) be a lattice ordered group belonging to the class \( \mathcal{C}_0 \). Assume that \( f \) is a cardinal property on \( \mathcal{C}_0 \) satisfying the conditions \((c_1)\) and \((c_2)\). For \( \alpha \in \mathcal{A} \) let \( B_\alpha \) be as above. Then \( G = (s) \prod_{\alpha \in \mathcal{A}} B_\alpha \). If, moreover, \( G \) is orthogonally complete, then \( G \) is an internal direct product of lattice ordered groups \( B_\alpha \) (\( \alpha \in \mathcal{A} \)).

This generalizes Theorem 1.15 of [4].

Let us now slightly modify the notation applied in 6.4 and in 5.10. In 6.4, we write now \( H_1^\alpha \) instead of \( H_1 \). Analogously, in 5.10 we write \( G_i^\alpha \) instead of \( G_i \) and \( I(\alpha) \) instead of \( I \); we get

**Lemma 6.7.** Let \( \alpha \in \mathcal{A} \). Then \( B_\alpha \) is a complete subdirect product of lattice ordered groups \( G_i^\alpha \) (\( i \in I(\alpha) \)).

In view of 6.6 and 6.7, for \( G \in \mathcal{C}_0 \) we obtain

\[
(2) \quad G = (s) \prod_{\alpha \in \mathcal{A}} \prod_{i \in I(\alpha)} G_i^\alpha.
\]
Lemma 6.8. For each $\alpha \in \mathcal{A}$ and each $i \in I(\alpha)$, the lattice ordered group $G_{i}^{\alpha}$ is $f$-homogeneous.

Proof. Let $\alpha \in \mathcal{A}$ and $i \in I$. In view of the definition of $G_{i}^{\alpha}$, there exists $a_{i}^{\alpha} \in A_{\alpha}$ such that

$$G_{i}^{\alpha} = \{a_{i}^{\alpha}\}^{\delta \delta}.$$  

According to the assertion 1.1 of [4], the interval $[0, a_{i}^{\alpha}]$ of $G$ is $f$-homogeneous. Hence 5.4 yields that $G_{i}^{\alpha}$ is $f$-homogeneous. □

From 6.6, (2) and 6.8 we obtain

Theorem 6.9. Let $G \neq \{0\}$ be a lattice ordered group belonging to the class $\mathcal{C}_{0}$. Assume that $f$ is a cardinal property on $\mathcal{C}_{0}$ satisfying the conditions (c1) and (c2). Then $G$ can be prepresented as a complete subdirect product (2), where all factors $G_{i}^{\alpha}$ are $f$-homogeneous. If, moreover, $G$ is orthogonally complete, then (2) is an internal direct product decomposition of $G$.

This generalizes theorem 1.21 of [4].

7. WEAKLY HOMOGENEOUS DIRECT FACTORS

In this section we apply the results of Section 6 for dealing with weak homogeneity of lattice ordered groups which belong to $\mathcal{C}_{0}$ and are not assumed to be unital.

Again, assume that $G \neq \{0\}$ is a lattice ordered group belonging to $\mathcal{C}_{0}$. Let $\mathcal{G}_{1}$ be as in Section 1.

Lemma 7.1. Let $G \in \mathcal{C}_{0}$. Then $G$ can be expressed as an internal direct product $A \times B$ such that

(i) $A$ is the complete subdirect product of linearly ordered groups belonging to $\mathcal{G}_{1}$;

(ii) if $0 < b \in B$, then the interval $[0, b]$ of $B$ fails to be a chain.

Proof. This is a consequence of 5.2, 5.8 and 5.10. □

From 7.1 we conclude that for proving the assertion (W) of Section 1 it suffices to deal with the lattice ordered group $B$. If $B = \{0\}$, then the assertion of (W) is valid for $G$; assume that $B \neq \{0\}$.

In view of 7.1 we have $\text{card}[0, b] \geq \aleph_{0}$ for each $0 < b \in B$. Moreover, it is easy to verify that $[0, b]$ contains an infinite orthogonal subset.
In [4], Section 3 the cardinal function $f_3$ was considered on the class of all bounded lattices defined by

$$f_3[a, b] = \max\{\text{card}[a, b], \aleph_0\}$$

for any non-trivial interval $[a, b]$; in the case $a = b$ we put $f_3[a, b] = 0$. We can apply $f_3$ to the class $\mathcal{C}_0$. If $[a, b]$ is a nontrivial interval of $B$, then we have

(1) \quad f_3[a, b] = \text{card}[a, b].

**Lemma 7.2.** Let $0 < b \in B$ and assume that the interval $[0, b]$ is $f_3$-homogeneous, $f_3[0, b] = \alpha$. Then $\alpha^\aleph_0 = \alpha$.

**Proof.** Since the interval $[0, b]$ has an infinite orthogonal subset we can apply the argument used in the proof of 3.5 in [4].

We say that $f_3$ satisfies $(c_2)$ with regard to $B$ if, whenever all the elements considered in $(c_2)$ belong to $B$, then the assertion of $(c_2)$ is valid.

**Lemma 7.3.** $f_3$ satisfies the condition $(c_2)$ with regard to $B$.

**Proof.** It suffices to apply 7.2 and the argument applied in the proof of 3.6 from [4].

According to 3.1 in [4], $f_3$ satisfies the condition $(c_1)$. From this and from 7.3 we conclude that we can apply the assertion of 6.9 to the lattice ordered group $G$ and to the cardinal property $f_3$. In view of (1), for intervals of $B$ the $f_3$-homogeneity is the same as weak homogeneity. Further, if $G$ is orthogonally complete, then $B$ is orthogonally complete as well. Hence we have

**Theorem 7.4.** Let $G \in \mathcal{C}_0$ and let $B$ be as in 7.1. Then $B$ can be represented as a complete subdirect product of weakly homogeneous lattice ordered groups. If, moreover, $G$ is orthogonally complete, then the just mentioned complete subdirect product turns out to be an internal direct product.

The assertion (W) of Section 1 is a consequence of 7.1 and 7.4.

We remark that if $G$ is a complete lattice ordered group and if $G_1$ is some of linearly ordered groups mentioned in the assertion (i) of 7.1, then $G_1$ is complete, thus $G_1 \in \{0, \mathbb{Z}, \mathbb{R}\}$. In the cases $G_1 = \{0\}$ or $G_1 = \mathbb{R}$ we obtain that $G_1$ is weakly homogeneous. Hence 7.2 and 7.4 yield a generalization of Theorem 3.7 of [4].
References


Author’s address: Ján Jakubík, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: kstefan@saske.sk.