Czechoslovak Mathematical Journal

Takao Komatsu

Hurwitz continued fractions with confluent hypergeometric functions

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 3, 919–932

Persistent URL: http://dml.cz/dmlcz/128216

Czechoslovak Mathematical Journal, 57 (132) (2007), 919–932

HURWITZ CONTINUED FRACTIONS WITH CONFLUENT HYPERGEOMETRIC FUNCTIONS

Takao Komatsu, Hirosaki

(Received July 19, 2005)

Abstract. Many new types of Hurwitz continued fractions have been studied by the author. In this paper we show that all of these closed forms can be expressed by using confluent hypergeometric functions ${}_0F_1(;c;z)$. In the application we study some new Hurwitz continued fractions whose closed form can be expressed by using confluent hypergeometric functions.

Keywords: Hurwitz continued fractions

MSC 2000: 11A55, 11J70, 33C10

1. Introduction

$\alpha = [a_0; a_1, a_2, \ldots]$ denotes the regular (or simple) continued fraction expansion of a real α, where

$$\alpha = a_0 + \theta_0, \quad a_0 = \lfloor \alpha \rfloor,$$
$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = \lfloor 1/\theta_{n-1} \rfloor \quad (n \geqslant 1).$$

Hurwitz continued fraction expansions have the form

$$\big[a_0; a_1, \ldots, a_n, \overline{Q_1(k), \ldots, Q_p(k)}\big]_{k=1}^{\infty}$$
$$= [a_0; a_1, \ldots, a_n, Q_1(1), \ldots, Q_p(1), Q_1(2), \ldots, Q_p(2), Q_1(3), \ldots],$$

where a_0 is an integer, $a_1, \ldots, a_n$ are positive integers, $Q_1, \ldots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \ldots$ and at

This research was supported in part by the Grant-in-Aid for Scientific research (C) (No. 18540006), the Japan Society for the Promotion of Science.

least one of the polynomials is not constant. Up to the present, some basic known examples are the following.

$$\mathrm{e}^{1/s} = \left[1; \overline{(2k-1)s-1, 1, 1}\right]_{k=1}^{\infty} \quad (s \in \mathbb{Z},\ s > 1).$$
$$a\mathrm{e}^{1/a} = \left[a+1; \overline{2a-1, 2k, 1}\right]_{k=1}^{\infty} \quad (a \in \mathbb{Z}_+).$$
$$\frac{1}{a}\mathrm{e}^{1/a} = \left[0; a-1, 2a, \overline{1, 2k, 2a-1}\right]_{k=1}^{\infty} \quad (a \in \mathbb{Z},\ a > 1).$$
$$\mathrm{e}^{2} = \left[7; \overline{3k-1, 1, 1, 3k, 12k+6}\right]_{k=1}^{\infty}.$$
$$\mathrm{e}^{2/s} = [1; \overline{3ks - \tfrac{1}{2}(5s+1), 12ks-6s, 3ks - \tfrac{1}{2}(s+1), 1, 1}]_{k=1}^{\infty} \quad (s \text{ odd},\ s \geqslant 3).$$
$$\sqrt{\frac{v}{u}} \tanh \frac{1}{\sqrt{uv}} = \left[0; \overline{(4k-3)u, (4k-1)v}\right]_{k=1}^{\infty} \quad (u, v \in \mathbb{Z}_+).$$
$$\frac{I_{(a/b)+1}(2/b)}{I_{a/b}(2/b)} = \left[0; \overline{a+kb}\right]_{k=1}^{\infty},$$

where $I_\lambda(z)$ are the modified Bessel functions of the first kind, defined by

$$I_\lambda(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\lambda+2n}}{n!\Gamma(\lambda+n+1)}.$$
$$\sqrt{\frac{v}{u}} \tan \frac{1}{\sqrt{uv}} = \left[0; u-1, \overline{1, (4k-1)v-2, 1, (4k+1)u-2}\right]_{k=1}^{\infty}.$$
$$\frac{J_{(a/b)+1}(2/b)}{J_{a/b}(2/b)} = \left[0; a+b-1, \overline{1, a+(k+1)b-2}\right]_{k=1}^{\infty},$$

where $J_\lambda(z)$ are the Bessel functions of the first kind, defined by

$$J_\lambda(z) = \left(\frac{z}{2}\right)^{\lambda} \sum_{n=0}^{\infty} \frac{(\mathrm{i}z/2)^{2n}}{n!\Gamma(\lambda+n+1)}.$$

It seems that each one of the above belongs to one of the types, e-type, tanh-type, tan-type and e^2-type. No concrete example where the degree of any polynomial exceeds 1 is known.

Recently, the author [4] obtained a generalized tanh-type Hurwitz continued fraction as

$$\left[0; \overline{u(a+(2k-1)b), v(a+2kb)}\right]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} (n!)^{-1} u^{-n-1} (vb)^{-n} \prod_{i=1}^{n+1} (a+bi)^{-1}}{\sum_{n=0}^{\infty} (n!)^{-1} (uvb)^{-n} \prod_{i=1}^{n} (a+bi)^{-1}},$$

which includes the cases of $\sqrt{v/u}\tanh 1/\sqrt{uv}$ and $I_{(a/b)+1}(2/b)/I_{a/b}(2/b)$. The author also obtained a generalized tan-type Hurwitz continued fraction as

$$\begin{aligned}&\big[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}\big]\\&\quad=\frac{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}},\end{aligned}$$

which includes the cases of $\sqrt{v/u}\tan 1/\sqrt{uv}$ and $J_{(a/b)+1}(2/b)/J_{a/b}(2/b)$.

In [6], the author constituted more general forms of Hurwitz continued fractions of e-type, namely, some extended forms of the continued fractions of $\mathrm{e}^{1/s}$, $a\mathrm{e}^{1/a}$ and $(1/a)\mathrm{e}^{1/a}$.

$$\begin{aligned}&\big[0; \overline{u(a+kb)-1, 1, v-1}\big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}b^{-n}(n!)^{-1}\Big((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\Big)}\end{aligned}$$

and

$$\begin{aligned}&\big[0; \overline{v-1, 1, u(a+kb)-1}\big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}b^{-n}(n!)^{-1}\Big(u^{-2n}v^{-2n-1}\prod_{i=1}^{n}(a+bi)^{-1}+u^{-2n-1}v^{-2n-2}\prod_{i=1}^{n+1}(a+bi)^{-1}\Big)}{\sum_{n=0}^{\infty}(uv)^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n}(a+bi)^{-1}}.\end{aligned}$$

There are still more known Hurwitz continued fractions which may not belong to any of the above categories. Most of them are easily derived from one of the basic types (see e.g. [4, Props. 1 and 2]).

2. Confluent hypergeometric functions

Some generalized Hurwitz continued fractions which the author obtained become elegant to look at with the aid of hypergeometric functions (see e.g. [9]). This was partly done by the author [5] without the complete form of Hurwitz continued fractions.

Using the notation of confluent hypergeometric limit functions defined by

$$ {}_0F_1(;c;z)=\sum_{n=0}^{\infty}\frac{1}{(c)_n}\frac{z^n}{n!}$$

with $(c)_n = c(c+1)\dots(c+n-1)$ $(n \geqslant 1)$ and $(c)_0 = 1$, we can write our generalized tanh-type and tan-type Hurwitz continued fractions as

$$(1)\qquad \left[0; \overline{u(a+(2k-1)b), v(a+2kb)}\right]_{k=1}^{\infty} = \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{uvb^2}\right)}{u(a+b){}_0F_1\left(; \frac{a}{b}+1; \frac{1}{uvb^2}\right)}$$

and

$$(2)\qquad \left[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}\right] = \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{-1}{uvb^2}\right)}{u(a+b){}_0F_1\left(; \frac{a}{b}+1; \frac{-1}{uvb^2}\right)},$$

respectively. These transformations were achieved by

$$\frac{\sum_{n=0}^{\infty}(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}} = \frac{\sum_{n=1}^{\infty}\frac{1}{ub(a/b+1)\cdot(a/b+2)_n}\frac{1}{n!}\left(\frac{1}{uvb^2}\right)^n}{\sum_{n=1}^{\infty}\frac{1}{(a/b+1)_n}\frac{1}{n!}\left(\frac{1}{uvb^2}\right)^n} = \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{uvb^2}\right)}{u(a+b){}_0F_1\left(; \frac{a}{b}+1; \frac{1}{uvb^2}\right)}.$$

Using the confluent hypergeometric limit functions, we can write the generalized e-type Hurwitz continued fractions as

$$(3)\qquad \left[0; \overline{u(a+kb)-1, 1, v-1}\right]_{k=1}^{\infty} = \frac{v{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{u^2v^2b^2}\right)}{uv(a+b){}_0F_1\left(; \frac{a}{b}+1; \frac{1}{u^2v^2b^2}\right) - {}_0F_1\left(; \frac{a}{b}+2; \frac{1}{u^2v^2b^2}\right)}$$

and

$$(4)\qquad \left[0; \overline{v-1, 1, u(a+kb)-1}\right]_{k=1}^{\infty} = \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{u^2v^2b^2}\right)}{uv^2(a+b){}_0F_1\left(; \frac{a}{b}+1; \frac{1}{u^2v^2b^2}\right)} + \frac{1}{v},$$

respectively.

In this paper we study some more general and new Hurwitz continued fractions by using the confluent hypergeometric functions.

3. Main results

Theorem 1.

$$\left[0; \overline{u(a+(2k-1)b)-1, 1, v-1, u'(a+2kb)-1, 1, v-1}\right]_{k=1}^{\infty}$$
$$= \frac{v\,{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{uu'v^2b^2}\right)}{uv(a+b)\,{}_0F_1\left(; \frac{a}{b}+1; \frac{1}{uu'v^2b^2}\right) - {}_0F_1\left(; \frac{a}{b}+2; \frac{1}{uu'v^2b^2}\right)},$$

where v is an integer with $v > 1$, and rational numbers u, u', a and b are chosen so that $u(a+(2k-1)b)-1$ and $u'(a+2kb)-1$ take positive integral values for $k = 1, 2, \ldots$.

Remark. If $u = u'$, this reduces to (3).

Theorem 2.

$$\left[0; \overline{v-1, 1, u(a+(2k-1)b)-1, v-1, 1, u'(a+2kb)-1}\right]_{k=1}^{\infty}$$
$$= \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{1}{uu'v^2b^2}\right)}{uv^2(a+b)\,{}_0F_1\left(; \frac{a}{b}+1; \frac{1}{uu'v^2b^2}\right)} + \frac{1}{v},$$

where v is an integer with $v > 1$, and rational numbers u, u', a and b are chosen so that $u(a+(2k-1)b)-1$ and $u'(a+2kb)-1$ take positive integral values for $k = 1, 2, \ldots$.

Remark. If $u = u'$, this reduces to (4).

Theorem 3.

$$\left[0; \overline{u(a+(2k-1)b)-1, 1, v-2, 1, u'(a+2kb)-1, v}\right]_{k=1}^{\infty}$$
$$= \frac{v\,{}_0F_1\left(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2}\right)}{uv(a+b)\,{}_0F_1\left(; \frac{a}{b}+1; \frac{-1}{uu'v^2b^2}\right) - {}_0F_1\left(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2}\right)},$$

where v is an integer with $v > 2$, and rational numbers u, u', a and b are chosen so that $u(a+(2k-1)b)-1$ and $u'(a+2kb)-1$ take positive integral values for $k = 1, 2, \ldots$.

Theorem 4.

$$\left[0; v-1, \overline{1, u(a+(2k-1)b)-1, v, u'(a+2kb)-1, 1, v-2}\right]_{k=1}^{\infty}$$
$$= \frac{{}_0F_1\left(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2}\right)}{uv^2(a+b)\,{}_0F_1\left(; \frac{a}{b}+1; \frac{-1}{uu'v^2b^2}\right)} + \frac{1}{v},$$

where v is an integer with $v > 2$, and rational numbers u, u', a and b are chosen so that $u(a+(2k-1)b)-1$ and $u'(a+2kb)-1$ take positive integral values for $k = 1, 2, \ldots$.

Theorem 5.

$$\big[0;1,\overline{u((2k-1)b+2),kb+1}\big]_{k=1}^{\infty}=\frac{{}_0F_1\big(;\frac{2}{b}+1;\frac{2}{ub^2}\big)}{{}_0F_1\big(;\frac{2}{b};\frac{2}{ub^2}\big)},$$

where b is a positive integer and u is a positive rational number so that both ub and $2u$ are positive integers.

Theorem 6.

$$\big[1;\overline{u((2k-1)b+2)-2,1,kb-1,1}\big]_{k=1}^{\infty}=\frac{{}_0F_1\big(;\frac{2}{b}+1;\frac{-2}{ub^2}\big)}{{}_0F_1\big(;\frac{2}{b};\frac{-2}{ub^2}\big)},$$

where b is a positive integer and u is a positive rational number so that both ub and $2u$ are positive integers.

4. Proof of the results

We use Hurwitz's method to obtain the continued fraction expansion $(a\alpha+b)/d$ from the continued fraction expansion of α. In most practical cases it is enough to consider the rational linear forms of α. According to Satz 4.1 [7, p. 111], which is essentially from Hurwitz [2] and Châtelet [1], it says

Lemma 1. *Let $[a_0;a_1,a_2,\dots]$ be the regular continued fraction of an irrational number α and denote its nth convergent by $p_n/q_n=[a_0;a_1,a_2,\dots,a_n]$. Moreover, let $\beta=(r_0\alpha+t_0)/s_0$, where r_0, s_0 and t_0 are integers with $r_0>0$, $s_0>0$ and $r_0s_0=N>1$. For an arbitrary index $\nu\geqslant 1$ we have*

$$\frac{r_0[a_0;a_1,\dots,a_{\nu-1}]+t_0}{s_0}=\frac{r_0p_{\nu-1}+t_0q_{\nu-1}}{s_0q_{\nu-1}}=[b_0;b_1,\dots,b_{\mu-1}]$$

where the index μ is adjusted so that $\mu\equiv\nu \pmod 2$. Denote its convergent by

$$\frac{p'_{\mu-1}}{q'_{\mu-1}}=[b_0;b_1,\dots,b_{\mu-1}].$$

Then three integers t_1, r_1 and s_1 are uniquely given satisfying the matrix formula

$$\begin{pmatrix}r_0&t_0\\0&s_0\end{pmatrix}\begin{pmatrix}p_{\nu-1}&p_{\nu-2}\\q_{\nu-1}&q_{\nu-2}\end{pmatrix}=\begin{pmatrix}p'_{\mu-1}&p'_{\mu-2}\\q'_{\mu-1}&q'_{\mu-2}\end{pmatrix}\begin{pmatrix}r_1&t_1\\0&s_1\end{pmatrix},$$

where $r_1>0$, $s_1>0$, $r_1s_1=N$, $-s_1\leqslant t_1\leqslant r_1$ and $\beta=[b_0;b_1,\dots,b_{\mu-1},\beta_\mu]$ with $\beta_\mu=(r_1\alpha_\nu+t_1)/s_1$.

P r o o f of Theorem 1. Replacing v by $u'v^2$ and taking its reciprocal in (1), we have

$$(\alpha :=)\ u(a+b)\cdot\frac{{}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2}\right)}{{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2}\right)}=\left[\overline{u(a+(2k-1)b),u'v^2(a+2kb)}\right]_{k=1}^{\infty}.$$

We shall find the continued fraction of $\alpha-1/v$ by applying Lemma 1. For $k=1,2,\dots$ we have

$$\begin{aligned}\frac{v[u(a+(2k-1)b)]-1}{v}&=u(a+(2k-1)b)-\frac{1}{v}\\&=[u(a+(2k-1)b)-1;1,v-1]\end{aligned}$$

and

$$\begin{aligned}&\begin{pmatrix}v&-1\\0&v\end{pmatrix}\begin{pmatrix}u(a+(2k-1)b)&1\\1&0\end{pmatrix}\\&\quad=\begin{pmatrix}uv(a+(2k-1)b)-1&u(a+(2k-1)b)\\v&1\end{pmatrix}\begin{pmatrix}1&-v\\0&v^2\end{pmatrix},\end{aligned}$$

thus

$$\begin{aligned}\frac{[u'v^2(a+2kb)]-v}{v^2}&=u'(a+2kb)-\frac{1}{v}\\&=[u'(a+2kb)-1;1,v-1]\end{aligned}$$

and

$$\begin{pmatrix}1&-v\\0&v^2\end{pmatrix}\begin{pmatrix}u'v^2(a+2kb)&1\\1&0\end{pmatrix}=\begin{pmatrix}u'v(a+2kb)-1&u'(a+2kb)\\v&1\end{pmatrix}\begin{pmatrix}v&-1\\0&v\end{pmatrix}.$$

Therefore, we obtain the desired continued fraction as

$$\frac{1}{\alpha-1/v}=\left[0;\overline{u(a+(2k-1)b)-1,1,v-1,u'(a+2kb)-1,1,v-1}\right]_{k=1}^{\infty}.$$

□

P r o o f of Theorem 2. Replacing u by uv^2 and v by u' in (1), we have

$$(\alpha :=)\ \frac{1}{uv^2(a+b)}\frac{{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2}\right)}{{}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2}\right)}=\left[0;\overline{uv^2(a+(2k-1)b),u'(a+2kb)}\right]_{k=1}^{\infty}.$$

We shall find the desired continued fraction as $\alpha+1/v$ by applying Lemma 1. First,

$$\frac{v[0]+1}{v}=[0;v-1,1]$$

and

$$\begin{pmatrix} v & 1 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ v & v-1 \end{pmatrix} \begin{pmatrix} 1 & v-v^2 \\ 0 & v^2 \end{pmatrix}.$$

Next, for $k = 1, 2, \ldots$ we have

$$\begin{aligned} \frac{[uv^2(a+(2k-1)b)] + v - v^2}{v^2} &= u(a+(2k-1)b) - 1 + \frac{1}{v} \\ &= [u(a+(2k-1)b) - 1; v-1, 1] \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} 1 & v-v^2 \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} uv^2(a+(2k-1)b) & 1 \\ 1 & 0 \end{pmatrix} \\ = &\begin{pmatrix} uv(a+(2k-1)b) - v + 1 & u(v-1)(a+(2k-1)b) - v + 2 \\ v & v-1 \end{pmatrix} \begin{pmatrix} v & 1-v \\ 0 & v \end{pmatrix}, \end{aligned}$$

thus

$$\begin{aligned} \frac{v[u'(a+2kb)] + 1 - v}{v} &= u'(a+2kb) - 1 + \frac{1}{v} \\ &= [u'(a+2kb) - 1; v-1, 1] \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} v & 1-v \\ 0 & v \end{pmatrix} \begin{pmatrix} u'(a+2kb) & 1 \\ 1 & 0 \end{pmatrix} \\ = &\begin{pmatrix} vu'(a+2kb) + 1 - v & (v-1)u'(a+2kb) + 2 - v \\ v & v-1 \end{pmatrix} \begin{pmatrix} 1 & v-v^2 \\ 0 & v^2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha + \frac{1}{v} &= \big[0; v-1, 1, \overline{u(a+(2k-1)b) - 1, v-1, 1, u'(a+2kb) - 1, v-1, 1}\big]_{k=1}^{\infty} \\ &= \big[0; \overline{v-1, 1, u(a+(2k-1)b) - 1, v-1, 1, u'(a+2kb) - 1}\big]_{k=1}^{\infty}. \end{aligned}$$

□

P r o o f of Theorem 3. Replacing v by $u'v^2$ and taking its reciprocal in (2), we have

$$\begin{aligned} (\alpha :=)\, u(a+b) \cdot \frac{{}_0F_1\big(; \frac{a}{b}+1; \frac{-1}{uu'v^2b^2}\big)}{{}_0F_1\big(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2}\big)} \\ = [u(a+b) - 1; \overline{1, u'v^2(a+2kb) - 2, 1, u(a+(2k+1)b) - 2}\,]_{k=1}^{\infty}. \end{aligned}$$

The desired continued fraction is obtained as $1/(\alpha - 1/v)$. □

Proof of Theorem 4. Replacing u by uv^2 and v by u' in (2), we have

$$\begin{aligned}(\alpha :=)\ &\frac{1}{uv^2(a+b)}\frac{{}_0F_1\left(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2}\right)}{{}_0F_1\left(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2}\right)}\\ &= \left[0; uv^2(a+b)-1, \overline{1, u'(a+2kb)-2, 1, uv^2(a+(2k+1)b)-2}\right]_{k=1}^{\infty}.\end{aligned}$$

We can obtain the desired continued fraction as $\alpha + 1/v$. □

Proof of Theorem 5. This proof can be also done in a similar manner to those of Theorems 1 to 4. However, we shall proceed in a more direct way. By (91.4) in [10, Chapter 18]

$$\frac{{}_0F_1(;c+1;z)}{{}_0F_1(;c;z)} = \frac{1}{1+}\,\frac{z/c(c+1)}{1}\,\frac{}{+}\,\frac{z/(c+1)(c+2)}{1}\,\frac{}{+}\,\frac{z/(c+2)(c+3)}{1}\,\frac{}{+\ldots}.$$

We transform this to a simple continued fraction by using the following formula in [3, p. 35, (2.3.23)] (cf. [5, Lemma 1]):

$$[a_0; a_1, a_2, \ldots] = a_0^* + \frac{b_1^*}{a_1^*+}\,\frac{b_2^*}{a_2^*+}\,\frac{b_3^*}{a_3^*+\ldots}$$

iff $a_0 = a_0^*$, $a_1 = a_1^*/b_1^*$ and for $k = 1, 2, \ldots$

$$a_{2k} = \frac{b_{2k-1}^* b_{2k-3}^* \ldots b_1^*}{b_{2k}^* b_{2k-2}^* \ldots b_2^*} a_{2k}^* \quad \text{and} \quad a_{2k+1} = \frac{b_{2k}^* b_{2k-2}^* \ldots b_2^*}{b_{2k+1}^* b_{2k-1}^* \ldots b_1^*} a_{2k+1}^*.$$

Then our case turns to

$$\frac{{}_0F_1(;c+1;z)}{{}_0F_1(;c;z)} = \left[0; 1, \frac{c(c+1)}{z}, \frac{c+2}{c}, \frac{c(c+3)}{z}, \frac{c+4}{c}, \frac{c(c+5)}{z}, \frac{c+6}{c}, \ldots\right].$$

If we set $c = 2/b$ and $z = 2/(ub^2)$, we get the desired result. □

Proof of Theorem 6. From Theorem 5 with the rule $[\ldots, a, -b, \gamma] = [\ldots, a-1, 1, b-1, -\gamma]$, we have

$$\begin{aligned}&\frac{{}_0F_1\left(;\frac{2}{b}+1;\frac{-2}{ub^2}\right)}{{}_0F_1\left(;\frac{2}{b};\frac{-2}{ub^2}\right)}\\
&= [0; 1, -(b+2)u, b+1, -(3b+2)u, 2b+1, -(5b+2)u, 3b+1, \ldots]\\
&= [0; 0, 1, (b+2)u-1, -(b+1), (3b+2)u, -(2b+1), (5b+2)u, -(3b+1), \ldots]\\
&= [1; (b+2)u-2, 1, b, -(3b+2)u, 2b+1, -(5b+2)u, 3b+1, \ldots]\\
&= [1; (b+2)u-2, 1, b-1, 1, (3b+2)u-1, -(2b+1), (5b+2)u, -(3b+1), \ldots]\\
&= \ldots\\
&= [1; (b+2)u-2, 1, b-1, 1, (3b+2)u-2, 1, 2b-1, 1, (5b+2)u-2, 1, 3b-1, 1, \ldots].\end{aligned}$$

□

5. Some variations on confluent hypergeometric limit functions

We shall study some variations which can be derived from the basic results (1) to (4). Consider the case where $u(a+b)$ is even and va is even in (1). By Raney's method ([8]), the continued fraction $[a_0; a_1, a_2, \ldots]$ corresponds to the matrix product

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots = R^{a_0}L^{a_1}R^{a_2}\cdots,$$

where

$$R^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L^a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

Using the *transition formulae*

$$\begin{aligned} AR &= R^c A, & A'L &= L^c A', \\ AL^c &= LA, & A'R^c &= RA', \\ ALR^{c-1} &= R^{c-1}LA', & A'RL^{c-1} &= L^{c-1}RA \end{aligned}$$

in the case where $c=2$, we have for $k=1,2,\ldots$

$$\begin{aligned} &\frac{1}{2}[u(a+(2k-1)b)-1; v(a+2kb), u(a+(2k+1)b), v(a+(2k+2)b), \ldots] \\ &\quad \leftrightarrow A'R^{u(a+(2k-1)b)-1}L^{v(a+2kb)}R^{u(a+(2k+1)b)}L^{v(a+(2k+2)b)}\cdots \\ &= R^{\frac{u(a+(2k-1)b)}{2}-1}A'RL^{v(a+2kb)}R^{u(a+(2k+1)b)}L^{v(a+(2k+2)b)}\cdots \\ &= R^{\frac{u(a+(2k-1)b)}{2}-1}LRAL^{v(a+2kb)-1}R^{u(a+(2k+1)b)}L^{v(a+(2k+2)b)}\cdots \\ &= R^{\frac{u(a+(2k-1)b)}{2}-1}LRL^{\frac{v(a+2kb)}{2}-1}ALR^{u(a+(2k+1)b)}L^{v(a+(2k+2)b)}\cdots \\ &= R^{\frac{u(a+(2k-1)b)}{2}-1}LRL^{\frac{v(a+2kb)}{2}-1}RLA'R^{u(a+(2k+1)b)-1}L^{v(a+(2k+2)b)}\cdots \\ &\quad \leftrightarrow \Big[\frac{u(a+(2k-1)b)}{2}-1; 1, 1, \frac{v(a+2kb)}{2}-1, 1, 1, \ldots\Big]. \end{aligned}$$

Therefore, if

$$\alpha = \frac{{}_0F_1\big(; \frac{a}{b}+2; \frac{1}{uvb^2}\big)}{u(a+b){}_0F_1\big(; \frac{a}{b}+1; \frac{1}{uvb^2}\big)},$$

then

$$\begin{aligned} \frac{1-\alpha}{1+\alpha} &= \frac{1}{1+\frac{1}{\frac{1}{2}(1/\alpha-1)}} = \Big[0; 1, \tfrac{1}{2}\big[u(a+b)-1; \overline{v(a+2kb), u(a+(2k+1)b)}\big]_{k=1}^{\infty}\Big] \\ &= \Big[0; 1, \overline{\tfrac{1}{2}u(a+(2k-1)b)-1, 1, 1, \tfrac{1}{2}v(a+2kb)-1, 1, 1}\Big]_{k=1}^{\infty}. \end{aligned}$$

In a similar manner, from (1) we have

Theorem 7.

$$\frac{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)-{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)+{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}$$

$$=\begin{cases}\Big[0;\overline{1,\frac{u(a+(6k-5)b)-1}{2},2v(a+(6k-4)b),\frac{u(a+(6k-3)b)-1}{2},1,1,\frac{v(a+(6k-2)b)-1}{2},}\\ \quad\overline{2u(a+(6k-1)b),\frac{v(a+6kb)-1}{2},1}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is odd and } va \textit{ is odd};\\ \Big[0;\overline{1,\frac{u(a+(4k-3)b)}{2}-1,1,1,\frac{v(a+(4k-2)b)-1}{2},2u(a+(4k-1)b),}\\ \quad\overline{\frac{v(a+4kb)-1}{2},1}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is even and } va \textit{ is odd};\\ \Big[0;\overline{1,\frac{u(a+(4k-3)b)-1}{2},2v(a+(4k-2)b),\frac{u(a+(4k-1)b)-1}{2},1,1,}\\ \quad\overline{\frac{v(a+4kb)}{2}-1,1}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is odd and } va \textit{ is even};\\ \Big[0;\overline{1,\frac{u(a+(2k-1)b)}{2}-1,1,1,\frac{v(a+2kb)}{2}-1,1}\Big]_{k=1}^{\infty}\\ \quad \textit{if } u(a+b) \textit{ is even and } va \textit{ is even}.\end{cases}$$

From (2) we have

Theorem 8.

$$\frac{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{-1}{uvb^2}\right)-{}_0F_1\left(;\frac{a}{b}+2;\frac{-1}{uvb^2}\right)}{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{-1}{uvb^2}\right)+{}_0F_1\left(;\frac{a}{b}+2;\frac{-1}{uvb^2}\right)}$$

$$=\begin{cases}\Big[0;1,\overline{\frac{u(a+(6k-5)b)-3}{2},1,2v(a+(6k-4)b)-2,1,\frac{u(a+(6k-3)b)-3}{2},2,}\\ \quad\overline{\frac{v(a+(6k-2)b)-3}{2},1,2u(a+(6k-1)b)-2,1,\frac{v(a+6kb)-3}{2},2}\Big]_{k=1}^{\infty}\\ \quad \textit{if } u(a+b) \textit{ is odd and } va \textit{ is odd};\\ \Big[0;1,\overline{\frac{u(a+(4k-3)b)}{2}-1,2,\frac{v(a+(4k-2)b)-3}{2},1,2u(a+(4k-1)b)-2,1,}\\ \quad\overline{\frac{v(a+4kb)-3}{2},2}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is even and } va \textit{ is odd};\\ \Big[0;1,\overline{\frac{u(a+(4k-3)b)-3}{2},1,2v(a+(4k-2)b)-2,1,\frac{u(a+(4k-1)b)-3}{2},2,}\\ \quad\overline{\frac{v(a+4kb)}{2}-1,2}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is odd and } va \textit{ is even};\\ \Big[0;1,\overline{\frac{u(a+(2k-1)b)}{2}-1,2,\frac{v(a+2kb)}{2}-1,2}\Big]_{k=1}^{\infty} \quad \textit{if } u(a+b) \textit{ is even and } va \textit{ is even}.\end{cases}$$

From (3) we have

Theorem 9.

$$\frac{uv(a+b)_0F_1\big(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}\big)-(v+1)_0F_1\big(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2}\big)}{uv(a+b)_0F_1\big(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}\big)+(v-1)_0F_1\big(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2}\big)}$$

$$=\begin{cases}
\Big[0;\overline{1,\frac{u(a+(4k-3)b)-3}{2},1,2v-1,\frac{u(a+(4k-2)b)}{2}-1,1,2v-1,\frac{u(a+(4k-1)b)-1}{2},2,}\\
\quad\overline{\frac{v}{2}-1,1,1,\frac{u(a+4kb)}{2}-1,2,\frac{v}{2}-1,1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is odd, } ua \textit{ is even, } v \textit{ is even;}\\
\Big[0;\overline{1,\frac{u(a+(4k-3)b)-3}{2},1,2v-1,\frac{u(a+(4k-2)b)}{2}-1,1,2v-1,\frac{u(a+(4k-1)b)-1}{2},}\\
\quad\overline{2,\frac{v-1}{2},2u(a+4kb)-1,1,\frac{v-3}{2},1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is odd, } ua \textit{ is even, } v \textit{ is odd;}\\
\Big[0;\overline{1,\frac{u(a+(2k-1)b)-3}{2},1,2v-1,\frac{u(a+2kb)-1}{2},2,\frac{v}{2}-1,1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is odd, } ua \textit{ is odd, } v \textit{ is even;}\\
\Big[0;\overline{1,\frac{a+(3k-2)b)-3}{2},1,2v-1,\frac{u(a+(3k-1)b)-1}{2},2,\frac{v-1}{2},2u(a+3kb)-1,}\\
\quad\overline{1,\frac{v-3}{2},1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is odd, } ua \textit{ is odd, } v \textit{ is odd;}\\
\Big[0;\overline{1,\frac{u(a+(4k-3)b)}{2}-1,2,\frac{v}{2}-1,1,1,\frac{u(a+(4k-2)b)-3}{2},1,2v-1,}\\
\quad\overline{\frac{u(a+(4k-1)b)}{2}-1,1,2v-1,\frac{u(a+4kb)-1}{2},2,\frac{v}{2}-1,1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is even, } ua \textit{ is odd, } v \textit{ is even;}\\
\Big[0;\overline{1,\frac{u(a+kb)}{2}-1,2,\frac{v}{2}-1,1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is even, } ua \textit{ is even, } v \textit{ is even;}\\
\Big[0;\overline{1,\frac{u(a+(2k-1)b)}{2}-1,2,\frac{v-1}{2},2u(a+2kb)-1,1,\frac{v-3}{2},1}\Big]_{k=1}^{\infty}\\
\qquad \textit{if } u(a+b) \textit{ is even, } v \textit{ is odd.}
\end{cases}$$

From (4) we have

Theorem 10.

$$\frac{uv(v-1)(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}\right)-{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2}\right)}{uv(v+1)(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{u^2v^2b^2}\right)+{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{u^2v^2b^2}\right)}$$

$$=\begin{cases}\left[0;\overline{1,\frac{v-3}{2},1,2u(a+(2k-1)b)-1,\frac{v-1}{2},2,\frac{u(a+2kb)}{2}-1,1}\right]_{k=1}^{\infty}\\ \quad \text{if } v \text{ is odd, } ua \text{ is even;}\\ \left[0;\overline{1,\frac{v-3}{2},1,2u(a+(3k-2)b)-1,\frac{v-1}{2},2,\frac{u(a+(3k-1)b)-1}{2},}\right.\\ \quad \left.\overline{2v-1,1,\frac{u(a+3kb)-3}{2},1}\right]_{k=1}^{\infty}\\ \quad \text{if } v \text{ is odd, } ua \text{ is odd, } b \text{ is even;}\\ \left[0;\overline{1,\frac{v-3}{2},1,2u(a+(4k-3)b)-1,\frac{v-1}{2},2,\frac{u(a+(4k-2)b)-1}{2},2v-1,1,}\right.\\ \quad \left.\overline{\frac{u(a+(4k-1)b)}{2}-1,2v-1,1,\frac{u(a+4kb)-3}{2},1}\right]_{k=1}^{\infty} \quad \text{if } v \text{ is odd, } ua \text{ is odd, } b \text{ is odd;}\\ \left[0;\overline{1,\frac{v}{2}-1,2,\frac{u(a+kb)}{2}-1,1}\right]_{k=1}^{\infty} \quad \text{if } v \text{ is even, } ua \text{ is even, } ub \text{ is even;}\\ \left[0;\overline{1,\frac{v}{2}-1,2,\frac{u(a+(4k-3)b)}{2}-1,1,1,\frac{v}{2}-1,2,\frac{u(a+(4k-2)b)-1}{2},2v-1,1,}\right.\\ \quad \left.\overline{\frac{u(a+(4k-1)b)}{2}-1,2v-1,1,\frac{u(a+4kb)-3}{2},1}\right]_{k=1}^{\infty}\\ \quad \text{if } v \text{ is even, } ua \text{ is odd, } ub \text{ is odd;}\\ \left[0;\overline{1,\frac{v}{2}-1,2,\frac{u(a+(2k-1)b)-1}{2},2v-1,1,\frac{u(a+2kb)-3}{2},1}\right]_{k=1}^{\infty}\\ \quad \text{if } v \text{ is even, } ua \text{ is odd, } ub \text{ is even;}\\ \left[0;\overline{1,\frac{v}{2}-1,2,\frac{u(a+(4k-3)b)-1}{2},2v-1,1,\frac{u(a+(4k-2)b)}{2}-1,2v-1,1,}\right.\\ \quad \left.\overline{\frac{u(a+(4k-1)b)-3}{2},1,1,\frac{v}{2}-1,2,\frac{u(a+4kb)}{2}-1,1}\right]_{k=1}^{\infty}\\ \quad \text{if } v \text{ is even, } ua \text{ is even, } ub \text{ is odd.}\end{cases}$$

It is possible to obtain some more general results. If we replace the first partial quotient 1 by any positive integer a_1, we consider the form $\left[0;a_1,\frac{1}{2}\left(\frac{1}{\alpha}-1\right)\right]$ for α with $0<\alpha<1$. For example, in the case where both $u(a+b)$ and va are even in (1) we have

$$\frac{u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)-{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}{a_1u(a+b){}_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)+(2-a_1){}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}$$

$$=\left[0;a_1,\overline{\frac{u(a+(2k-1)b)}{2}-1,1,1,\frac{v(a+2kb)}{2}-1,1,1}\right]_{k=1}^{\infty}.$$

If we consider the form $\left[0; a_1, 2\left(\frac{1}{\alpha}-1\right)\right]$ for α, we can obtain e.g. in the case where va is even in (1),

$$\frac{u(a+b)_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)-{}_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}{a_1u(a+b)_0F_1\left(;\frac{a}{b}+1;\frac{1}{uvb^2}\right)+\left(\frac{1}{2}-a_1\right)_0F_1\left(;\frac{a}{b}+2;\frac{1}{uvb^2}\right)}$$
$$=\left[0; a_1, 2u(a+b)-2, \overline{\frac{v(a+2kb)}{2}, 2u(a+(2k+1)b)}\right]_{k=1}^{\infty}.$$

Acknowledgement. This work was partly done while the author was visiting the Department of Pure Mathematics and Mathematical Statistics, University of Cambridge. He is grateful to Professor Alan Baker for his warm hospitality. The author thanks the referee for his/her careful reading of the manuscript.

References

[1] *A. Châtelet*: Contribution a la théorie des fractions continues arithmétiques. Bull. Soc. Math. France *40* (1912), 1–25. zbl

[2] *A. Hurwitz*: Über die Kettenbrüche, deren Teilnenner arithmetische Reihen bilden. Vierteljahrsschrift d. Naturforsch. Gesellschaft in Zürich, Jahrg. 41, 1896; Mathematische Werke, Band II. Birkhäuser-Verlag, Basel, 1963, pp. 276–302. zbl

[3] *W. B. Jones, W. J. Thron*: Continued Fractions: Analytic Theory and Applications (Encyclopedia of mathematics and its applications, Vol. 11). Addison-Wesley, Reading, 1980. zbl

[4] *T. Komatsu*: On Hurwitzian and Tasoev's continued fractions. Acta Arith. *107* (2003), 161–177. zbl

[5] *T. Komatsu*: Simple continued fraction expansions of some values of certain hypergeometric functions. Tsukuba J. Math. *27* (2003), 161–173. zbl

[6] *T. Komatsu*: Hurwitz and Tasoev continued fractions. Monatsh. Math. *145* (2005), 47–60. zbl

[7] *O. Perron*: Die Lehre von den Kettenbrüchen, Band I. Teubner, Stuttgart, 1954. zbl

[8] *G. N. Raney*: On continued fractions and finite automata. Math. Ann. *206* (1973), 265–283. zbl

[9] *L. J. Slater*: Generalized hypergeometric functions. Cambridge Univ. Press, Cambridge, 1966. zbl

[10] *H. S. Wall*: Analytic Theory of Continued Fractions. D. van Nostrand Company, Toronto, 1948. zbl

Author's address: Takao Komatsu, Department of Mathematical Sciences, Faculty of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan, e-mail: komatsu@cc.hirosaki-u.ac.jp.