Bahmann Yousefi; S. Haghkhah

Hypercyclicity of special operators on Hilbert function spaces

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 3, 1035–1041

Persistent URL: http://dml.cz/dmlcz/128224

Czechoslovak Mathematical Journal, 57 (132) (2007), 1035–1041

HYPERCYCLICITY OF SPECIAL OPERATORS ON HILBERT FUNCTION SPACES

B. Yousefi, S. Haghkhah, Shiraz

(Received September 20, 2005)

Abstract. In this paper we give some sufficient conditions for the adjoint of a weighted composition operator on a Hilbert space of analytic functions to be hypercyclic.

Keywords: multiplier, orbit, hypercyclic vector, multiplication operator, weighted composition operator

MSC 2000: 47B37, 47B33

Introduction

Let H be a Hilbert space of functions analytic on a plane domain G such that for each λ in G the linear functional of evaluation at λ given by $f \mapsto f(\lambda)$ is a bounded linear functional on H. By the Riesz representation theorem there is a vector K_λ in H such that $f(\lambda) = \langle f, K_\lambda \rangle$. We call K_λ the reproducing kernel at λ.

Let T be a bounded linear operator on H. For $x \in H$, the orbit of x under T is the set of images of x under the successive iterates of T:

$$\operatorname{orb}(T, x) = \{x, Tx, T^2x, \ldots\}.$$

The vector x is called hypercyclic for T if $\operatorname{orb}(T, x)$ is dense in H. Also a hypercyclic operator is one that has a hypercyclic vector.

The first example of a hypercyclic operator on a Hilbert space was constructed by Rolewicz in 1969 [12]. He showed that if B is the backward shift on $\ell^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$.

This paper is a part of the second author's Doctoral thesis written at Shiraz University under the direction of the first author.

A complex-valued function ψ on G is called a multiplier of H if $\psi H \subset H$. The operator of multiplication by ψ is denoted by M_ψ and is given by $f \mapsto \psi f$. By the closed graph theorem M_ψ is bounded. The collection of all multipliers is denoted by $M(H)$. Each multiplier is a bounded analytic function on G. In fact, $\|\varphi\|_G \leqslant \|M_\varphi\|$ ([14]).

If w is a multiplier of H and φ is a mapping from G into G such that $f \circ \varphi \in H$ for all $f \in H$, then C_φ (defined on H by $C_\varphi f = f \circ \varphi$) and $M_w C_\varphi$ are called the composition and the weighted composition operator, respectively. We define the iterates $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$ (n times). Note that $C_{\varphi_n} = C_\varphi^n$ for all n. In this paper we investigate the hypercyclicity of the adjoint of a weighted composition operator acting on a Hilbert space of analytic functions. For some sources on hypercyclic topic see [1]–[13], [15], [16].

Main results

A nice criterion, namely the Hypercyclicity Criterion is used in the proof of our main theorem. It was developed independently by Kitai [10], Gethner and Shapiro [6]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [13], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3].

The formulation of the Hypercyclicity Criterion in the following theorem was given by J. Bes in his PhD. thesis [1] (see also [2]).

The Hypercyclicity Criterion Theorem. Suppose X is a separable Banach space and T is a continuous linear mapping on X. If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that

1. $T^{n_k} y \to 0$ for every $y \in Y$, and
2. there exist functions $S_{n_k}\colon Z \to X$ such that for every $z \in Z$, $S_{n_k} z \to 0$ and $T^{n_k} S_{n_k} z \to z$,

then T is hypercyclic.

Throughout this section let H be a Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that H contains constants and the functional of evaluation at λ is bounded for all λ in $\mathbb{D}$. Further, let φ be an analytic univalent map from $\mathbb{D}$ onto $\mathbb{D}$. By φ_n^{-1} we mean the nth iterate of φ^{-1}.

Theorem. *Suppose that the composition operator C_φ is bounded on H and w is a nonconstant multiplier of H such that the sets $\{\lambda \in \mathbb{D}\colon \sup_n |w \circ \varphi_n(\lambda)| < 1\}$ and $\{\lambda \in \mathbb{D}\colon \inf_n |w \circ \varphi_n^{-1}(\lambda)| > 1\}$ have limit points in $\mathbb{D}$. Then the adjoint of the weighted composition operator $M_w C_\varphi$ is hypercyclic.*

P r o o f. First we note that if $\lambda \in \mathbb{D}$ and $f \in H$, then we get

$$\langle f, M_w^* K_\lambda \rangle = \langle wf, K_\lambda \rangle = \langle f, \overline{w(\lambda)} K_\lambda \rangle,$$

which implies that $M_w^* K_\lambda = \overline{w(\lambda)} K_\lambda$. Also

$$\left\langle f, C_\varphi^* K_\lambda \right\rangle = \langle f \circ \varphi, K_\lambda \rangle = f(\varphi(\lambda)) = \left\langle f, K_{\varphi(\lambda)} \right\rangle,$$

hence $C_\varphi^* K_\lambda = K_{\varphi(\lambda)}$. Thus we have

$$(M_w C_\varphi)^* K_\lambda = C_\varphi^* (M_w^* K_\lambda) = \overline{w(\lambda)} C_\varphi^* K_\lambda = \overline{w(\lambda)} K_{\varphi(\lambda)}.$$

Put $A = M_w C_\varphi$ and $\varphi_0 = I$ where I is the identity mapping on $\mathbb{D}$. Then for all $n \in \mathbb{N}$ and all λ in $\mathbb{D}$ we get

$$(A^*)^n K_\lambda = \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\lambda))} \right) K_{\varphi_n(\lambda)}.$$

Put

$$E = \{\lambda \in \mathbb{D} \colon \sup_n |w(\varphi_n(\lambda))| < 1\}$$

and

$$H_E = \operatorname{span}\{K_\lambda \colon \lambda \in E\}.$$

The set H_E is dense in H, because if $f \in H$ and $\langle f, K_\lambda \rangle = 0$ for all λ in E, then $f(\lambda) = 0$ for all λ in E. So by virtue of the hypothesis of the theorem, the zeros of f have a limit point in $\mathbb{D}$, which implies that $f \equiv 0$ on $\mathbb{D}$. Thus H_E is dense in H.

Note that if $\lambda \in E$, then there exists a number α such that $0 < \alpha < 1$ and $\sup_n |w(\varphi_n(\lambda))| < \alpha < 1$. Thus $|w(\varphi_n(\lambda))| < \alpha$ for all n and so we have

$$\prod_{i=0}^{n-1} |w(\varphi_i(\lambda))| \leqslant \prod_{i=0}^{n-1} \alpha = \alpha^n.$$

Since $\lim_{n \to \infty} \alpha^n = 0$, we get

$$\prod_{i=0}^{\infty} |w(\varphi_i(\lambda))| = 0$$

and so we have $\lim_n (A^*)^n K_\lambda = 0$. Thus $(A^*)^n \to 0$ pointwise on H_E which is dense in H.

Now we want to find a right inverse of A^* on a dense subset of H. To see this put

$$F = \{\lambda \in \mathbb{D} \colon \inf_n |w(\varphi_n^{-1}(\lambda))| > 1\}.$$

By a method similar to that we used to prove that H_E is dense in H, we can see that the set

$$H_F = \operatorname{span}\{K_\lambda \colon \lambda \in F\}$$

is dense in H, since F has a limit point in $\mathbb{D}$. To find the desired right inverse of A^*, first consider the special case when the collection of linear functionals of point evaluations $\{K_\lambda \colon \lambda \in F\}$ is linearly independent. Note that in the next definition there is no possibility of dividing by zero.

Define $B \colon H_F \to H$ by extending the definition

$$BK_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)} \quad (\lambda \in F)$$

linearly to H_F (it is good to note that if $\lambda \in F$, then $\varphi^{-1}(\lambda) \in F$ and indeed B maps H_F to H_F). Now we clearly get

$$B^2 K_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} (\overline{w(\varphi^{-1}(\varphi^{-1}(\lambda)))})^{-1} K_{\varphi^{-1}(\varphi^{-1}(\lambda))}$$

for all λ in F. Continuing in this manner we can see that

$$B^n K_\lambda = \left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}} \right) K_{\varphi_n^{-1}(\lambda)},$$

where φ_i^{-1} is the ith iterate of φ^{-1} and $n \in \mathbb{N}$. By the definition of B we have

$$A^* B K_\lambda = A^* ((\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)}) = K_{\varphi(\varphi^{-1}(\lambda))} = K_\lambda$$

for all λ in F. Thus $A^* B$ is identity on the dense subset H_F of H.

Now we want to show that $B^n \to 0$ pointwise on H_F. Note that if $\lambda \in F$, then there exists a number $\beta > 1$ such that

$$\inf_n |w(\varphi_n^{-1}(\lambda))| > \beta > 1.$$

Thus $|w(\varphi_n^{-1}(\lambda))| > \beta > 1$ for all n and so we have

$$\prod_{i=1}^{n} |w(\varphi_i^{-1}(\lambda))|^{-1} \leqslant \left(\frac{1}{\beta}\right)^n.$$

Since $0 < 1/\beta < 1$, we obtain that $\lim\limits_{n \to \infty} (1/\beta)^n = 0$ and so

$$\lim_n \left(\prod_{i=1}^{n} |w(\varphi_i^{-1}(\lambda))|^{-1} \right) K_{\varphi_n^{-1}(\lambda)} = 0.$$

This implies that $B^n \to 0$ pointwise on H_F which is dense in H. Thus by the Hypercyclicity Criterion Theorem or by Corollary 1.5 in [7, p. 235], $A^* = (M_w C_\varphi)^*$ has a hypercyclic vector.

In the case when the linear functionals of point evaluations are not linearly independent, we use a standard method: consider a countable dense subset $F_1 = \{\lambda_n\colon\ n \geqslant 1\}$ of F and inductively choose a subsequence $\{z_n\}$ as follows. Let $z_1 = \lambda_1$. Define

$$F_2 = F_1 \setminus \{\lambda \in F_1\colon\ K_\lambda \in \operatorname{span}\{K_{z_1}\}\}.$$

Denote the first element of F_2 by z_2 and define

$$F_3 = F_2 \setminus \{\lambda \in F_2\colon\ K_\lambda \in \operatorname{span}\{K_{z_1}, K_{z_2}\}\}.$$

Continuing in this manner, we obtain a subset $G = \{z_n\}_n$ of F for which the set

$$H_G = \operatorname{span}\{K_\lambda\colon\ \lambda \in G\}$$

is dense in H with linearly independent linear functionals of point evaluations $\{K_\lambda\colon \lambda \in G\}$. Now for each $n \in \mathbb{N}$, define mappings $S_n\colon H_G \to H$ by extending the definition

$$S_n K_\lambda = \left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) K_{\varphi_n^{-1}(\lambda)} \quad (\lambda \in G)$$

linearly to H_G.

Note that if we substitute $\varphi_n^{-1}(\lambda)$ instead of λ in the formula obtained earlier for $(A^*)^n K_\lambda$, we get

$$\begin{aligned}(A^*)^n K_{\varphi_n^{-1}(\lambda)} &= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\varphi_n^{-1}(\lambda)))}\right) K_{\varphi_n(\varphi_n^{-1}(\lambda))} \\ &= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_{n-i}^{-1}(\lambda))}\right) K_{\varphi_n \circ \varphi_n^{-1}(\lambda)} \\ &= \prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))} K_\lambda\end{aligned}$$

for all λ in G.

By the definition of S_n we have

$$\begin{aligned}(A^*)^n S_n K_\lambda &= (A^*)^n \left(\left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) K_{\varphi_n^{-1}(\lambda)} \right) \\ &= \left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) (A^*)^n K_{\varphi_n^{-1}(\lambda)} = K_\lambda\end{aligned}$$

for all λ in G. Thus for all $n \in \mathbb{N}$, $(A^*)^n S_n$ is identity on the dense subset H_G of H. Now, exactly as proved before we have that $B^n \to 0$ pointwise on H_F, we can see that $S_n \to 0$ pointwise on H_G which is dense in H. Thus the conditions of the Hypercyclicity Criterion Theorem are satisfied and so the proof is complete. □

Corollary. *Suppose that w is a nonconstant multiplier of H such that* $\operatorname{ran} w$ *intersects the unit circle. Then the adjoint of the multiplication operator M_w is hypercyclic.*

Proof. In the above theorem let φ be identity. Then $\varphi_n(\lambda) = \lambda$ and $\varphi_n^{-1}(\lambda) = \lambda$ for all λ in $\mathbb{D}$. Further, we note that the condition $1 \in H$ implies that $w \in H$ and so w is analytic on the open unit disc $\mathbb{D}$. Now by the Open Mapping Theorem $w(\mathbb{D})$ is open. But $\operatorname{ran} w = w(\mathbb{D})$ intersects the unit circle, thus the sets

$$\{\lambda \in \mathbb{D}\colon\ |w(\lambda)| < 1\}$$

and

$$\{\lambda \in \mathbb{D}\colon\ |w(\lambda)| > 1\}$$

are nonempty open sets in $\mathbb{D}$ and so clearly have limit points in $\mathbb{D}$. Now we can apply the result of the Theorem and so the proof of the Corollary is complete. □

Remark 1. Note that in the above theorem we restrict ourselves for simplicity to the open unit disc but it remains true if we substitute the open unit disc $\mathbb{D}$ by a connected open subset Ω of $\mathbb{C}^n$ where $n \in \mathbb{N}$. In this case H is a Hilbert space of complex valued analytic functions on Ω such that H contains constants and the functional of evaluation at λ is bounded for all λ in Ω. Also, φ is an analytic univalent map from Ω onto Ω and w is an analytic complex valued function on Ω.

Remark 2. Let B, F, G, H_F, H_G and S_n be defined as in the proof of the main theorem. Note that if we define the mapping $B\colon\ H_G \to H$ exactly as it is defined (in the proof of the theorem) on H_F, then for defining $B^2\colon\ H_G \to H$ we should have $B(H_G) \subset H_G$, which is not since we do not know that whether $\varphi^{-1}(\lambda) \in G$ whenever $\lambda \in G$. For this reason we have to define the mappings S_n on H_G instead of the operators B^n.

Acknowledgment.
The authors thank the referee for interesting comments and helpful suggestions concerning the paper.

References

[1] *J. Bes*: Three problems on hypercyclic operators. PhD. Thesis. Kent State University, 1998.
[2] *J. Bes, A. Peris*: Hereditarily hypercyclic operators. J. Funct. Anal. *167* (1999), 94–112. zbl
[3] *P. S. Bourdon*: Orbits of hyponormal operators. Mich. Math. J. *44* (1997), 345–353. zbl
[4] *P. S. Bourdon, J. H. Shapiro*: Cyclic Phenomena for Composition Operators. Memoirs of the Am. Math. Soc. 125. Am. Math. Soc., Providence, 1997. zbl
[5] *P. S. Bourdon, J. H. Shapiro*: Hypercyclic operators that commute with the Bergman backward shift. Trans. Am. Math. Soc *352* (2000), 5293–5316. zbl
[6] *R. M. Gethner, J. H. Shapiro*: Universal vectors for operators on spaces of holomorphic functions. Proc. Am. Math. Soc. *100* (1987), 281–288. zbl
[7] *G. Godefroy, J. H. Shapiro*: Operators with dense, invariant cyclic vector manifolds. J. Funct. Anal. *98* (1991), 229–269. zbl
[8] *K. Goswin, G. Erdmann*: Universal families and hypercyclic operators. Bull. Am. Math. Soc. *35* (1999), 345–381. zbl
[9] *D. A. Herrero*: Limits of hypercyclic and supercyclic operators. J. Funct. Anal. *99* (1991), 179–190. zbl
[10] *C. Kitai*: Invariant closed sets for linear operators. Thesis. University of Toronto, Toronto, 1982.
[11] *C. Read*: The invariant subspace problem for a class of Banach spaces. 2. Hypercyclic operators. Isr. J. Math. *63* (1988), 1–40. zbl
[12] *S. Rolewicz*: On orbits of elements. Stud. Math. *32* (1969), 17–22. zbl
[13] *H. N. Salas*: Hypercyclic weighted shifts. Trans. Am. Math. Soc. *347* (1995), 993–1004. zbl
[14] *A. L. Shields, L. J. Wallen*: The commutants of certain Hilbert space operators. Indiana Univ. Math. J. *20* (1971), 777–788. zbl
[15] *B. Yousefi, H. Rezaei*: Hypercyclicity on the algebra of Hilbert-Schmidt operators. Result. Math. *46* (2004), 174–180. zbl
[16] *B. Yousefi, H. Rezaei*: Some necessary and sufficient conditions for Hypercyclicity Criterion. Proc. Indian Acad. Sci. (Math. Sci.) *115* (2005), 209–216. zbl

Authors' address: B. Yousefi, S. Haghkhah, Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran, e-mail: `byousefi@shirazu.ac.ir`.