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HIGHER ORDER JET INVOLUTION

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Abstract. We introduce an exchange natural isomorphism between iterated higher order jet functors depending on a classical linear connection on the base manifold. As an application we study the prolongation of higher order connections to jet bundles.

Keywords: jet prolongation, classical linear connection, higher order connection

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1. Preliminaries

It is well known that the canonical involution of the iterated tangent bundle interchanges both the projections of TTN into TN. In general, consider an arbitrary couple F and G of product preserving functors on the category $\mathcal{M}f$ of smooth manifolds and all smooth maps and denote by p^F and p^G the bundle projections. By [8], there is a natural equivalence

$$\kappa^{F,G}\colon\ FG \to GF \tag{1}$$

such that for every smooth manifold N the following diagram commutes

$$\begin{tikzcd} FGN \arrow[rr, "\kappa_N^{F,G}"] \arrow[dr, "F(p_N^G)"'] & & GFN \arrow[dl, "p_{FN}^G"] \\ & FN & \end{tikzcd} \tag{2}$$

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In particular, denoting by T^r_m the functor of m-dimensional velocities of order r defined by $T^r_m N = J^r_0(\mathbb{R}^m, N)$, we have the natural equivalence

$$\kappa^{T^r_m, T^s_m}\colon\ T^r_m T^s_m \to T^s_m T^r_m \tag{3}$$

which interchanges the related projections.

In the geometric approach to theoretic physics and higher order mechanics it is useful to replace $T^r_m N$ by rth order jet prolongations $J^r Y$ of an arbitrary fibered manifold $Y \to M$. Using such a point of view, there is a problem of the existence of an isomorphism

$$J^r J^s Y \to J^s J^r Y. \tag{4}$$

The first author and I. Kolář [2] have recently proved that the only natural transformation $J^r J^s \to J^r J^s$ is the identity. We remark that for $r = s = 1$ the latter result was proved also by I. Kolář and M. Modugno, [6]. Further, in [3] we have proved that there is no natural exchange isomorphism (4).

On the other hand, M. Modugno [12] has introduced the natural exchange isomorphism $\mathrm{ex}_\Lambda\colon\ J^1 J^1 Y \to J^1 J^1 Y$ depending on a classical linear connection Λ on the base manifold M, see also [8]. Moreover, in [3] we have introduced the following general concept of an involution of iterated bundle functors defined on the category $\mathscr{F}\mathscr{M}_m$ of fibered manifolds with m-dimensional bases and of fibered manifold morphisms covering local diffeomorphisms.

Definition 1. Let F and G be two bundle functors on $\mathscr{F}\mathscr{M}_m$ and denote by $p^F_Y\colon\ FY \to Y$, $p^G_Y\colon\ GY \to Y$ the bundle projections. A natural equivalence $A\colon$ $FG \to GF$ is called an involution if

$$p^G_{FY} \circ A_Y = F(p^G_Y).$$

So the involution interchanges the projections $p^G_{FY}\colon\ GFY \to FY$ and $F(p^G_Y)\colon$ $FGY \to FY$. It turns out that this property is essential in the applications. In particular, using a suitable involution, one can introduce new geometric constructions of connections, see Proposition 10 below. By [12], the map $\mathrm{ex}_\Lambda\colon\ J^1 J^1 Y \to J^1 J^1 Y$ interchanges both canonical projections $p^{J^1}_{J^1 Y}$ and $J^1(p^{J^1}_Y)$ of $J^1 J^1 Y$ into $J^1 Y$. Hence ex_Λ is the involution depending on a linear connection Λ. The present paper is devoted to the following problems:

Problem 1. To introduce an involution of iterated holonomic jet functors $A^{r,s}_\Lambda\colon$ $J^r J^s \to J^s J^r$ for any r, s, depending on a classical linear connection Λ on the base manifold M.

Problem 2. To introduce an involution of iterated nonholonomic jet functors $\tilde{A}^{r,s}_{\Lambda}\colon \tilde{J}^r\tilde{J}^s \to \tilde{J}^s\tilde{J}^r$ for any r, s, depending on a classical linear connection Λ on the base manifold M.

Problem 3. To introduce the prolongation of higher order connections to jet bundles.

The structure of the paper is as follows. In Section 2 we define a linear isomorphism

$$J^r_0J^s(\mathbb{R}^{m,n}) \to J^s_0J^r(\mathbb{R}^{m,n}),$$

where $\mathbb{R}^{m,n}$ means the product fibered manifold $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Using the theory of Weil algebras we show that this isomorphism corresponds to the canonical natural equivalence $\kappa^{T^r_m,T^s_m}$ of iterated higher order velocities functors. Section 3 is devoted to the solution of Problem 1. We also describe some geometric properties of $A^{r,s}_{\Lambda}$. In Section 4 we solve Problem 2. Further, in Section 5 we define an involution $J^r_vJ^s \to J^sJ^r_v$ depending on a linear connection Λ, where J^r_v is the vertical r-jet functor. Finally, Section 6 is devoted to applications of our involutions to the prolongation of higher order connections. In particular, given an sth order connection $\Gamma\colon Y \to J^sY$, we introduce an sth order connection $A^r(\Lambda,\Gamma)$ on $J^rY \to M$. For $s=1$ we obtain in such a way a geometric construction of a connection on $J^rY \to M$ by means of a connection Γ on $Y \to M$ and a linear connection Λ on M.

In what follows we use the terminology and notation from the book [8]. We denote by $\mathscr{FM} \supset \mathscr{FM}_m$ the category of fibered manifolds and fiber respecting mappings and by $\mathscr{FM}_{m,n}$ the subcategory of $\mathscr{FM}_m$ with n-dimensional fibres and local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

2. Linear isomorphism $J^r_0J^s(\mathbb{R}^{m,n}) \to J^s_0J^r(\mathbb{R}^{m,n})$

In what follows we identify sections of $\mathbb{R}^{m,n}$ with maps $\mathbb{R}^m \to \mathbb{R}^n$. We also use the following notation

$$j^r_0j^s(f(x,\underline{x})) = j^r_0(x \to j^s_x(\underline{x} \to f(x,\underline{x}))) \in J^r_0J^s(\mathbb{R}^{m,n}).$$

Definition 2. Define a linear isomorphism

$$A^{r,s}_{m,n}\colon\ J^r_0J^s(\mathbb{R}^{m,n}) \to J^s_0J^r(\mathbb{R}^{m,n}) \tag{5}$$

by

$$A^{r,s}_{m,n}(j^r_0j^s(f(x,\underline{x}))) = j^s_0j^r(f(\underline{x}-x,\underline{x})).$$

Using standard arguments one can easily show that the definition of $A^{r,s}_{m,n}$ is correct. For, if $|\beta| \geqslant r$ or $|\alpha| \geqslant s$, then we have

$$\begin{aligned} A^{r,s}_{m,n}(j^r_0 j^s(f(x,\underline{x}) + x^\beta(\underline{x}-x)^\alpha g(x,\underline{x}))) \\ &= j^s_0 j^r(f(\underline{x}-x,\underline{x}) + (\underline{x}-x)^\beta x^\alpha g(\underline{x}-x,\underline{x})) \\ &= j^s_0 j^r(f(\underline{x}-x,\underline{x})) \\ &= A^{r,s}_{m,n}(j^r_0 j^s(f(x,\underline{x}))). \end{aligned}$$

We first prove the following invariance condition

Proposition 1. *Let $\Phi\colon \mathbb{R}^{m,n} \to \mathbb{R}^{m,k}$ be an $\mathscr{F}\mathscr{M}_m$-map of the form*

$$\Phi(x,y) = (B(x), \varphi(x,y)),$$

where $B\colon \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism and $\varphi\colon \mathbb{R}^{m,n} \to \mathbb{R}^k$. Then we have

$$A^{r,s}_{m,k}(J^r J^s \Phi(v)) = J^s J^r \Phi(A^{r,s}_{m,n}(v)) \tag{6}$$

for any $v \in J^r_0 J^s(\mathbb{R}^{m,n})$.

Proof. We can write

$$\begin{aligned} A^{r,s}_{m,k}(J^r J^s \Phi(j^r_0 j^s(f(x,\underline{x}))) &= A^{r,s}_{m,k}(j^r_0 j^s(\varphi(B^{-1}(\underline{x}), f(B^{-1}(\underline{x}), B^{-1}(\underline{x}))))) \\ &= j^s_0 j^r(\varphi(B^{-1}(\underline{x}), f(B^{-1}(\underline{x}-x), B^{-1}(\underline{x})))) \\ &= j^s_0 j^r(\varphi(B^{-1}(\underline{x}), f(B^{-1}(\underline{x}) - B^{-1}(x), B^{-1}(\underline{x})))) \\ &= J^s J^r \Phi(j^s_0 j^r(f(\underline{x}-x,\underline{x}))) \\ &= J^s J^r \Phi(A^{r,s}_{m,n}(j^r_0 j^s(f(x,\underline{x})))). \end{aligned}$$

□

We recall that every product preserving bundle functor F on $\mathscr{M}f$ is a Weil functor $F = T^A$, where $A = F\mathbb{R}$ is the corresponding Weil algebra, [8]. Moreover, the natural transformations $T^A \to T^B$ of two such functors are in a canonical bijection with algebra homomorphisms $A \to B$ and the iteration $T^A \circ T^B$ corresponds to the tensor product $A \otimes B$ of Weil algebras. By Proposition 35.5 from [8], every Weil algebra A is a finite dimensional quotient of an algebra of germs $C^\infty_0(\mathbb{R}^n)$ of smooth functions $\mathbb{R}^n \to \mathbb{R}$ at zero for some n. In particular, the Weil algebra of the functor T^r_m of m-dimensional velocities of order r is of the form

$$\mathbb{D}^r_m = T^r_m \mathbb{R} = C^\infty_0(\mathbb{R}^m)/\mathscr{D},$$

where $\mathscr{D}$ is the ideal generated by x^α for $|\alpha| = r+1$. Then the Weil algebra of the iteration $T^r_m \circ T^s_m$ is of the form

$$\mathbb{D}^r_m \otimes \mathbb{D}^s_m = C^\infty_0(\mathbb{R}^m \times \mathbb{R}^m)/\mathscr{C},$$

where $\mathscr{C}$ is the ideal generated by x^α and $\underline{x}^\beta$ for $|\alpha| = r+1$ and $|\beta| = s+1$. Consider now a functor $F^{r,s}_m$ on $\mathscr{M}f$ defined by

$$F^{r,s}_m N = J^r_0 J^s(\mathbb{R}^m \times N),$$

where $\mathbb{R}^m \times N$ is the trivial bundle over $\mathbb{R}^m$. Clearly, $F^{r,s}_m$ preserves products. By [8], this is a Weil functor determined by the Weil algebra

$$F^{r,s}_m \mathbb{R} = J^r_0 J^s(\mathbb{R}^{m,1}). \tag{7}$$

Lemma 1. *There is an isomorphism of Weil algebras*

$$\varphi^{r,s}_m \colon\ J^r_0 J^s(\mathbb{R}^{m,1}) \to \mathbb{D}^r_m \otimes \mathbb{D}^s_m. \tag{8}$$

P r o o f. Obviously, the Weil algebra (7) is of the form $J^r_0 J^s(\mathbb{R}^{m,1}) = C^\infty_0(\mathbb{R}^m \times \mathbb{R}^m)/\mathscr{B}^{r,s}_m$, where $\mathscr{B}^{r,s}_m$ is the ideal generated by x^α and $(\underline{x} - x)^\beta$ for $|\alpha| = r+1$ and $|\beta| = s+1$. Then the isomorphism (8) is defined by the pullback with respect to the difeomorphism $x^i \mapsto x^i$, $(\underline{x}^j - x^j) \mapsto \underline{x}^j$, $i, j = 1, \ldots, m$. $\square$

From Lemma 1 it follows directly

Proposition 2. *Let $\varphi^{r,s}_m$ be the isomorphism (8) of Weil algebras. Then the composition*

$$\varphi^{s,r}_m \circ A^{r,s}_{m,1} \circ (\varphi^{r,s}_m)^{-1} \colon\ \mathbb{D}^r_m \otimes \mathbb{D}^s_m \to \mathbb{D}^s_m \otimes \mathbb{D}^r_m$$

is just the canonical exchange isomorphism $\mathbb{D}^r_m \otimes \mathbb{D}^s_m \to \mathbb{D}^s_m \otimes \mathbb{D}^r_m$.

The isomorphism (8) of Weil algebras induces a natural equivalence $\overline{\varphi}^{r,s}_m \colon\ F^{r,s}_m \to T^r_m \circ T^s_m$ of the corresponding Weil functors. So we have

Proposition 3. *There is an isomorphism*

$$(\overline{\varphi}^{r,s}_m)_{\mathbb{R}^n}\colon\ J^r_0 J^s(\mathbb{R}^{m,n}) \longrightarrow T^r_m(T^s_m\mathbb{R}^n)$$

and the following diagram commutes

$$\begin{tikzcd}
J^r_0 J^s(\mathbb{R}^{m,n}) \arrow[r, "A^{r,s}_{m,n}"] \arrow[d, "(\overline{\varphi}^{r,s}_m)_{\mathbb{R}^n}"'] & J^s_0 J^r(\mathbb{R}^{m,n}) \arrow[d, "(\overline{\varphi}^{s,r}_m)_{\mathbb{R}^n}"] \\
T^r_m(T^s_m\mathbb{R}^n) \arrow[r, "(\kappa^{T^r_m,T^s_m})_{\mathbb{R}^n}"] & T^s_m(T^r_m\mathbb{R}^n)
\end{tikzcd}$$

Using such a point of view, the linear isomorphism (5) corresponds to the canonical exchange isomorphism (3) of iterated higher order velocities functors. Moreover, we have $T^r_m(T^s_m\mathbb{R}^n) = (\mathbb{D}^r_m \otimes \mathbb{D}^s_m)^n$, which yields an identification

$$J^r_0 J^s(\mathbb{R}^{m,n}) \cong (\mathbb{D}^r_m \otimes \mathbb{D}^s_m)^n.$$

Denoting by $\eta\colon\ \mathbb{D}^r_m \otimes \mathbb{D}^s_m \to \mathbb{D}^s_m \otimes \mathbb{D}^r_m$ the canonical exchange isomorphism of Weil algebras, the linear isomorphism $A^{r,s}_{m,n}\colon\ J^r_0 J^s(\mathbb{R}^{m,n}) \to J^s_0 J^r(\mathbb{R}^{m,n})$ is of the form

$$A^{r,s}_{m,n} = (\eta \times \ldots \times \eta)\colon\ (\mathbb{D}^r_m \otimes \mathbb{D}^s_m)^n \to (\mathbb{D}^s_m \otimes \mathbb{D}^r_m)^n.$$

3. Solution of Problem 1

Let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object and let Λ be a classical linear connection on the base manifold M. Further, let $v \in J^r_z J^s Y$, $z \in M$. Choose an arbitrary fibered coordinate system $\Psi = (\underline{\psi}, \psi)\colon\ Y \to \mathbb{R}^{m,n}$ such that $\underline{\psi}\colon\ M \to \mathbb{R}^m$ is a normal coordinate system of Λ with centre z, $\underline{\psi}(z) = 0$.

Definition 3. We define a map $(A^{r,s}_\Lambda)_Y\colon\ J^r J^s Y \to J^s J^r Y$ by

$$(A^{r,s}_\Lambda)_Y(v) = J^s J^r \Psi^{-1}(A^{r,s}_{m,n}(J^r J^s \Psi(v))). \tag{9}$$

If $\Psi' = (\underline{\psi}', \psi')\colon\ Y \to \mathbb{R}^{m,n}$ is another fibered coordinate system such that $\underline{\psi}'$ is a normal coordinate system of Λ with centre z, $\underline{\psi}'(z) = 0$, then

$$\Psi' \circ \Psi^{-1}(x, y) = (B(x), \varphi(x, y))$$

for some linear isomorphism B of $\mathbb{R}^m$. Using the invariance condition (6) from Proposition 1 one can show in the standard way that the definition of $A^{r,s}_\Lambda$ does not depend on the choice of Ψ with the above property. So $A^{r,s}_\Lambda$ is well-defined globally (and then it is $\mathcal{FM}_m$-natural). One evaluates directly the following analogy of (2).

Proposition 4. *The following diagram commutes*

$$\begin{tikzcd} J^rJ^sY \arrow[rr, "(A^{r,s}_\Lambda)_Y"] \arrow[dr, "J^r(p^{J^s}_Y)"'] & & J^sJ^rY \arrow[dl, "p^{J^s}_{J^rY}"] \\ & J^rY & \end{tikzcd}$$

Moreover, we have

$$(A^{s,r}_\Lambda)_Y \circ (A^{r,s}_\Lambda)_Y = \mathrm{id}_{J^rJ^sY}$$

Corollary 1. $A^{r,s}_\Lambda\colon J^rJ^s \to J^sJ^r$ *is an involution depending on a linear connection* Λ.

Let $\mathrm{ex}_\Lambda\colon J^1J^1 \to J^1J^1$ be the involution introduced by M. Modugno in [12]. By [6], the only two natural transformations $J^1J^1Y \to J^1J^1Y$ depending on a linear symmetric connection Λ on the base manifold M are the identity and ex_Λ. So we have

Proposition 5. *Let* Λ *be a symmetric linear connection on the base manifold* M*. Then the natural involution* $A^{1,1}_\Lambda\colon J^1J^1 \to J^1J^1$ *coincides with* ex_Λ.

Let Λ^k_{ij} be the coordinates of Λ. Denoting by x^i, y^p the canonical coordinates on Y, the induced coordinates on J^1Y are denoted by $y^p_i = \partial y^p/\partial x^i$. Finally, on $J^1(J^1Y \to M)$ we have the coordinates x^i, y^p, y^p_i, $Y^p_i = \partial y^p/\partial x^i, y^p_{ij} = \partial y^p_i/\partial x^j$. By [6], the coordinate form of ex_Λ is

$$\overline{y}^p_i = Y^p_i, \quad \overline{Y}^p_i = y^p_i, \quad \overline{y}^p_{ij} = y^p_{ji} + (y^p_k - Y^p_k)\Lambda^k_{ji}. \tag{10}$$

Denote by QP^1M the bundle of classical linear connections on M and write $\Sigma\colon QP^1M \to TM \otimes \Lambda^2T^*M \subset TM \otimes T^*M \otimes T^*M$ for the torsion tensor. By [8], $J^1Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$. So we have a mapping

$$\sigma\colon J^1J^1Y \to VY \otimes T^*M, \quad A \mapsto p^{J^1}_{J^1Y}(A) - J^1(p^{J^1}_Y)(A)$$

and the contraction on TM yields a mapping

$$\langle\sigma, \Sigma(\Lambda)\rangle\colon J^1J^1Y \to VY \otimes T^*M \otimes T^*M \subset VJ^1Y \otimes T^*M.$$

Clearly, the last space is the vector bundle associated with the affine bundle $p^{J^1}_{J^1Y}\colon J^1J^1Y \to J^1Y$.

Proposition 6. *We have*

$$A_\Lambda^{1,1} = \mathrm{ex}_\Lambda + \tfrac{1}{2}\langle \sigma, \Sigma(\Lambda)\rangle.$$

In particular, if Λ is torsion free, then $A_\Lambda^{1,1} = \mathrm{ex}_\Lambda$.

P r o o f. Let Λ be a connection on $\mathbb{R}^m$ such that the usual coordinate system on $\mathbb{R}^m$ is a normal one for Λ with centre 0. Then $\Lambda_{ij}^k(0) + \Lambda_{ji}^k(0) = 0$ for all $i, j, k = 1, \ldots, m$. By the definition, the map $A_\Lambda^{1,1}$ over zero is given by

$$\overline{y}_i^p = Y_i^p, \quad \overline{Y}_i^p = y_i^p, \quad \overline{y}_{ij}^p = y_{ji}^p + (y_k^p - Y_k^p)(\Lambda_{ji}^k(0) + \tfrac{1}{2}(\Lambda_{ij}^k(0) - \Lambda_{ji}^k(0))).$$

□

So the coordinate expression of $A_\Lambda^{1,1}$ is

$$\overline{y}_i^p = Y_i^p, \quad \overline{Y}_i^p = y_i^p, \quad \overline{y}_{ij}^p = y_{ji}^p + (y_k^p - Y_k^p)\Lambda_{ji}^k + \tfrac{1}{2}(y_k^p - Y_k^p)(\Lambda_{ij}^k - \Lambda_{ji}^k) \tag{11}$$

for an arbitrary linear connection Λ. If Λ is torsion-free, then the equations of $A_\Lambda^{1,1}$ are given by (10).

Remark 1. It turns out that the problem of classification all natural transformations $J^r J^s \to J^s J^r$ depending on a linear connection Λ is very difficult. Up till now this problem was solved only for $r = s = 1$, see [6]. On the basis of methods from [8] and [10] we prepare a solution of this problem for $r = 2$ and $s = 1$.

4. Solution of Problem 2

In the theory of higher order jets it is useful to distinguish between nonholonomic, semiholonomic and holonomic ones, see e.g. [5], [7], [11]. Given a fibered manifold $Y \to M$, we denote by $\tilde{J}^r Y$, $\overline{J}^r Y$ or $J^r Y$ the rth order nonholonomic, semiholonomic or holonomic prolongation, respectively. Clearly, for $r = 1$ all such spaces coincide, while for $r > 1$ we have $J^r Y \subset \overline{J}^r Y \subset \tilde{J}^r Y$. We recall that $\tilde{J}^r Y$ is defined by iteration

$$\tilde{J}^1 Y = J^1 Y, \quad \tilde{J}^r Y = J^1(\tilde{J}^{r-1} Y \to M),$$

which yields a natural identification $\tilde{J}^r(\tilde{J}^s Y) = \tilde{J}^{r+s} Y$. We introduce an involution depending on a classical linear connection Λ on the base manifold M

$$\tilde{A}_\Lambda^{r,s} \colon \tilde{J}^r \tilde{J}^s \to \tilde{J}^s \tilde{J}^r \tag{12}$$

by the following induction. First, we have the involution $(\tilde{A}^{1,1}_\Lambda)_Y := (A^{1,1}_\Lambda)_Y\colon \tilde{J}^1\tilde{J}^1Y \to \tilde{J}^1\tilde{J}^1Y$. Then we can define $(\tilde{A}^{r,1}_\Lambda)_Y\colon \tilde{J}^r\tilde{J}^1Y \to \tilde{J}^1\tilde{J}^rY$ by

$$(\tilde{A}^{r,1}_\Lambda)_Y := (\tilde{A}^{1,1}_\Lambda)_{\tilde{J}^{r-1}Y} \circ J^1(\tilde{A}^{r-1,1}_\Lambda)_Y,$$

where $(\tilde{A}^{r-1,1}_\Lambda)_Y\colon \tilde{J}^{r-1}\tilde{J}^1Y \to \tilde{J}^1\tilde{J}^{r-1}Y$. Finally, we define the map (12) by

$$(\tilde{A}^{r,s}_\Lambda)_Y := J^1(\tilde{A}^{r,s-1}_\Lambda)_Y \circ (\tilde{A}^{r,1}_\Lambda)_{\tilde{J}^{s-1}Y}.$$

Clearly, we have

Proposition 7. $\tilde{A}^{r,s}_\Lambda\colon \tilde{J}^r\tilde{J}^s \to \tilde{J}^s\tilde{J}^r$ *is an involution depending on a linear connection* Λ.

Starting from the equations (11) of $\tilde{A}^{1,1}_\Lambda$ and using differentiation, one can evaluate directly the coordinate expression of $\tilde{A}^{r,s}_\Lambda$ for any r, s.

Remark 2. One can also prove that $\tilde{A}^{2,1}_\Lambda$ does not send J^2J^1 onto J^1J^2. In general, the involution $A^{r,s}_\Lambda$ is not the restriction of $\tilde{A}^{r,s}_\Lambda$ to the holonomic jets J^rJ^s.

5. Involution $J^r_vJ^s \to J^sJ^r_v$

For any $\mathcal{FM}_m$-object $Y \to M$ we have its vertical r-jet prolongation

$$J^r_vY = \{j^r_x\sigma \big| \sigma\colon M \to Y_x,\ x \in M\}$$

over Y. Any $\mathcal{FM}_m$-map $f\colon Y_1 \to Y_2$ over $\underline{f}\colon M_1 \to M_2$ induces a fibered map $J^r_vf\colon J^r_vY_1 \to J^r_vY_2$ covering f such that $J^r_vf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f\circ\sigma\circ\underline{f}^{-1})$, $j^r_x\sigma \in J^r_vY_1$. Then the correspondence $J^r_v\colon \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor. Quite similarly to Section 2 we use the following notations

$$\begin{aligned} j^r_0j^s(f(x,\underline{x})) &= j^r_0(x \to j^s_0(\underline{x} \to f(x,\underline{x}))) \in (J^r_v)_0J^s(\mathbb{R}^{m,n}),\\ j^s_0j^r(f(x,\underline{x})) &= j^s_0(x \to j^r_x(\underline{x} \to f(x,\underline{x}))) \in J^s_0J^r_v(\mathbb{R}^{m,n}). \end{aligned}$$

By [3], there is no $\mathcal{FM}_m$-natural exchange isomorphism $J^r_vJ^s \to J^sJ^r_v$.

Definition 4. Define a linear isomorphism

$$C^{r,s}_{m,n}\colon (J^r_v)_0J^s(\mathbb{R}^{m,n}) \to J^s_0J^r_v(\mathbb{R}^{m,n}) \tag{13}$$

by

$$C^{r,s}_{m,n}(j^r_0j^s(f(x,\underline{x}))) = j^s_0j^r(f(\underline{x}-x,x)).$$

Using standard arguments one can easily show that the definition of $C^{r,s}_{m,n}$ is correct. For, if $|\alpha| \geqslant r$ or $|\beta| \geqslant s$, then we have

$$C^{r,s}_{m,n}(j^r_0 j^s(x^\alpha \underline{x}^\beta g(x,\underline{x}))) = j^s_0 j^r((\underline{x}-x)^\alpha x^\beta g(\underline{x}-x,x)) = 0.$$

Proposition 8. *Let $\Phi\colon \mathbb{R}^{m,n} \to \mathbb{R}^{m,k}$ be an $\mathcal{FM}_m$-map of the form*

$$\Phi(x,y) = (B(x), \varphi(x,y)),$$

where $B\colon \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism and $\varphi\colon \mathbb{R}^{m,n} \to \mathbb{R}^k$. Then we have

$$C^{r,s}_{m,k}(J^r_v J^s \Phi(v)) = J^s J^r_v \Phi(C^{r,s}_{m,n}(v))$$

for any $v \in (J^r_v)_0 J^s(\mathbb{R}^{m,n})$.

Proof. We can write

$$\begin{aligned}
C^{r,s}_{m,k}(J^r_v J^s \Phi(j^r_0 j^s(f(x,\underline{x}))) &= C^{r,s}_{m,k}(j^r_0 j^s(\varphi(B^{-1}(\underline{x}), f(B^{-1}(x), B^{-1}(\underline{x}))))) \\
&= j^s_0 j^r(\varphi(B^{-1}(x), f(B^{-1}(\underline{x}-x), B^{-1}(x)))) \\
&= j^s_0 j^r(\varphi(B^{-1}(x), f(B^{-1}(\underline{x}) - B^{-1}(x), B^{-1}(x)))) \\
&= J^s J^r_v \Phi(j^s_0 j^r(f(\underline{x}-x, x))) \\
&= J^s J^r_v \Phi(C^{r,s}_{m,n}(j^r_0 j^s(f(x,\underline{x})))).
\end{aligned}$$

□

Now let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object, Λ be a classical linear connection on the base manifold M and let $v \in (J^r_v)_z J^s Y$, $z \in M$. Moreover, take a fibered coordinate system $\Psi = (\underline{\psi}, \psi)\colon Y \to \mathbb{R}^{m,n}$ such that $\underline{\psi}\colon M \to \mathbb{R}^m$ is a normal coordinate system of Λ with centre z, $\underline{\psi}(z) = 0$.

Definition 5. We define a map $(C^{r,s}_\Lambda)_Y\colon J^r_v J^s Y \to J^s J^r_v Y$ by

$$(C^{r,s}_\Lambda)_Y(v) = J^s J^r_v \Psi^{-1}(C^{r,s}_{m,n}(J^r_v J^s \Psi(v))). \tag{14}$$

Using Proposition 8 we prove that $(C^{r,s}_\Lambda)_Y$ is well defined globally and is an $\mathcal{FM}_m$-natural isomorphism.

Denoting by $\alpha^r_Y\colon J^r_v Y \to Y$ the jet projection, one can construct the jet extension $J^s(\alpha^r_Y)\colon J^s J^r_v Y \to J^s Y$. We obtain easily

Proposition 9. *The following diagrams commute*

$$\begin{tikzcd}
J^r_v J^s Y \arrow[rr, "(C^{r,s}_\Lambda)_Y"] \arrow[dr, "J^r_v(p^{J^s}_Y)"'] & & J^s J^r_v Y \arrow[dl, "p^{J^s}_{J^r_v Y}"] \\
& J^r_v Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
J^r_v J^s Y \arrow[rr, "(C^{r,s}_\Lambda)_Y"] \arrow[dr, "\alpha^r_{J^s Y}"'] & & J^s J^r_v Y \arrow[dl, "J^s \alpha^r_Y"] \\
& J^s Y &
\end{tikzcd}$$

Corollary 2. $C^{r,s}_\Lambda \colon J^r_v J^s \to J^s J^r_v$ *is an involution depending on a linear connection* Λ.

6. Solution of Problem 3

In general, an rth order nonholonomic connection on a fibered manifold $Y \to M$ is a section $\Gamma\colon Y \to \tilde{J}^r Y$. Such a connection is called semiholonomic or holonomic if it has values in $\overline{J}^r Y$ or in $J^r Y$, respectively. We recall that the theory of higher order connections was introduced by C. Ehresmann [4] on a Lie groupoid. Further, the Ehresmann's theory was extended to the case of an arbitrary fibered manifold by I. Kolář. It has been pointed out recently that higher order connections are useful for numerous problems in differential geometry, see e.g. [9], [13]. The rest of this section will be devoted to the prolongation of holonomic connections to higher order jet bundles. The following assertion follows directly from the definition of an involution.

Proposition 10. *Let J be any of the functors J^r, $\tilde{J}^r$, $\overline{J}^r$ and let F be a bundle functor on $\mathscr{F}\mathscr{M}_m$ such that there is an involution $A\colon FJ \to JF$. If $\Gamma\colon Y \to JY$ is a higher order connection on $Y \to M$, then the composition $A_Y \circ F\Gamma$ is a connection of the same type on $FY \to M$.*

By [3], the involution $A\colon FJ \to JF$ from the last proposition exists only for $F = V^A$, where V^A is a vertical Weil functor determined by a Weil algebra A. Using the involution $V^A \tilde{J}^r \to \tilde{J}^r V^A$, A. Cabras and I. Kolář have introduced the prolongation of higher order nonholonomic connections to vertical Weil bundles, [1]. On the other hand, if $F \neq V^A$, then there is no involution $FJ \to JF$. That is why we have defined involutions depending on a linear connection Λ. From Corollary 1 it follows

Proposition 11. *Let $\Gamma\colon Y \to J^sY$ be an sth order holonomic connection on $Y \to M$. Then*

$$A^r(\Lambda,\Gamma) := (A^{r,s}_\Lambda)_Y \circ J^r\Gamma\colon\ J^rY \to J^sJ^rY$$

is an sth order holonomic connection on $J^rY \to M$.

Corollary 3. *Let $\Gamma\colon Y \to J^1Y$ be a connection on $Y \to M$. Then $A^r(\Lambda,\Gamma)\colon J^rY \to J^1J^rY$ is a connection on $J^rY \to M$.*

Quite analogously, from Corollary 2 we obtain

Proposition 12. *Let $\Gamma\colon Y \to J^sY$ be an sth order holonomic connection on $Y \to M$. Then*

$$C^r(\Lambda,\Gamma) := (C^{r,s}_\Lambda)_Y \circ J^r_v\Gamma\colon\ J^r_vY \to J^sJ^r_vY$$

is an sth order holonomic connection on $J^r_vY \to M$.

Definition 6. A vertical rth order connection on $Y \to M$ is a section $\Gamma\colon Y \to J^r_vY$ of $J^r_vY \to Y$.

Proposition 9 also yields

Proposition 13. *Let $\Gamma\colon Y \to J^r_vY$ be a vertical rth order connection on $Y \to M$. Then*

$$(C^s)^{-1}(\Lambda,\Gamma) := (C^{r,s}_\Lambda)^{-1}_Y \circ J^s\Gamma\colon\ J^sY \to J^r_vJ^sY$$

is a vertical rth order connection on $J^sY \to M$.

Remark 3. Using Proposition 10 and the involution (12), one can also introduce the prolongation of nonholonomic connections $\tilde\Gamma\colon Y \to \tilde J^sY$ to nonholonomic jet bundles $\tilde J^rY \to M$.

Taking into account the natural equivalence (1) on $\mathscr{M}f$, one can ask about the existence of a similar natural equivalence for fiber product preserving functors F and G on $\mathscr{F}\mathscr{M}_m$. If $F = V^A$ is a vertical Weil functor, then the first author and I. Kolář [2] have introduced an involution $V^AG \to GV^A$ for an arbitrary fiber product preserving functor G on $\mathscr{F}\mathscr{M}_m$. Moreover, in [3] we have proved that there is no involution $FG \to GF$ for an arbitrary couple F and G of higher order jet functors. On the other hand, in this paper we have constructed higher order jet involutions $A^{r,s}_\Lambda$ and $\tilde A^{r,s}_\Lambda$ depending on a linear connection Λ. At the end we formulate an

Open problem. To introduce an involution $A^{F,G}_\Lambda\colon FG \to GF$ depending on a linear connection Λ on the base manifold M for an arbitrary couple F and G of fiber product preserving functors on $\mathscr{F}\mathscr{M}_m$.

References

[1] *A. Cabras, I. Kolář*: Prolongation of second order connections to vertical Weil bundles. Arch. Math., Brno *37* (2001), 333–347. zbl

[2] *M. Doupovec, I. Kolář*: Iteration of fiber product preserving bundle functors. Monatsh. Math. *134* (2001), 39–50. zbl

[3] *M. Doupovec, W. M. Mikulski*: On involutions of iterated bundle functors. Colloq. Math *106* (2006), 135–145.

[4] *C. Ehresmann*: Sur les connexions d'ordre supérieur. Atti del V. Cong. del'Unione Mat. Italiana, 1955, Roma Cremonese (1956), 344–346.

[5] *C. Ehresmann*: Extension du calcul des jets aux jets non holonomes. C. R. Acad. Sci., Paris *239* (1954), 1762–1764. zbl

[6] *I. Kolář, M. Modugno*: Natural maps on the iterated jet prolongation of a fibered manifold. Ann. Mat. Pura Appl., IV. Ser. *158* (1991), 151–165. zbl

[7] *I. Kolář*: A general point of view to nonholonomic jet bundles. Cah. Topol. Géom. Différ. Catég. *44* (2003), 149–160. zbl

[8] *I. Kolář, P. W. Michor, and J. Slovák*: Natural Operations in Differential Geometry. Springer-Verlag, Berlin, 1993. zbl

[9] *I. Kolář, G. Virsik*: Connections in first principal prolongations. Suppl. Rend. Circ. Mat. Palermo, II. Ser. *43* (1996), 163–171. zbl

[10] *D. Krupka, J. Janyška*: Lectures on Differential Invariants. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Mathematica 1. University J. E. Purkyně, Brno, 1990. zbl

[11] *P. Libermann*: Introduction to the theory of semi-holonomic jets. Arch. Math. (Brno) *33* (1996), 173–189. zbl

[12] *M. Modugno*: Jet involution and prolongation of connections. Čas. Pěst. Mat. *114* (1989), 356–365. zbl

[13] *G. Virsik*: On the holonomity of higher order connections. Cah. Topol. Géom. Difér. Catég. *12* (1971), 197–212. zbl

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