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ON THE ORDER OF CERTAIN CLOSE TO REGULAR GRAPHS WITHOUT A MATCHING OF GIVEN SIZE

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Abstract. A graph G is a $\{d, d+k\}$-graph, if one vertex has degree $d+k$ and the remaining vertices of G have degree d. In the special case of $k = 0$, the graph G is d-regular. Let $k, p \geqslant 0$ and $d, n \geqslant 1$ be integers such that n and p are of the same parity. If G is a connected $\{d, d+k\}$-graph of order n without a matching M of size $2|M| = n - p$, then we show in this paper the following: If $d = 2$, then $k \geqslant 2(p+2)$ and

(i) $n \geqslant k + p + 6$.

If $d \geqslant 3$ is odd and t an integer with $1 \leqslant t \leqslant p + 2$, then

(ii) $n \geqslant d + k + 1$ for $k \geqslant d(p+2)$,

(iii) $n \geqslant d(p+3) + 2t + 1$ for $d(p+2-t) + t \leqslant k \leqslant d(p+3-t) + t - 3$,

(iv) $n \geqslant d(p+3) + 2p + 7$ for $k \leqslant p$.

If $d \geqslant 4$ is even, then

(v) $n \geqslant d + k + 2 - \eta$ for $k \geqslant d(p+3) + p + 4 + \eta$,

(vi) $n \geqslant d + k + p + 2 - 2t = d(p+4) + p + 6$ for $k = d(p+3) + 4 + 2t$ and $p \geqslant 1$,

(vii) $n \geqslant d + k + p + 4$ for $d(p+2) \leqslant k \leqslant d(p+3) + 2$,

(viii) $n \geqslant d(p+3) + p + 4$ for $k \leqslant d(p+2) - 2$,

where $0 \leqslant t \leqslant \frac{1}{2}p - 1$ and $\eta = 0$ for even p and $0 \leqslant t \leqslant \frac{1}{2}(p-1)$ and $\eta = 1$ for odd p.

The special case $k = p = 0$ of this result was done by Wallis [6] in 1981, and the case $p = 0$ was proved by Caccetta and Mardiyono [2] in 1994. Examples show that the given bounds (i)–(viii) are best possible.

Keywords: matching, maximum matching, close to regular graph

MSC 2000: 05C70

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [3]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x in the graph G. We denote by K_n the complete graph of order n. A graph G is a $\{d, d+k\}$-*graph*, if one vertex has degree $d + k$ and the remaining vertices of G have degree d. In the special case of $k = 0$, we speak of a *d-regular graph*. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G - A)$ the number of odd components in the subgraph $G - A$.

The proof of our main theorem is based on the following generalization of Tutte's famous 1-factor theorem [4] by Berge [1] in 1958, and we call it the Theorem of Tutte-Berge (for a proof see, e.g., [5]).

Theorem of Tutte-Berge (Berge [1], 1958). *Let G be a graph of order n. If M is a maximum matching of G, then*

$$n - 2|M| = \max_{A \subseteq V(G)} \{q(G - A) - |A|\}.$$

Theorem 2. *Let $k, p \geqslant 0$ and $d, n \geqslant 1$ be integers such that n and p are of the same parity. If G is a connected $\{d, d+k\}$-graph of order n without a matching M of size $2|M| = n - p$, then the following holds:*
If $d = 2$, then $k \geqslant 2(p+2)$ and
(i) $n \geqslant k + p + 6$.
If $d \geqslant 3$ is odd and t an integer with $1 \leqslant t \leqslant p + 2$, then
(ii) $n \geqslant d + k + 1$ *for* $k \geqslant d(p+2)$,
(iii) $n \geqslant d(p+3) + 2t + 1$ *for* $d(p+2-t) + t \leqslant k \leqslant d(p+3-t) + t - 3$,
(iv) $n \geqslant d(p+3) + 2p + 7$ *for* $k \leqslant p$.
If $d \geqslant 4$ is even, then
(v) $n \geqslant d + k + 2 - \eta$ *for* $k \geqslant d(p+3) + p + 4 + \eta$,
(vi) $n \geqslant d + k + p + 2 - 2t = d(p+4) + p + 6$ *for* $k = d(p+3) + 4 + 2t$ *and* $p \geqslant 1$,
(vii) $n \geqslant d + k + p + 4$ *for* $d(p+2) \leqslant k \leqslant d(p+3) + 2$,
(viii) $n \geqslant d(p+3) + p + 4$ *for* $k \leqslant d(p+2) - 2$,
where $0 \leqslant t \leqslant \frac{1}{2}p - 1$ *and* $\eta = 0$ *for even* p *and* $0 \leqslant t \leqslant \frac{1}{2}(p-1)$ *and* $\eta = 1$ *for odd* p.

Proof. The bounds (ii) and (v) are immediate. By the hypotheses and the Theorem of Tutte-Berge, it follows that there exists a non-empty set $A \subset V(G)$ such that $q(G - A) \geqslant |A| + p + 1$. However, since n and p are of the same parity, it is straightforward to verify that this even leads to the better bound

$$q(G - A) \geqslant |A| + p + 2. \tag{1}$$

(i): Since $d = 2$ is even, k is even, and hence each odd component of $G - A$ is connected by an even number of edges with A. If $u \in V(G)$ with $d_G(u) = k + 2$, then we observe that

$$(2) \qquad 2q(G-A) \leqslant 2|A| + k \quad \text{when } u \in A,$$

$$(3) \qquad 2q(G-A) \leqslant 2|A| \quad \text{when } u \notin A.$$

If $u \notin A$, then the inequalities (1) and (3) yield the contradiction $2|A| \geqslant 2|A| + 2(p+2)$.

Thus $u \in A$, and (1) and (2) lead to $k \geqslant 2q(G-A) - 2|A| \geqslant 2(p+2)$, as desired. Now, suppose to the contrary that there exists such a graph with $n \leqslant k+p+5$. Since $d_G(u) = k+2$, we deduce that $n = k+3+r$ with $0 \leqslant r \leqslant p+2$. If we define by α the number of vertices in A not adjacent with u, and by β the number of vertices in $G-A$ not adjacent with u, then we observe that $r = \alpha + \beta$. Since every vertex of $G-A$ has degree 2, each odd component of $G-A$ is a path. Hence each odd component of $G-A$ with at least three vertices contains at least one vertex not adjacent with u. The definition of β thus shows that $G-A$ has at most β odd components of order three or more and therefore at least $q(G-A) - \beta$ components of order one. This implies that there are at least $q(G-A) - \beta$ edges from the components of order one to $A - \{u\}$. But since $u \in A$ is adjacent to $|A| - 1 - \alpha$ vertices in A, there can be at most $|A| - 1 - \alpha + 2\alpha = |A| - 1 + \alpha$ edges going out of $A - \{u\}$ and so $q(G-A) - \beta \leqslant |A| - 1 + \alpha$. According to (1), we obtain

$$|A| + p + 2 - \beta \leqslant q(G-A) - \beta \leqslant |A| - 1 + \alpha.$$

This leads to the contradiction $p + 3 \leqslant \alpha + \beta = r \leqslant p + 2$, and the proof of (i) is complete.

(iii) and (iv) Let $u \in V(G)$ such that $d_G(u) = k + d$. The hypotheses that d is odd and that n and p are of the same parity, show that k, n, and p are of the same parity. Since (ii) is valid, it remains to investigate the case of $k \leqslant d(p+2) - 2$. Now, suppose to the contrary that there exists such a graph with

(a) $n \leqslant d(p+3) + 2t - 1$ for $d(p+2-t) + t \leqslant k \leqslant d(p+3-t) + t - 3$ with $1 \leqslant t \leqslant p+2$,

(b) $n \leqslant d(p+3) + 2p + 5$ for $k \leqslant p$.

The odd components of $G-A$ are classified into three groups according to order. We let:

$$\alpha_1 := \text{the number of odd components of } G-A \text{ of order at most } d-2,$$

$$\alpha_2 := \text{the number of odd components of } G-A \text{ of order } d,$$

$$\alpha_3 := \text{the number of odd components of } G-A \text{ of order at least } d+2.$$

This leads to

$$n \geqslant |A| + \alpha_1 + \alpha_2 d + \alpha_3(d+2) \tag{4}$$

and (1) yields

$$\alpha_1 + \alpha_2 + \alpha_3 = q(G-A) \geqslant |A| + p + 2. \tag{5}$$

It is easy to verify that there are at least d edges of G joining each odd component of $G - A$ of order at most d with A. Since G is connected, we deduce that

$$d(\alpha_1 + \alpha_2) + \alpha_3 \leqslant d|A| + k, \tag{6}$$

$$d(\alpha_1 + \alpha_2) + \alpha_3 \leqslant d|A| \quad \text{when } u \notin A. \tag{7}$$

In the case $\alpha_3 \geqslant p + 3$, the inequality (4) yields the following contradiction to assumption (a) as well as to assumption (b).

$$\begin{aligned} n &\geqslant |A| + \alpha_1 + \alpha_2 d + \alpha_3(d+2) \\ &\geqslant 1 + (p+3)(d+2) \\ &= (p+3)d + 2p + 7. \end{aligned}$$

If $\alpha_3 \leqslant p+2$, then (5) leads to $d(\alpha_1+\alpha_2) \geqslant d(|A|+p+2-\alpha_3)$. In the case that $u \notin A$, the inequality (7) gives $d(|A|+p+2-\alpha_3) \leqslant d|A| - \alpha_3$ and thus $d(p+2) \leqslant (d-1)\alpha_3$, a contradiction to $\alpha_3 \leqslant p+2$. It follwows that $u \in A$. Combining (5) and (6), we obtain $d(|A|+p+2-\alpha_3) \leqslant d|A| + k - \alpha_3$ and so

$$k \geqslant d(p+2) - \alpha_3(d-1). \tag{8}$$

Because of $\alpha_3 \leqslant p+2$, we conclude that $k \geqslant p+2$. This means that (iv) is proved. For the proof of (iii) we distinguish different cases.

Case 1. Assume that $\alpha_3 = p+2$. The inequality (5) shows that $\alpha_1 + \alpha_2 \geqslant |A| + p + 2 - \alpha_3 \geqslant 1$. Hence there exists at least one odd component U of $G - A$ with at most d vertices. Since $N(x) \subseteq V(U) \cup A$ for $x \in V(U)$, we observe that $|A| + |V(U)| \geqslant d+1$. This leads to the following contradiction to assumption (a):

$$\begin{aligned} n &\geqslant |A| + |V(U)| + \alpha_3(d+2) \\ &\geqslant d + 1 + (p+2)(d+2) \\ &= (p+3)d + 2p + 5 \\ &\geqslant (p+3)d + 2t + 1. \end{aligned}$$

Case 2. Assume that $\alpha_3 \leqslant p+1$ and $p+2 \leqslant k \leqslant (p+2)d-2$. The inequality (8) is equivalent with

$$\alpha_3 \geqslant \frac{d(p+2)-k}{d-1}. \tag{9}$$

Combining this with the condition $\alpha_3 \leqslant p+1$, we find that $k \geqslant d+p+1$. This shows that $t = p+2$ is not possible. Hence we assume in the following that $1 \leqslant t \leqslant p+1$. Furthermore, the inequality (9) and the hypothesis $k \leqslant d(p+3-t)+t-3$ leads to

$$\alpha_3 \geqslant \frac{d(p+2)-d(p+3-t)-t+3}{d-1} = t - \frac{d-3}{d-1} > t-1$$

and thus $1 \leqslant t \leqslant \alpha_3 \leqslant p+1$. If s is an integer with $\alpha_3 = p+1-s$, then we observe that $0 \leqslant s \leqslant p+1-t$. We deduce from (5) that

$$\alpha_1 + \alpha_2 \geqslant |A| + p + 2 - p - 1 + s = |A| + s + 1 \geqslant s+2. \tag{10}$$

Subcase 2.1. Assume that $\alpha_2 \geqslant s+2$. The inequality (4) implies the following contradiction to assumption (a):

$$\begin{aligned} n &\geqslant |A| + \alpha_1 + \alpha_2 d + \alpha_3(d+2) \\ &\geqslant 1 + (s+2)d + (p+1-s)(d+2) \\ &= (p+3)d + 2(p+1-s) + 1 \\ &\geqslant (p+3)d + 2t + 1. \end{aligned}$$

Subcase 2.2. Assume that $\alpha_2 = s+1$. In view of (10), we conclude that $\alpha_1 \geqslant |A| \geqslant 1$. Hence there exists at least one odd component U of $G-A$ with at most $d-2$ vertices. It follows that $|A| + |V(U)| \geqslant d+1$, and this leads to

$$\begin{aligned} n &\geqslant |A| + |V(U)| + \alpha_2 d + \alpha_3(d+2) \\ &\geqslant d + 1 + (s+1)d + (p+1-s)(d+2) \\ &= (p+3)d + 2(p+1-s) + 1 \\ &\geqslant (p+3)d + 2t + 1, \end{aligned}$$

a contradiction to assumption (a).

Subcase 2.3. Assume that $\alpha_2 \leqslant s$. Let $\alpha_2 = s-r$ with an integer $0 \leqslant r \leqslant s$. According to (5), we have

$$\alpha_1 \geqslant |A| + p + 2 - \alpha_2 - \alpha_3 = |A| + r + 1. \tag{11}$$

In addition, there are at least $d-1$ edges of G joining each odd component of $G-A$ of order at most $d-2$ with $A-\{u\}$. Applying (11), we obtain

$$d(|A|-1) \geqslant \alpha_1(d-1) \geqslant (|A|+r+1)(d-1).$$

This yields $|A| \geqslant (r+2)d-r-1$ and (11) implies $\alpha_1 \geqslant |A|+r+1 \geqslant (r+2)d$. Combining the last inequalities with (4), we arrive at

$$\begin{aligned} n &\geqslant |A|+\alpha_1+\alpha_2 d+\alpha_3(d+2) \\ &\geqslant (r+2)d-r-1+(r+2)d+(s-r)d+(p+1-s)(d+2) \\ &= (p+r+5)d+2p-2s-r+1 \\ &\geqslant (p+r+5)d+2p-2(p+1-t)-r+1 \\ &= (p+r+5)d+2t-r-1 \\ &\geqslant (p+3)d+2t+1, \end{aligned}$$

a contradiction to assumption (a). Since we have discussed all possible cases, the proof of (iii) is complete.

(vi)–(viii) Let $u \in V(G)$ such that $d_G(u)=k+d$. The hypothesis that d is even implies that k is also even. Since (v) is valid, it remains to investigate the case of $k \leqslant d(p+3)+p+2+\eta$.

Now we call an odd component of $G-A$ large if it has more than d vertices and small otherwise. If we denote by β_1 and β_2 the number small and large components, respectively, then we deduce that

$$n \geqslant |A|+\beta_1+(d+1)\beta_2. \tag{12}$$

In addition, (1) yields

$$\beta_1+\beta_2 = q(G-A) \geqslant |A|+p+2. \tag{13}$$

It is easy to verify that there are at least d edges of G joining each small component of $G-A$ with A. Since G is connected, there are at least 2 edges of G joining each large component of $G-A$ with A. We therefore deduce that

$$d\beta_1+2\beta_2 \leqslant d|A|+k, \tag{14}$$

$$d\beta_1+2\beta_2 \leqslant d|A| \quad \text{when } u \notin A. \tag{15}$$

(viii) Let $k \leqslant d(p+2)-2$ and suppose to the contrary that there exists such a graph with

$$n \leqslant d(p+3)+p+2. \tag{16}$$

If $\beta_2 \geqslant p+3$, then (12) leads to the following contradiction to the assumption (16):

$$\begin{aligned} n &\geqslant |A| + \beta_1 + (d+1)\beta_2 \\ &\geqslant 1 + (d+1)(p+3) \\ &= d(p+3) + p + 4. \end{aligned}$$

If $\beta_2 = p+2$, then the inequality (13) shows that $\beta_1 \geqslant |A| \geqslant 1$. Hence there exists at least one odd component U of $G - A$ with at most $d-1$ vertices. It follows that $|A| + |V(U)| \geqslant d+1$, and this leads to

$$\begin{aligned} n &\geqslant |A| + |V(U)| + (d+1)\beta_2 \\ &\geqslant d + 1 + (d+1)(p+2) \\ &= d(p+3) + p + 3, \end{aligned}$$

a contradiction to the assumption (16).

If $\beta_2 \leqslant p+1$, then it follows from (13) that $d\beta_1 \geqslant d(|A|+1)$. In the case that $u \notin A$, inequality (15) yields the contradiction

$$d(|A|+1) \leqslant d\beta_1 + 2\beta_2 \leqslant d|A|.$$

Assume next that $u \in A$.

If $\beta_2 = 0$, then (13) gives $\beta_1 \geqslant |A| + p + 2$ and thus (14) leads to

$$d|A| + k \geqslant d\beta_1 \geqslant d(|A| + p + 2).$$

This implies $k \geqslant d(p+2)$, a contradiction to the hypothesis $k \leqslant d(p+2) - 2$.

There it remains the case of $1 \leqslant \beta_2 \leqslant p+1$. Let $\beta_2 = s+1$ with an integer $0 \leqslant s \leqslant p$. We deduce from (13) the inequality

$$\beta_1 \geqslant |A| + p + 1 - s. \tag{17}$$

If we count the edges between $G - A$ and $A - \{u\}$, then we obtain the inequality chain

$$\begin{aligned} d(|A|-1) &\geqslant (d-1)\beta_1 + \beta_2 \\ &\geqslant (d-1)(|A| + p + 1 - s) + s + 1. \end{aligned}$$

This leads to $|A| \geqslant d(p+2-s) - p + 2s$. Applying (12), (17), and the hypothesis $d \geqslant 4$, we arrive at the following contradiction to our assumption (16):

$$\begin{aligned} n &\geqslant |A| + \beta_1 + (d+1)\beta_2 \\ &\geqslant |A| + |A| + p + 1 - s + (d+1)(s+1) \\ &\geqslant 2d(p+2-s) - 2p + 4s + p + 1 - s + d(s+1) + s + 1 \\ &= d(p+3) + p + 4 + (p-s)(d-4) + 2p + 2d - 2 \\ &\geqslant d(p+3) + p + 4. \end{aligned}$$

(vii) Let $d(p+2) \leqslant k \leqslant d(p+3)+2$ and suppose to the contrary that there exists such a graph with

$$n \leqslant d + k + p + 2.$$

Since $n \geqslant d + k + 2 - \eta$, we can assume that

$$n = d + k + p + 2 - 2s$$

with an integer s such that $0 \leqslant s \leqslant \frac{1}{2}(p+1)$ when p is odd and $0 \leqslant s \leqslant \frac{1}{2}p$ when p is even. Hence there exist $p+1-2s$ vertices in G which are not adjacent with u.

Assume that $u \notin A$. The inequality (13) implies that $G - A$ contains at least $p+3$ odd components. Because of $u \notin A$, we conclude that u is non-adjacent with at least $p+3$ vertices of G. However, this gives the contradiction

$$d_G(u) \leqslant n - p - 3 = d + k - 1 - 2s < d + k.$$

Assume next that $u \in A$. Let $\alpha \leqslant p + 1 - 2s$ be the number of vertices in A not adjacent with u. If we count the number of edges between $G - A$ and $A - \{u\}$, then we obtain

$$\begin{aligned} (d-1)\beta_1 + \beta_2 &\leqslant (|A|-1)(d-1) + \alpha \\ &\leqslant (|A|-1)(d-1) + p + 1 - 2s. \end{aligned}$$

This inequality chain shows that

$$\beta_1 \leqslant |A| - 1 + \frac{p+1-2s-\beta_2}{d-1}.$$

Therefore (13) leads to

$$|A| + p + 2 - \beta_2 \leqslant \beta_1 \leqslant |A| - 1 + \frac{p+1-2s-\beta_2}{d-1}.$$

This yields

$$\beta_2 \geqslant p + 3 + \frac{2+2s}{d-2}$$

and thus $\beta_2 \geqslant p + 4$. Applying (12), we arrive at

$$\begin{aligned} d + k + p + 2 - 2s = n &\geqslant |A| + \beta_1 + (d+1)\beta_2 \\ &\geqslant 1 + (d+1)(p+4). \end{aligned}$$

This implies $k \geqslant d(p+3)+3+2s$, a contradiction to the hypothesis $k \leqslant d(p+3)+2$.

(vi) Let $p \geqslant 1$ and $k = d(p+3) + 4 + 2t$ with $0 \leqslant t \leqslant \frac{1}{2}p - 1$ when p is even and $0 \leqslant t \leqslant \frac{1}{2}(p-1)$ when p is odd. Suppose to the contrary that there exists such a graph with

$$n \leqslant d + k + p - 2t.$$

Let $n = d + k + p - 2r$ with an integer r such that $t \leqslant r \leqslant \frac{1}{2}p - 1$ when p is even and $t \leqslant r \leqslant \frac{1}{2}(p-1)$ when p is odd. If we define $r = s - 1$, then we obtain $n = d + k + p + 2 - 2s$ with $t + 1 \leqslant s \leqslant \frac{1}{2}p$ when p is even and $t + 1 \leqslant s \leqslant \frac{1}{2}(p+1)$ when p is odd. Analogously to the proof of (vii), we arrive at the contradiction

$$\begin{aligned} k &\geqslant d(p+3) + 3 + 2s \\ &= d(p+3) + 3 + 2(r+1) \\ &= d(p+3) + 5 + 2r \\ &\geqslant d(p+3) + 5 + 2t. \end{aligned}$$

Since we have discussed all possible cases, the proof of Theorem 2 is complete. □

For $p = k = 0$, the statements (iv) and (viii) of Theorem 2 immediately lead to the following 1981 result by Wallis [6].

Corollary 3 (Wallis [6], 1981)**.** *If G is a d-regular graph of order n with no perfect matching and no odd component, then*
(i) $n \geqslant 3d + 7$ *when* $d \geqslant 3$ *is odd,*
(ii) $n \geqslant 3d + 4$ *when* $d \geqslant 4$ *is even.*

For $p = 0$ and $k \geqslant 1$, the statements (i), (ii), (iii), (v), (vii), and (viii) of Theorem 2 yield the following 1994 result by Caccetta and Mardiyono [2].

Corollary 4 (Caccetta, Mardiyono [2], 1994). *If G is a connected $\{d, d+k\}$-graph of even order n without a perfect matching, then the following holds:*

(i) *If $d = 2$ then $k \geqslant 4$ and $n \geqslant k + 6$.*

If $d \geqslant 3$ is odd, then

(ii) $n \geqslant d + k + 1$ *for* $k \geqslant 2d$,

(iii) $n \geqslant 3d + 3$ *for* $d + 1 \leqslant k \leqslant 2d - 2$,

(iv) $n \geqslant 3d + 5$ *for* $2 \leqslant k \leqslant d - 1$.

If $d \geqslant 4$ is even, then

(v) $n \geqslant d + k + 2$ *for* $k \geqslant 3d + 4$,

(vi) $n \geqslant d + k + 4$ *for* $2d \leqslant k \leqslant 3d + 2$,

(vii) $n \geqslant 3d + 4$ *for* $2 \leqslant k \leqslant 2d - 2$.

The following examples show that the various bounds in Theorem 2 are best possible.

Example 5. Let $p \geqslant 0$ and $k \geqslant 2(p + 2)$ be integers such that k is even. In addition, let $P_i = x_1^i x_2^i x_3^i$ for $i = 1, 2, \ldots, p + 3$ and $W_j = y_1^j y_2^j$ for $j = 1, 2, \ldots, \frac{1}{2}(k - 2(p + 2))$ be $p + 3$ paths of length two and $\frac{1}{2}(k - 2(p + 2))$ paths of length one, respectively. If u is a further vertex, then we define the graph G as the disjoint union of $P_1, P_2, \ldots, P_{p+3}$ and $W_1, W_2, \ldots, W_{\frac{1}{2}(k-2(p+2))}$ together with the edge sets $\{ux_1^i\colon 1 \leqslant i \leqslant p + 3\}$, $\{ux_3^i\colon 1 \leqslant i \leqslant p + 3\}$, $\{uy_1^j\colon 1 \leqslant j \leqslant \frac{1}{2}(k - 2(p + 2))\}$, $\{uy_2^j\colon 1 \leqslant j \leqslant \frac{1}{2}(k - 2(p + 2))\}$. The resulting $\{2, 2 + k\}$-graph G is connected of order $n = k + p + 6$ without a matching M of size $2|M| = n - p = k + 6$. This shows that Theorem 2 (i) is best possible.

In the next examples we make use of the following notations.

Let $R(n, m)$ be an m-regular graph of order n.

Let $H(n_1, n_2; d, d-1)$ be a graph of order $n_1 + n_2$ with n_1 vertices of degree d and n_2 vertices of degree $d - 1$.

Example 6. Let $d \geqslant 3$, $k \geqslant 0$ and $p \geqslant 0$ be integers such that d is odd and k and p are of the same parity.

Case 1. Let $k \geqslant d(p + 2)$, and let G_0 consist of the disjoint union of $p + 2$ copies of the complete graph K_d and a graph $R(k - d(p+1), d-1)$. If u is a further vertex, then we join u with the $k + d$ vertices of G_0 having degree $d - 1$. The resulting $\{d, d + k\}$-graph G is connected of order $n = k + p + 1$ without a matching M of size $2|M| = n - p$. This shows that Theorem 2 (ii) is best possible.

Case 2. Let $k = d(p + 2 - t) + t + 2s$ with $0 \leqslant s \leqslant \frac{1}{2}(d - 3)$ and $1 \leqslant t \leqslant p + 2$. In addition, let G_0 consist of the disjoint union of $p + 3 - t$ copies of the complete graph K_d and $t-1$ copies of $H(d+1, 1; d, d-1)$ and a graph $H(d+1-2s, 2s+1; d, d-1)$. If u is a further vertex, then we join u with the $k+d$ vertices of G_0 having degree $d-1$.

The resulting $\{d, d+k\}$-graph G is connected of order $n = d(p+3) + 2t + 1$ without a matching M of size $2|M| = n - p$. This shows that Theorem 2 (iii) is best possible.

Case 3. Let $k \leqslant p$ and $d \geqslant p+3-k$. In addition, let G_0 consist of the disjoint union of $p+2$ copies of $H(d+1, 1; d, d-1)$ and a graph $H(p+4-k, d+k-p-2; d, d-1)$. If u is a further vertex, then we join u with the $k+d$ vertices of G_0 having degree $d-1$. The resulting $\{d, d+k\}$-graph G is connected of order $n = d(p+3) + 2p + 7$ without a matching M of size $2|M| = n - p$. This shows that Theorem 2 (iv) is best possible.

Example 7. Let $d \geqslant 4$, $k \geqslant 0$ and $p \geqslant 0$ be integers such that d and k are even. In addition, let $\eta = 1$ when p is odd and $\eta = 0$ when p is even.

Case 1. Let $k \geqslant d(p+3) + p + 4 + \eta$, and let G_0 consist of the disjoint union of $p+3$ copies of $H(d, 1; d-1, d-2)$ and a graph $H(k-d(p+3), d-(p+3)-\eta; d-1, d-2)$. If u and v are two further vertices, then we join u with all vertices of G_0 and v with all vertices of G_0 having degree $d-2$. If p is odd, then we add also the edge uv. The resulting $\{d, d+k\}$-graph G is connected of order $n = k + d + 2 - \eta$ without a matching M of size $2|M| = n - p$. Thus Theorem 2 (v) is best possible.

Case 2. Let $p \geqslant 1$ and $k = d(p+3) + 4 + 2t$ with $0 \leqslant t \leqslant \frac{1}{2}p - 1$ when p is even and $0 \leqslant t \leqslant \frac{1}{2}(p-1)$ when p is odd and $d \geqslant 2t+4$. In addition, let G_0 consist of the disjoint union of $p-2t+\eta$ copies of $H(1, d; d, d-1)$ and $3+2t-\eta$ copies of $H(d, 1; d-1, d-2)$ and a graph $H(4+2t, d-3-2t; d-1, d-2)$. If u and v are two further vertices, then we join u with all vertices of G_0 having degree less than d and v with all vertices of G_0 having degree $d-2$. If p is odd, then we add also the edge uv. The resulting $\{d, d+k\}$-graph G is connected of order $n = d + k + p + 2 - 2t = d(p+4) + p + 6$ without a matching M of size $2|M| = n - p$. Thus Theorem 2 (vi) is best possible.

Case 3. Let $d(p+2) \leqslant k \leqslant d(p+3) + 2$, and let G_0 consist of the disjoint union of $p+2$ copies of $H(1, d; d, d-1)$ and a graph $H(1, k-d(p+1); d, d-1)$. If u is a further vertex, then we join u with the $k+d$ vertices of G_0 having degree $d-1$. The resulting $\{d, d+k\}$-graph G is connected of order $n = d + k + p + 4$ without a matching M of size $2|M| = n - p$. Thus Theorem 2 (vii) is best possible.

Case 4. Let $k \leqslant d(p+2) - 2$.

Subcase 4.1. Let $d(p+1) + 2 \leqslant k \leqslant d(p+2) - 2$, and let G_0 consist of $p+2$ copies of $H(1, d; d, d-1)$ and a graph $H(d(p+2) - k + 1, k - d(p+1); d, d-1)$. If u is a further vertex, then we join u with the $k+d$ vertices of G_0 having degree $d-1$. The resulting $\{d, d+k\}$-graph G is connected of order $n = d(p+3) + p + 4$ without a matching M of size $2|M| = n - p$. Thus Theorem 2 (viii) is best possible in this case.

Subcase 4.2. Let $k \leqslant d(p+1)$. Assume that $d + k \geqslant 2(p+3)$. In addition, let G_1 consist of $p+3$ copies of $H(d-1, 2; d, d-1)$. The graph G_0 originates from G_1 by deleting a matching of size $\frac{1}{2}(d + k - 2(p+3))$ such that each vertex in G_0 has degree at least $d-1$. If u is a further vertex, then we join u with the $k+d$ vertices of G_0 having degree $d-1$. The resulting $\{d, d+k\}$-graph G is connected of order $n = d(p+3)+p+4$ without a matching M of size $2|M| = n-p$. Thus Theorem 2 (viii) is best possible in this case.

References

[1] *C. Berge*: Sur le couplage maximum d'un graphe. C. R. Acad. Sci. Paris *247* (1958), 258–259. (In French.) zbl

[2] *L. Caccetta, S. Mardiyono*: On the existence of almost-regular-graphs without one-factors. Australas. J. Comb. *9* (1994), 243–260. zbl

[3] *G. Chartrand, L. Lesniak*: Graphs and Digraphs, 3rd Edition. Chapman and Hall, London, 1996. zbl

[4] *W. T. Tutte*: The factorization of linear graphs. J. Lond. Math. Soc. *22* (1947), 107–111. zbl

[5] *L. Volkmann*: Foundations of Graph Theory. Springer-Verlag, Wien-New York, 1996. (In German.) zbl

[6] *W. D. Wallis*: The smallest regular graphs without one-factors. Ars Comb. *11* (1981), 295–300. zbl

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