Czechoslovak Mathematical Journal

Ali Abkar

On the integral representation of superbiharmonic functions

*Czechoslovak Mathematical Journal*, Vol. 57 (2007), No. 3, 877–883

Persistent URL: `http://dml.cz/dmlcz/128213`

**Czechoslovak Mathematical Journal, 57 (132) (2007), 877–883**

# ON THE INTEGRAL REPRESENTATION OF SUPERBIHARMONIC FUNCTIONS

Ali Abkar, Tehran

(Received June 23, 2005)

*Abstract.* We consider a nonnegative superbiharmonic function $w$ satisfying some growth condition near the boundary of the unit disk in the complex plane. We shall find an integral representation formula for $w$ in terms of the biharmonic Green function and a multiple of the Poisson kernel. This generalizes a Riesz-type formula already found by the author for superbihamonic functions $w$ satisfying the condition $0 \leqslant w(z) \leqslant C(1-|z|)$ in the unit disk. As an application we shall see that the polynomials are dense in weighted Bergman spaces whose weights are superbiharmonic and satisfy the stated growth condition near the boundary.

*Keywords*: superbiharmonic function, biharmonic Green function, weighted Bergman space

*MSC 2000*: 31A30

## 1. Introduction

We denote by $\mathbb{C}$ the complex plane, by $\mathbb{D}$ the open unit disk $\{z \in \mathbb{C}\colon\ |z|<1\}$, and by $\mathbb{T}$ the unit circle $\{z \in \mathbb{C}\colon\ |z|=1\}$. The Laplace operator in the complex plane is denoted by

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big), \qquad z = x + \mathrm{i}y.$$

We write $\mathrm{d}A(z) = \pi^{-1}\,\mathrm{d}x\,\mathrm{d}y$ for the normalized Lebesgue area measure on the unit disk. Similarly for $z = \mathrm{e}^{\mathrm{i}\theta}$, we write $\mathrm{d}\sigma(z) = (2\pi)^{-1}\,\mathrm{d}\theta$ for the normalized arc length measure on the unit circle.

Let $u$ be a locally integrable complex-valued function on the unit disk. We say that $u$ is *biharmonic* provided that $\Delta^2 u = 0$ in the sense of distributions. We say that $u$

Research supported in part by IPM under the grant number 83310011.

is *superbiharmonic* if $u$ is real-valued and $\Delta^2 u \geqslant 0$ in the sense of distributions, or equivalently, $\Delta^2 u$ is a locally finite positive Borel measure on $\mathbb{D}$. As we pointed out earlier, our representation formula involves the biharmonic Green function for the operator $\Delta^2$ in the unit disk; that is, the function given by

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 + \bigl(1-|z|^2\bigr)\bigl(1-|\zeta|^2\bigr), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

Let $w$ be a superbiharmonic function which satisfies the growth condition

$$0 \leqslant w(z) \leqslant C(1-|z|), \qquad z \in \mathbb{D}.$$

It was shown in [1] that $w$ enjoys the representation

$$w(z) = \int_{\mathbb{D}} \Gamma(\zeta,z)\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta) + \int_{\mathbb{T}} H(\zeta,z)\partial_{n(\zeta)} w(\zeta)\,\mathrm{d}\sigma(\zeta), \qquad z \in \mathbb{D},$$

where $H(\zeta,z)$ has the form

$$H(\zeta,z) = (1-|z|^2)\frac{1-|\zeta z|^2}{|1-\bar{z}\zeta|^2}, \qquad (\zeta,z) \in \overline{\mathbb{D}}\times\mathbb{D},$$

and $\partial_{n(\zeta)}$ stands for the normal derivative (in the sense of distributions) with respect to $\zeta$. It is easy to see that

$$H(\zeta,z) = (1-|z|^2)P(z,\zeta) = (1-|z|^2)\frac{1-|z|^2}{|\zeta-z|^2}, \qquad (\zeta,z) \in \mathbb{T}\times\mathbb{D},$$

where $P(z,\zeta)$ denotes the Poisson kernel for the unit disk. Note that the above representation formula can be written as

$$w(z) = \int_{\mathbb{D}} \Gamma(\zeta,z)\,\mathrm{d}\mu(\zeta) + \int_{\mathbb{T}} H(\zeta,z)\,\mathrm{d}\nu(\zeta), \qquad z \in \mathbb{D},$$

where $\mathrm{d}\mu$ is a positive Borel measure on $\mathbb{D}$ and $\mathrm{d}\nu$ is a positive Borel measure on $\mathbb{T}$; indeed, $\mathrm{d}\mu$ is the measure induced by $\Delta^2 u$ and $\mathrm{d}\nu$ is understood as the weak-star limit of a family of positive continuous functions on the unit circle.

In this paper, we replace the above growth condition on $w$ by the following weaker one; the superbiharmonic function $w$ satisfies

$$0 \leqslant w(z) \leqslant g(z)(1-|z|), \qquad z \in \mathbb{D},$$

where $g$ is a function with the property

$$\sup_{0<r<1} \int_{\mathbb{T}} g(rz)\,\mathrm{d}\sigma(z) < +\infty.$$

We shall see that under above conditions, the same representation formula holds true. Indeed, we can express the conditions on $w$ in the following concise form; $w$ is a nonnegative superbiharmonic function satisfying

$$\sup_{0<r<1} \frac{1}{1-r} \int_{\mathbb{T}} w(rz)\,\mathrm{d}\sigma(z) < +\infty.$$

A nonnegative function on the unit disk is known as a *weight function.* Weight functions satisfying the growth condition above play a prominent role in the theory of weighted Bergman spaces. It turns out that the polynomials are dense in such weighted Bergman spaces. This fact in turn can be used for establishing Beurling's theorem for zero-based invariant subspaces in the Bergman spaces (see [2] for details).

## 2. The integral representation

We begin with the following lemma.

**Lemma 2.1.** *Let $w$ be a nonnegative superbiharmonic function satisfying the condition*

$$\sup_{0<r<1} \frac{1}{1-r} \int_{\mathbb{T}} w(rz)\,\mathrm{d}\sigma(z) < +\infty.$$

*Then for every $z \in \mathbb{D}$ we have*

$$v(z) := \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta) < +\infty,$$

*moreover, for every integer $n$,*

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \frac{\bar{z}^n v(rz)}{1-r^2}\,\mathrm{d}\sigma(z) = 0.$$

Proof. For $0 < r < 1$, we put $w_r(z) = w(rz)$. Consider the fraction $\frac{(1-|z|^2)^3}{1-r^2|z|^2}$ which vanishes along with its normal derivative on the boundary of the unit disk. Since $\Delta^2 w_r(z) = r^4(\Delta^2 w)(rz)$, we obtain by two successive applications of Green's formula

$$\begin{aligned}(1)\qquad r^4 \int_{\mathbb{D}} \frac{(1-|z|^2)^3}{1-r^2|z|^2} (\Delta^2 w)(rz)\,\mathrm{d}A(z) = &\int_{\mathbb{D}} \Delta^2\Big(\frac{(1-|z|^2)^3}{1-r^2|z|^2}\Big) w(rz)\,\mathrm{d}A(z)\\ &+ \frac{6}{1-r^2}\int_{\mathbb{T}} w(rz)\,\mathrm{d}\sigma(z).\end{aligned}$$

According to our assumption on $w$ we know that the second term on the right-hand side of (1) remains bounded as $r \to 1^-$. As for the first term on the right we see that

$$\Delta^2 \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2} = 4(1-r^2)^3|z|^2 \frac{-r^8|z|^8+5r^6|z|^6-10r^4|z|^4+9r^2|z|^2+9}{\left(1-r^2|z|^2\right)^5} + 4(r^4-3r^2+3).$$

This leads to

$$\left|\Delta_z^2 \frac{\left(1-|z|^2\right)^3}{1-r^2|z|^2}\right| \leqslant 36 \frac{(1-r^2)^3}{\left(1-r^2|z|^2\right)^5} + 12,$$

from which it follows that

$$\begin{aligned} (2) \qquad & \left|\int_{\mathbb{D}} \Delta^2\Big(\frac{(1-|z|^2)^3}{1-r^2|z|^2}\Big) w(rz)\,\mathrm{d}A(z)\right| \\ & \leqslant 36(1-r^2)^3 \int_{\mathbb{D}} \frac{w(rz)}{(1-r^2|z|^2)^5}\,\mathrm{d}A(z) + 12\int_{\mathbb{D}} w(rz)\,\mathrm{d}A(z). \end{aligned}$$

That the second term on the right-hand side of (2) remains bounded as $r \to 1^-$ is clear. As for the first term on the right, we note that

$$\frac{1}{2\pi}\int_0^{2\pi} w(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leqslant C(1-r^2),$$

from which it follows that

$$\begin{aligned} (3) \qquad & (1-r^2)^3 \int_{\mathbb{D}} \frac{w(rz)}{\left(1-r^2|z|^2\right)^5}\,\mathrm{d}A(z) \\ & = (1-r^2)^3 \frac{1}{\pi}\int_0^1 \frac{s\,\mathrm{d}s}{(1-r^2s^2)^5}\int_0^{2\pi} w(rs\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \\ & \leqslant C(1-r^2)^3 \int_0^1 \frac{2s\,\mathrm{d}s}{(1-r^2s^2)^4} \\ & = \frac{C}{3}((1-r^2)+(1-r^2)^2+1) \leqslant C. \end{aligned}$$

It now follows from (1), (2) and (3) that for a new constant—we denote it again by $C$—

$$r^4 \int_{\mathbb{D}} \frac{(1-|z|^2)^3}{1-r^2|z|^2} (\Delta^2 w)(rz)\,\mathrm{d}A(z) \leqslant C.$$

We now use Fatou's lemma to conclude that

$$\int_{\mathbb{D}} (1-|z|^2)^2 \Delta^2 w(z)\,\mathrm{d}A(z) \leqslant C.$$

Finally, we know that (see [1], Lemma 2.2)

$$0 < \Gamma(z,\zeta) \leqslant \frac{1}{2}(1+|z|)^2(1-|\zeta|^2)^2, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

from which it follows that

$$\int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta) \leqslant 2\int_{\mathbb{D}} (1-|\zeta|^2)^2\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta) < +\infty.$$

As for the second part of the lemma, we set

$$C(n,r) := \int_{\mathbb{T}} \frac{\bar{z}^n v(rz)}{1-r^2}\,\mathrm{d}\sigma(z).$$

It is easy to see that

$$0 \leqslant |C(n,r)| \leqslant C(0,r).$$

Therefore it suffices to show that $C(0,r) \to 0$, as $r \to 1^-$. To this end, we recall (see [1], Lemma 2.2) that

$$0 < \Gamma(z,\zeta) \leqslant \frac{\left(1-|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{|1-\bar{\zeta}z|^2}, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

It now follows from Fubini's theorem that

$$\begin{aligned} 0 \leqslant C(0,r) &\leqslant \int_{\mathbb{D}}\int_{\mathbb{T}} \frac{\left(1-r^2|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{\left(1-r^2\right)|1-rz\bar{\zeta}|^2}\Delta^2 w(\zeta)\,\mathrm{d}\sigma(z)\,\mathrm{d}A(\zeta) \\ &= \left(\int_{\mathbb{D}} \left(1-|\zeta|^2\right)^2\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta)\right)\int_{\mathbb{T}} \frac{1-r^2}{|1-rz\bar{\zeta}|^2}\,\mathrm{d}\sigma(z) \\ &= \int_{\mathbb{D}} \frac{1-r^2}{1-r^2|\zeta|^2}\left(1-|\zeta|^2\right)^2\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta). \end{aligned}$$

But the last integral tends to zero as $r \to 1^-$; this follows from the dominated convergence theorem and the first part of the lemma. □

**Theorem 2.2.** *Let $w$ be a nonnegative superbiharmonic function satisfying the condition*

$$\sup_{0<r<1} \frac{1}{1-r}\int_{\mathbb{T}} w(rz)\,\mathrm{d}\sigma(z) < +\infty.$$

*Then there exists a unique positive measure $\nu$ on the unit circle such that*

$$w(z) = \int_{\mathbb{D}} \Gamma(\zeta,z)\Delta^2 w(\zeta)\,\mathrm{d}A(\zeta) + \int_{\mathbb{T}} H(\zeta,z)\,\mathrm{d}\nu(\zeta), \qquad z \in \mathbb{D}.$$

P r o o f. We consider the function $v$ defined in the statement of the above lemma. Since $w-v$ is biharmonic, it follows from the Almansi representation formula (see [3], Lemma 3.1) that there exist two real-valued harmonic functions $h$ and $u$ such that

$$w(z)-v(z)=u(z)+\bigl(1-|z|^2\bigr)h(z), \qquad z\in\mathbb{D}.$$

For a fixed $0<r<1$, recall that $f_r(z)=f(rz)$, so that

$$w(rz)-v(rz)=u_r(z)+(1-r^2)h_r(z), \qquad z\in\mathbb{T}.$$

Writing the $n$-th Fourier coefficient of both sides we obtain

$$\begin{aligned}\int_{\mathbb{T}}\bar{z}^n\bigl(w(rz)-v(rz)\bigr)\,\mathrm{d}\sigma(z)&=\hat{u}_r(n)+(1-r^2)\hat{h}_r(n)\\&=r^{|n|}\hat{u}(n)+(1-r^2)r^{|n|}\hat{h}(n).\end{aligned}$$

We now let $r\to 1^-$ and use the second part of the above lemma together with our assumption on $w$ to conclude that $\hat{u}(n)=0$, for every integer $n$. This means that

$$w(z)-v(z)=(1-|z|^2)h(z), \qquad z\in\mathbb{D}.$$

Therefore

$$\int_{\mathbb{T}}|h_r(z)|\,\mathrm{d}\sigma(z)\leqslant\int_{\mathbb{T}}\frac{w(rz)}{1-r^2}\,\mathrm{d}\sigma(z)+\int_{\mathbb{T}}\frac{v(rz)}{1-r^2}\,\mathrm{d}\sigma(z).$$

This together with the second part of the lemma and our assumption implies that

$$\sup_{0<r<1}\|h_r\|_{L^1(\mathbb{T})}<+\infty.$$

Therefore we can find (see [4]) a unique real-valued Borel measure $\nu$ on $\mathbb{T}$—as the weak-star limit of the measures $\mathrm{d}\nu_r=h_r\,\mathrm{d}\sigma$—such that

$$h(z)=\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\nu(\zeta) \qquad z\in\mathbb{D}.$$

Finally, we see that

$$w(z)-v(z)=\bigl(1-|z|^2\bigr)h(z)=\int_{\mathbb{T}}\frac{\bigl(1-|z|^2\bigr)^2}{|\zeta-z|^2}\,\mathrm{d}\nu(\zeta)=\int_{\mathbb{T}}H(\zeta,z)\,\mathrm{d}\nu(\zeta).$$

The last thing to be proved is the fact that the measure $\nu$ is positive. Let $p(z)$ be a nonnegative trigonometric polynomial on the unit circle. According to the second part of the lemma we have

$$\int_{\mathbb{T}}p(z)\,\mathrm{d}\nu(z)=\lim_{r\to 1^-}\int_{\mathbb{T}}p(z)h_r(z)\,\mathrm{d}\sigma(z)=\lim_{r\to 1^-}\int_{\mathbb{T}}p(z)\frac{w(rz)}{1-r^2}\,\mathrm{d}\sigma(z)\geqslant 0,$$

from which the result follows. $\square$

## 3. An application

Let $w$ be a weight function on the unit disk. We say that a function $f$, analytic in $\mathbb{D}$, belongs to the *weighted Bergman space* $L^p_a(\mathbb{D}, w)$, $0 < p < +\infty$, provided that

$$\|f\|^p_{L^p_a(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^p w(z)\,\mathrm{d}A(z) < +\infty.$$

If $1 \leqslant p < +\infty$, it is well-known that $L^p_a(\mathbb{D})$ is a Banach space, and for $0 < p < 1$, it is a quasi-Banach space. It is now time to state the main result of this section.

**Theorem 3.1.** *Let $w$ be a superbiharmonic weight function satisfying the condition*

$$\sup_{0<r<1} \frac{1}{1-r} \int_{\mathbb{T}} w(rz)\,\mathrm{d}\sigma(z) < +\infty.$$

*Then the polynomials are dense in the weighted Bergman space $L^p_a(\mathbb{D}, w)$ for $0 < p < +\infty$.*

Proof. As in [1], Proposition 3.3, we first show that $r \mapsto rw(z/r)$ is an increasing function of $r$ for $|z| < r < 1$. The rest of the proof proceeds as in [1], Theorem 3.5. $\square$

*References*


[1] *A. Abkar*: Application of a Riesz-type formula to weighted Bergman spaces. Proc. Amer. Math. Soc. *131* (2003), 155–164. zbl

[2] *A. Abkar*: Norm approximation by polynomials in some weighted Bergman spaces. J. Func. Anal. *191* (2002), 224–240. zbl

[3] *H. Hedenmalm*: A computation of Green function for the weighted biharmonic operators $\Delta|z|^{-2\alpha}\Delta$ with $\alpha > -1$. Duke Math. J. *75* (1994), 51–78. zbl

[4] *K. Hoffman*: Banach Spaces of Analytic Functions. Dover Publications, Inc. New York, 1988. zbl



*Author's addresses*: Ali Abkar, Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran, e-mail: `abkar@khayam.ut.ac.ir` and Mathematics Department, Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-5746, Iran.