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LJ-SPACES

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Abstract. In this paper LJ-spaces are introduced and studied. They are a common generalization of Lindelöf spaces and J-spaces researched by E. Michael. A space X is called an LJ-space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is Lindelöf. Semi-strong LJ-spaces and strong LJ-spaces are also defined and investigated. It is demonstrated that the three spaces are different and have interesting properties and behaviors.

Keywords: LJ-spaces, Lindelöf, J-spaces, L-map, (countably) compact, perfect map, order topology, connected, topological linear spaces

MSC 2000: 54D20, 54D30, 54F05, 54F65

1. Introduction

The Jordan curve theorem is one of the classical theorems of mathematics; it says that if C is a simple closed curve in the plane $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ has precisely two components W_1 and W_2, of which C is the common boundary [M].

Generalizing these properties, E. Michael [3] introduced and studied the following J-spaces.

A space X is *a J-space* if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact.

A compact space is a J-space, but a J-space need not be compact.

We wonder whether in the definition of the J-space, "A or B is compact" is equivalent to "A or B is Lindelöf". If not, what properties would the following space have?

Definition 1. A space X is an LJ-space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is Lindelöf.

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Obviously, both the Lindelöf spaces and J-spaces are LJ-spaces. In this note, we show that the LJ-space is different from the J-space or the Lindelöf space. Related spaces—strong LJ-spaces and semi-strong LJ-spaces are also introduced and studied. That the three classes of spaces are different is demonstrated by examples; their characterizations and relationships are investigated. They have interesting properties and behavior.

Throughout the note, spaces are topological spaces which are Hausdorff. A space X is Lindelöf if every open cover of X has a countable subcover. All maps are continuous. A map $f\colon X \to Y$ is boundary-perfect ([3]) if f is closed and $\partial(f^{-1}(y))$ is compact for any $y \in Y$. For a subset A of the space X, we reserve ∂A and A° for the boundary and interior of A respectively, and the symbols $\mathbb{R}$ and $\mathbb{Z}^+$ for the sets of all real numbers and all non-negative integers respectively. Further, $\mathbb{R}^+ = \{x \in \mathbb{R}\colon x \geqslant 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R}\colon x \leqslant 0\}$. The cardinality of a set A is denoted by $|A|$. As a space, every ordinal has the usual order topology unless specifically stated otherwise. Other terms and symbols will be found in [1].

2. Definitions and implications

The following two spaces are related to J-spaces. A space X is *a strong J-space* [3] if every compact $K \subset X$ is contained in a compact subset M of X such that $X \setminus M$ is connected. A space X is *a semi-strong J-space* [3] if every compact $K \subset X$ is contained in a compact subset M of X such that $M \cup C = X$ for some connected $C \subset X \setminus K$. In [3], it is shown that the following implications hold while the inverses are not true:

compactness $\Rightarrow$ strong $J \Rightarrow$ semi-strong J $\Rightarrow$ J.

We are naturally interested in the properties introduced below.

Definition 2. A space X is *a strong LJ-space* if every compact $K \subset X$ is contained in a closed Lindelöf $L \subset X$ such that $X \setminus L$ is connected.

Definition 3. A space X is *a semi-strong LJ-space* if every compact $K \subset X$ is contained in a closed Lindelöf $L \subset X$ such that $L \cup C = X$ for some connected $C \subset X \setminus K$.

Clearly, Lindelöf spaces are strong LJ-spaces and LJ-spaces. So $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}^n$ ($n > 1$), the real line $\mathbb{R}$ and the Sorgenfrey line S are strong LJ-spaces. In [3], it is shown that $\mathbb{R}^+$, $\mathbb{R}^-$ and $\mathbb{R}^n$ ($n > 1$) are also strong J-spaces while $\mathbb{R}$ is not a J-space.

Proposition 1. *The Sorgenfrey line S is not a J-space.*

Proof. The closed cover $\{(-\infty,0],\ [0,\infty)\}$ of S satisfies that $(-\infty,0]\cap[0,\infty)=\{0\}$ is compact, but neither $(-\infty,0]$ nor $[0,\infty)$ is compact. $\square$

It was shown that every topological linear space $X\neq\mathbb{R}$ is a strong J-space (Proposition 2.6 of [3]). Since strong J-spaces are strong LJ-spaces and the real line $\mathbb{R}$ is a strong LJ-space, we have

Proposition 2. *All topological linear spaces are strong LJ-spaces.*

The long line Z (see [8] and [3]) (that is, $Z=[0,\omega_1)\times[0,1)$ with the order topology generated by the lexicographical order) is connected, non-compact, countably compact and locally compact.

Proposition 3.

(1) *The long line Z is a strong J-space (so a strong LJ-space), but not a Lindelöf space.*

(2) *The product $\{0,1\}\times Z$ is not an LJ-space.*

Proof. Let $K\subset Z$ be compact. Then K is bounded and so there exists an $\alpha_0\in[0,\omega_1)$ such that $K\subset[\langle 0,0\rangle,\langle\alpha_0,0\rangle]$. Then $L=[\langle 0,0\rangle,\langle\alpha_0,0\rangle]$ is compact and $Z\setminus L$ is connected. Thus Z is a strong J-space. Clearly, Z is not Lindelöf.

(2) Put $A=\{0\}\times Z$, $B=\{1\}\times Z$. Then the closed cover $\{A,B\}$ of $\{0,1\}\times Z$ is the desired one. $\square$

Proposition 4.

(1) *$[0,\omega_1)$ is a J-space (so an LJ-space), but not a semi-strong LJ-space. Moreover, any closed subsapce of $[0,\omega_1)$ is a J-space.*

(2) *The product $[0,\omega_1)\times[0,\omega_1)$ is not an LJ-space (so not a J-space).*

Proof. (1) Let $\{A,B\}$ be a closed cover of $[0,\omega_1)$ and let $A\cap B$ be compact. Then A or B is bounded in $[0,\omega_1)$. In fact, assume that both A and B are unbounded in $[0,\omega_1)$; then $A\cap B$ is unbounded, which contradicts the compactness of $A\cap B$. Without loss of generality, we assume that A is bounded in $[0,\omega_1)$, then there exists a $\beta\in[0,\omega_1)$ such that $A\subset[0,\beta]$. Thus A is compact since $[0,\beta]$ is compact. So $[0,\omega_1)$ is a J-space.

Let us note that if $A\subset[0,\omega_1)$ with $|A|\geqslant 2$, then A is not connected. For the compact $K=\{0\}\subset[0,\omega_1)$, if $L\supset K$ is closed, Lindelöf, and $C\subset([0,\omega_1)\setminus K)$ is connected, then $L\cup C\neq[0,\omega_1)$, so the LJ-space $[0,\omega_1)$ is not a semi-strong LJ-space.

Let F be a closed subspace of $[0,\omega_1)$. If F is compact, then it is a J-space. If F is not compact, then F is also a J-space since F and $[0,\omega_1)$ are homeomorphic.

(2) Put $A = \{0\} \times [0, \omega_1)$, $B = [1, \omega_1) \times [0, \omega_1)$, then $\{A, B\}$ is a closed cover of $[0, \omega_1) \times [0, \omega_1)$ with $A \cap B$ compact, however, neither A nor B is Lindelöf. □

In Example 5 we present an ω_1-broom space $Y(\omega_1)$ and show that it is a semi-strong LJ-space and has interesting properties, but it is not a strong LJ-space.

Theorem 1. *Let X be a space and let us consider the following assertions:*

(A) *X is a strong LJ-space;* (a) *X is a strong J-space;*

(B) *X is a semi-strong LJ-space;* (b) *X is a semi-strong J-space;*

(C) *X is an LJ-space;* (c) *X is a J-space.*

Then (A) $\Rightarrow$ (B) $\Rightarrow$ (C), (a) $\Rightarrow$ (A), (b) $\Rightarrow$ (B), (c) $\Rightarrow$ (C) and the implications are not reversible.

P r o o f. (A) $\Rightarrow$ (B) is clear. To show (B) $\Rightarrow$ (C), let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact. Then there is a closed Lindelöf $L \subset X$ and a connected $C \subset X \setminus (A \cap B)$ such that $A \cap B \subset L$ and $L \cup C = X$. Since $\{A \cap C, B \cap C\}$ is a disjoint closed cover of the connected set C, so $A \cap C = \emptyset$ or $B \cap C = \emptyset$. Thus $A \subset X \setminus C \subset L$ or $B \subset X \setminus C \subset L$ is Lindelöf. So X is an LJ-space.

The other implications are obvious.

The Sorgenfrey line S satisfies the conditions (A), (B) and (C), but by Proposition 1, it does not satisfy (c), (b) or (a). (C) $\not\Rightarrow$ (B) follows by Proposition 4 (1); (B) $\not\Rightarrow$ (A) follows by Example 5; (A) $\not\Rightarrow$ Lindelöf follows by Proposition 3 (1). □

3. Internal characterizations

Proposition 5. *The following conditions are equivalent for a space X.*

(1) *X is a strong LJ-space (or a strong J-space).*

(2) *If $\mathscr{W}$ is a disjoint open cover of $X \setminus K$ with K compact, then there is a $W \in \mathscr{W}$ and a connected open $C \subset W$ such that $X \setminus C$ is Lindelöf (compact, respectively).*

(3) *Same as (2), but with $|\mathscr{W}| = 2$.*

P r o o f. (1) $\Rightarrow$ (2). By (1), X has a connected open $C \subset X \setminus K$ with $X \setminus C$ Lindelöf (compact). So $C \subset \cup\mathscr{W}$. Since C is connected and $\mathscr{W}$ is disjoint and open, we have a $W \in \mathscr{W}$ such that $C \subset W$. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. □

Proposition 6. *The following conditions are equivalent for a space X.*

(1) *X is a semi-strong LJ-space (a semi-strong J-space).*

(2) *If $\mathscr{W}$ is a disjoint open cover of $X \setminus K$ with K compact, then there is a $W \in \mathscr{W}$ and a connected $C \subset W$ such that $\overline{X \setminus C}$ is Lindelöf (compact).*

(3) *Same as (2), but with $|\mathscr{W}| = 2$.*

Proof. (1) $\Rightarrow$ (2). By (1), X has a connected $C \subset X \setminus K$ and a closed Lindelöf (a compact) $L \supset K$ with $C \cup L = X$. So $C \subset W$ for a $W \in \mathscr{W}$ since $C \subset \cup\mathscr{W}$ and $\overline{X \setminus C} \subset L$ is Lindelöf (compact). (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. $\square$

Lemma 1. *If B is a closed non-Lindelöf subset of X and $C \subset B$ is Lindelöf, then there is a closed non-Lindelöf $D \subset B$ with $D \cap C = \emptyset$.*

Proof. Let $\mathscr{U}$ be an open cover of B with no countable subcover. Pick a countable $\mathscr{F} \subset \mathscr{U}$ covering C. Then $D = B \setminus \bigcup \mathscr{F}$ has the required properties. $\square$

Theorem 2. *The following conditions are equivalent for a space X.*

(1) *X is an LJ-space.*

(2) *For any $A \subset X$ with compact ∂A, $\overline{A}$ or $\overline{X - A}$ is Lindelöf.*

(3) *If A and B are disjoint closed subsets of X with ∂A or ∂B compact, then A or B is Lindelöf.*

(4) *If $\mathscr{W}$ is a disjoint open cover of $X \setminus K$ with K compact, then $X \setminus W$ is Lindelöf for some $W \in \mathscr{W}$.*

(5) *Same as (4), but with $|\mathscr{W}| = 2$.*

Proof. (1) $\Rightarrow$ (2) is clear since $\partial A = \overline{A} \cap \overline{X - A}$ and $\{\overline{A}, \overline{X - A}\}$ covers X.

(2) $\Rightarrow$ (3). Let A, B be disjoint closed subsets of X and let ∂A be compact, then A or $\overline{X \setminus A}$ is Lindelöf by (2). Since $B \subset \overline{(X \setminus A)}$, A or B is Lindelöf.

(3) $\Rightarrow$ (1). Let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact. Suppose B is not Lindelöf. By Lemma 1 there is a closed non-Lindelöf $D \subset B$ with $D \cap (A \cap B) = \emptyset$. Thus A and D are disjoint closed subsets of X and $\partial A \subset A \cap B$ is compact. So A or D is Lindelöf. Since D is not Lindelöf, A must be Lindelöf.

(1) $\Leftrightarrow$ (5) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (4). Assume (5). If for some $W_0 \in \mathscr{W}$, $W_0 \cup K$ is not Lindelöf, that is, $\{W_0, W^*\}$, where $W^* = \cup\{W \in \mathscr{W} : W \neq W_0\}$, has W^* such that $X \setminus W^* = W_0 \cup K$ is not Lindelöf, so by (5), $X \setminus W_0$ is Lindelöf. If for any $W \in \mathscr{W}$, $W \cup K$ is Lindelöf, then $\overline{W} \subset W \cup K$ is Lindelöf and X is Lindelöf. To show this, for any open cover $\mathscr{U}$ of X take a finite $\mathscr{F} \subset \mathscr{U}$ covering K. Put $U = \bigcup \mathscr{F}$. It is enough to show that $\mathscr{W}' = \{W \in \mathscr{W} : W \not\subset U\}$ is countable. Suppose not. Then $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ with $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$, $\mathscr{W}_1 \cap \mathscr{W}'$ and $\mathscr{W}_2 \cap \mathscr{W}'$ both uncountable. Let $V_i = \bigcup \mathscr{W}_i$ $(i = 1, 2)$. Then $\{V_1, V_2\}$ is a disjoint open cover of $X \setminus K$. By (5), $X \setminus V_1$ or $X \setminus V_2$ is Lindelöf.

Let $X \setminus V_2$ be Lindelöf. Then $\overline{V_1} \subset V_1 \cup K = X \setminus V_2$ is Lindelöf and so $C = \overline{V_1} \setminus U$ is also Lindelöf. Put $\mathscr{W}'_1 = \mathscr{W}_1 \cap \mathscr{W}'$. Then $\mathscr{W}'_1$ covers C and each $W \in \mathscr{W}'_1$ intersects C. This is a contradiction since C is Lindelöf and $\mathscr{W}'_1$ is uncountable and disjoint. So for any $W \in \mathscr{W}$, $X \setminus W$ is Lindelöf. □

Theorem 3. *Let $\{X_1, X_2\}$ be a closed cover of X with $X_1 \cap X_2$ compact, then the following conditions are equivalent.*

(1) *X is an (resp. a semi-strong, a strong) LJ-space.*

(2) *One of X_1 and X_2 is Lindelöf and the other is an (resp. a semi-strong, a strong, respectively) LJ-space.*

Proof. (a) *For the LJ-space.* (1) $\Rightarrow$ (2). By (1), X_1 or X_2 is Lindelöf. Let X_2 be Lindelöf. Let $\{A, B\}$ be a closed cover of X_1 with $A \cap B$ compact. Then X has a closed cover $\{A, B \cup X_2\}$ with $A \cap (B \cup X_2)$ compact. Hence A or $B \cup X_2$ is Lindelöf and so A or B is Lindelöf. (2) $\Rightarrow$ (1). Let X_2 be Lindelöf, X_1 an LJ-space and $\{A, B\}$ a closed cover of X with $A \cap B$ compact. Put $A_i = A \cap X_i$ and $B_i = B \cap X_i$ $(i = 1, 2)$. Then $\{A_1, B_1\}$ is a closed cover of X_1 with $A_1 \cap B_1$ compact. So A_1 or B_1 is Lindelöf. Let B_1 be Lindelöf. Then $B = B_1 \cup B_2$ is also Lindelöf.

(b) *For the semi-strong LJ-space.* (1) $\Rightarrow$ (2). By (1) and Theorem 1 ((B) $\Rightarrow$ (C)), let X_2 be Lindelöf and $K_1 \subset X_1$ compact. Then $K = K_1 \cup (X_1 \cap X_2)$ is compact. So $K \subset L$ for a closed Lindelöf $L \subset X$ and $L \cup C = X$ for a connected $C \subset X \setminus K$. Let $L_1 = L \cap X_1$. Put $M_i = C \cap X_i$, $i = 1, 2$, then $C = M_1 \cup M_2$. So $M_1 = \emptyset$ or $M_2 = \emptyset$ since C is connected. If $M_2 = \emptyset$, then $C \cup L_1 = X_1$ with $C \subset X_1$ and the Lindelöf $L_1 \supset K_1$. If $M_1 = \emptyset$, then the Lindelöf $X_1 = L_1$ is a semi-strong LJ-space. (2) $\Rightarrow$ (1). Let X_2 be Lindelöf, $K \subset X$ be compact and $K_1 = K \cap X_1$. Then X_1 has a closed Lindelöf $L_1 \supset K_1$ and a connected $C \subset X_1 \setminus K_1$ such that $L_1 \cup C = X_1$. Put $L = L_1 \cup X_2$, then is closed Lindelöf, $L \supset K$, $L \cup C = X$ and $C \subset X \setminus K$.

(c) *For the strong LJ-space.* (1) $\Rightarrow$ (2). By (1), let X_2 be Lindelöf. Let $K_1 \subset X_1$ be compact. Then $K = K_1 \cup (X_1 \cap X_2)$ is compact, so $K \subset L$ for a closed Lindelöf $L \subset X$ with $X \setminus L$ connected. Put $L_1 = L \cap X_1$, $M_i = (X \setminus L) \cap X_i$, $i = 1, 2$, then $X \setminus L = M_1 \cup M_2$. So $M_1 = \emptyset$ or $M_2 = \emptyset$. If $M_2 = \emptyset$, then $X_1 \setminus L_1 = X \setminus L$ is connected with $L_1 \subset X_1$ Lindelöf and $L_1 \supset K_1$. If $M_1 = \emptyset$, then the Lindelöf $X_1 = L_1$ is a strong LJ-space. (2) $\Rightarrow$ (1). Let X_2 be Lindelöf and X_1 a strong LJ-space. Let $K \subset X$ be compact. Then $K_1 = (K \cup X_2) \cap X_1$ is compact, so $K_1 \subset L_1$ for a closed Lindelöf $L_1 \subset X_1$ with $X_1 \setminus L_1$ connected. Put $L = L_1 \cup X_2$, then $L \supset K$ is Lindelöf and $X \setminus L = X_1 \setminus L_1$ is connected. □

Corollary 1. *Let $A \subset X$ be closed with ∂A compact. Then if X is an (a semi-strong, a strong) LJ-space, so is A.*

P r o o f. Put $X_1 = A$, $X_2 = \overline{X \setminus A}$. Then the conclusion follows from Theorem 3. □

Corollary 2. *Let $\{X_1, X_2\}$ be a closed cover of X with X_2 Lindelöf. Then*
(1) *if X_1 is an (a semi-strong) LJ-space, so is X.*
(2) *if X_1 is a strong LJ-space with $\partial(X_1)$ compact, so is X.*

P r o o f. (1) See Theorem 3 (case (a), (2) ⇒ (1) and case (b), (2) ⇒ (1)).

(2) Let $K \subset X$ be compact and $K_1 = (K \cap X_1) \cup \partial(X_1)$. Then X_1 has a closed Lindelöf $L_1 \supset K_1$ with $X_1 \setminus L_1$ connected. Put $B = X_2 \setminus X_1{}^{\circ}$ and $L = L_1 \cup B$, then the closed Lindelöf $L \supset K$ and $X \setminus L = X_1 \setminus L_1$ is connected. □

Corollary 3. *Let $X = E \cup U$ with U open in X and $\overline{U}$ compact. Then if E is an (a semi-strong, a strong) LJ-space, so is X.*

P r o o f. The closed $A = X \setminus U \subset E$ has a compact boundary in X and thus in E, so A is an LJ-space by Corollary 1 since E is an LJ-space. X has a closed cover $\{A, \overline{U}\}$ with $\overline{U}$ compact, so by Theorem 3, X is an LJ-space. The proofs of the other cases are similar. □

Remark 1. (1) (a) If $\{X_1, X_2\}$ is a closed cover of X with $X_1 \cap X_2$ compact, then X is a semi-strong J-space iff one of X_1 and X_2 is compact and the other is a semi-strong J-space (since semi-strong $J \Rightarrow J$, the proof is similar to Theorem 3 (b)). (b) Corollaries 1 and 3 are also true for a semi-strong J-space (this follows from (a)).

(2) In Theorem 3 and Corollary 2, the "Lindelöf" cannot be removed. In fact, the long line Z is a strong J-space, but not a Lindelöf one (see Proposition 3), but the topological sum $Z \oplus Z$ is not an LJ-space.

(3) In Corollary 1, the "∂A compact" cannot be omitted (see Theorem 6(2)).

Proposition 7. *Let E be a component of X. If X is a (semi-)strong LJ-space, so is E. Moreover, if a closed subset A is a union of components of X, so is A.*

P r o o f. Let $K \subset E$ be compact, then X has a closed Lindelöf $L \supset K$ with $X \setminus L$ connected since X is a strong LJ-space. If $L \supset E$, then the Lindelöf E is a strong LJ-space. If $L \not\supset E$, then the connected set $X \setminus L$ intersects E and hence $X \setminus L \subset E$. So E has a closed Lindelöf $L' = L \cap E \supset K$ and $E \setminus L' = X \setminus L$ is connected. The proof for a semi-strong LJ-space is similar. □

Theorem 4. *Let $\{X_1, X_2\}$ be a closed cover of X with $X_1 \cap X_2$ non-Lindelöf. If X_1 and X_2 are (semi-strong) LJ-spaces, so is X.*

P r o o f. To show that X is an LJ-space, let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact. For $i = 1, 2$, let $A_i = A \cap X_i$ and $B_i = B \cap X_i$. Then $\{A_i, B_i\}$ is a closed cover of the LJ-space X_i with $A_i \cap B_i$ compact, so either A_i or B_i is Lindelöf. Note that $X_1 \cap X_2 = (A_1 \cup B_1) \cap (A_2 \cup B_2) \subset (A \cap B) \cup B_1 \cup A_2$. If B_1 is Lindelöf, A_2 cannot be Lindelöf since $A \cap B$ is compact while $X_1 \cap X_2$ is not Lindelöf. Hence B_2 is Lindelöf, so $B = B_1 \cup B_2$ is also Lindelöf. The case for A_1 being Lindelöf is similar.

To show that X is a semi-strong LJ-space, let $K \subset X$ be compact and $K_i = K \cap X_i$ for $i = 1, 2$. Then K_i is compact, and so there is a closed Lindelöf $L_i \supset K_i$ in X_i and connected $C_i \subset X_i \setminus K_i$ with $C_i \cup L_i = X_i$ for $i = 1, 2$. Let $L = L_1 \cup L_2$ and $C = C_1 \cup C_2$. Clearly $L \supset K$ is closed Lindelöf and $C \cup L = X$. Since $X_1 \cap X_2$ is non-Lindelöf, $(X_1 \cap X_2) \setminus L \neq \emptyset$. Also $X_i \setminus L \subset X_i \setminus L_i \subset C_i$ for $i = 1, 2$, so $(X_1 \cap X_2) \setminus L \subset (C_1 \cap C_2)$. Hence $C_1 \cap C_2 \neq \emptyset$ and thus C is connected. Clearly $C \subset X \setminus K$. □

Remark 2. Theorem 4 is not true for strong LJ-spaces (see Example 5(2)) and is not reversible (in fact, the semi-strong LJ-space Y in Example 5 has a closed cover $\{Y, F\}$ with $Y \cap F = F$ non-Lindelöf. Y is a semi-strong LJ-space, but F is not an LJ-space since it is discrete and uncountable). In Theorem 4, the assumption that $X_1 \cap X_2$ is non-Lindelöf is also needed (see Remark 1 (1)).

4. External characterizations

To characterize the LJ-space, we introduce the notion of an L-map.

Definition 4. A map $f\colon X \to Y$ is an L-map if f is closed and $f^{-1}(y)$ is Lindelöf for any $y \in Y$.

Clearly, a perfect map is an L-map and is boundary-perfect (for the definition, see Introduction). A boundary-perfect map need not be an L-map (see the map g in Remark 6). Example 1 shows that an L-map need not be perfect or boundary-perfect.

Theorem 5. *The following conditions are equivalent for a space X.*

(1) *X is an LJ-space.*
(2) *If a closed $f\colon X \to Y$ has $\partial(f^{-1}(y_0))$ compact and $f^{-1}(y_0)$ non-Lindelöf for a $y_0 \in Y$, then $f^{-1}(y)$ is Lindelöf for any $y \in Y \setminus \{y_0\}$.*
(3) *Every boundary-perfect map $f\colon X \to Y$ onto a non-Lindelöf space Y is an L-map.*

P r o o f. (1) $\Rightarrow$ (2). For any $y \in Y \setminus \{y_0\}$, $A_0 = f^{-1}(y_0)$ and $A = f^{-1}(y)$ are disjoint closed subsets of X with $\partial(A_0)$ compact. Since $A_0 = f^{-1}(y_0)$ is not Lindelöf, by Theorem 2, $A = f^{-1}(y)$ is Lindelöf.

(2) $\Rightarrow$ (1). Let A_1, A_2 be disjoint closed subsets of X with $\partial(A_1)$ or $\partial(A_2)$ compact. Suppose that $\partial(A_1)$ is compact. Let Y be the quotient space obtained from X by identifying A_i with a point y_i for $i = 1, 2$, and let $f\colon X \to Y$ be the quotient map. Clearly f is closed and $\partial(A_1) = \partial(f^{-1}(y_1))$ is compact. If $A_1 = f^{-1}(y_1)$ is not Lindelöf, then since $y_2 \in Y \setminus \{y_1\}$, by (2), $A_2 = f^{-1}(y_2)$ is Lindelöf. So by Theorem 2, X is an LJ-space.

(1) $\Rightarrow$ (3). Let $f\colon X \to Y$ be as in the assumption and $y \in Y$. Since $\partial(f^{-1}(y))$ is compact, by Theorem 2, $f^{-1}(y)$ or $\overline{X - f^{-1}(y)}$ is Lindelöf. But $\overline{X - f^{-1}(y)}$ is not Lindelöf because Y is not Lindelöf, so $f^{-1}(y)$ is Lindelöf. Hence f is an L-map.

(3) $\Rightarrow$ (1). Let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact and let $Y = X/B$, let $f\colon X \to Y$ be the quotient map and $y_0 = f(B)$. Then f is closed, and if $y \in Y$, then $\partial(f^{-1}(y))$ is compact. So f is boundary-perfect. If Y is non-Lindelöf, then f is an L-map by the given condition, so $B = f^{-1}(y_0)$ is Lindelöf. If Y is Lindelöf, then the closed $f(A)$ is also Lindelöf. Then $f|_A\colon A \to f(A)$ is perfect. Hence $A = f|_A^{-1}(f(A))$ is Lindelöf. □

Remark 3. Theorem 5 is false if the assumption that Y is non-Lindelöf is omitted. Indeed, $f\colon X \to Y$, where X is the non-Lindelöf LJ-space Z in Proposition 3 and Y is a singleton, is such an example.

Corollary 4. *Every closed map $f\colon X \to Y$ from a paracompact LJ-space X onto a non-Lindelöf q-space Y is an L-map.*

P r o o f. This follows from Theorem 5 and the result that every closed map $f\colon X \to Y$ from a paracompact space X onto a q-space Y is boundary-perfect (see [4]). □

Remark 4. (1) Example 2 shows that the paracompactness of X in Corollary 4 cannot be omitted.

(2) In Corollary 4 the assumption that X is an LJ-space cannot be deleted. In fact, let $\mathbb{R}$ be discrete, $X = \mathbb{R} \times \mathbb{R}$ and $Y = \mathbb{R}$. Let $f\colon X \to Y$ be the projection, then f is a closed map, but not an L-map.

Proposition 8. *Let $f\colon X \to Y$ be a perfect map onto Y. Then*

(1) *if X is an (a semi-strong) LJ-space, so is Y.*

(2) *when f is open, if X is a strong LJ-space, so is Y.*

P r o o f. (1) is obvious since the inverse image of a compact set is compact for a perfect map.

(2) Let $K \subset Y$ be compact. Then X has a closed Lindelöf $L' \supset f^{-1}(K)$ with $X \setminus L'$ connected. Put $L = Y \setminus f(X \setminus L')$, then $L \supset K$ and $Y \setminus L = f(X \setminus L')$ is connected. Since $f^{-1}(L) \subset L'$, $f^{-1}(L)$ is Lindelöf and thus L is also Lindelöf. □

Remark 5. In Proposition 8 (2) the "open" cannot be omitted (see Example 5 (4)).

Recall that a continuous map $f\colon X \to Y$ is monotone if all fibers $f^{-1}(y)$ are connected.

Proposition 9. *Let $f\colon X \to Y$ be a monotone L-map onto Y. Then, if Y is an (a semi-strong, a strong) LJ-space, so is X.*

Proof. (a) Let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact. Then $\{f(A), f(B)\}$ is a closed cover of Y. By Lemma 5.5 of [3], $f(A) \cap f(B) = f(A \cap B)$ is compact. So $f(A)$ or $f(B)$ is Lindelöf. Thus $f^{-1}(f(A))$ or $f^{-1}(f(B))$ is Lindelöf since f is an L-map. So A or B is Lindelöf and X is an LJ-space.

(b) Let $K \subset X$ be compact. Then Y has a closed Lindelöf $L' \supset f(K)$ and a connected $C' \subset Y \setminus f(K)$ with $C' \cup L' = Y$. Then $L = f^{-1}(L')$ is closed Lindelöf. Since f is closed and monotone, $C = f^{-1}(C')$ is connected by Theorem 6.1.29 of [1]. Clearly $L \supset K$, $C \subset X \setminus K$ and $L \cup C = X$. Thus X is a semi-strong LJ-space.

(c) Let $K \subset X$ be compact. Then Y has a closed Lindelöf $L \supset f(K)$ with $Y \setminus L$ connected. So $f^{-1}(L) \supset K$ is Lindelöf and $X \setminus f^{-1}(L) = f^{-1}(Y \setminus L)$ is connected since f is closed and monotone. So X is a strong LJ-space. □

Remark 6. (1) Let $f\colon X \to Y$ be a monotone perfect map onto Y. Then, if Y is a semi-strong J-space, so is X (the proof is similar to the case (b) of Proposition 9).

(2) In Proposition 9 the "monotone" cannot be deleted: let Y be the long line Z which is a connected, non-Lindelöf, strong J-space and let $X = Y \oplus Y$. Then the obvious map $f\colon X \to Y$ is perfect, but clearly X is not an LJ-space. Also, the assumption in Proposition 9 that f is an L-map cannot be omitted or replaced by f being boundary-perfect. Indeed, if X is as above and E is a two-point space, then the obvious map $g\colon X \to E$ is boundary-perfect and monotone with each $f^{-1}(e)(e \in E)$ being a strong J-space, but X is not an LJ-space.

Proposition 10. *The following conditions are equivalent for a space Y.*

(1) *Y is an (a semi-strong, a strong) LJ-space.*

(2) *$Y \times Z$ is an (a semi-strong, a strong) LJ-space for every connected and compact space Z.*

(3) *$Y \times Z$ is an (a semi-strong, a strong) LJ-space for some connected and compact space Z.*

P r o o f. (1) $\Rightarrow$ (2) is by Proposition 9 with $X = Y \times Z$ and $f\colon X \to Y$ the projection. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) is by Proposition 8 with $X = Y \times Z$ and $f\colon X \to Y$ the projection. □

Proposition 11. *Each of the following conditions implies that $Y \times Z$ is an (a semi-strong, a strong) LJ-space.*

(1) *Y and Z are connected (semi-strong, strong) LJ-spaces.*

(2) *Y is a connected, non-compact (semi-strong, strong) LJ-space and Z is connected.*

P r o o f. (1) If Y or Z is compact, this follows from Proposition 10. If neither Y nor Z is compact, by Proposition 2.5 of [3], $Y \times Z$ is a strong J-space and it follows from Theorem 1.

(2) If Z is compact, this follows from Proposition 10. If Z is not compact, it follows from Propositions 2.5 of [3] and Theorem 1. □

Remark 7. (1) Propositions 10 and 11 are true for semi-strong J-spaces (by Proposition 8.5 of [3] and Remark 6 (1)).

(2) In (1) and in Proposition 10 (2), (3) (Proposition 5.7(b), (c) of [3]), the connectedness cannot be omitted: by Proposition 3 (2), the long line Z is a strong J-space, but $Z \times \{0, 1\}$ is not an LJ-space.

5. Relationships

Recall a space X is called *hereditarily disconnected* if X does not contain any connected subsets of cardinality larger than one.

Theorem 6. *Let* (A), (B), (C), (a), (b) *and* (c) *be the same as in Theorem* 1. *Then*

(1) *for a locally connected space X; (A)$\Leftrightarrow$(B)$\Leftrightarrow$(C).*

(2) *none of the six properties is productive (additive, preserved by the quotient mapping, hereditary with respect to closed subspaces);*

(3) *for a countably compact space X, $(A) \Leftrightarrow (a)$, $(B) \Leftrightarrow (b)$, $(C) \Leftrightarrow (c)$, $(D) \Leftrightarrow (d)$ and $(E) \Leftrightarrow (e)$;*

(4) *for a hereditarily disconnected space X, "X is Lindelöf" $\Leftrightarrow (A) \Leftrightarrow (B)$ and "X is compact" $\Leftrightarrow (a) \Leftrightarrow (b)$.*

P r o o f. (1) $(A) \Rightarrow (B) \Rightarrow (C)$ follows by Theorem 1. $(C) \Rightarrow (A)$. Let $K \subset X$ be compact. Since X is locally connected, there is a disjoint open cover $\mathscr{W}$ of $X \setminus K$

with each $W \in \mathscr{W}$ connected. By Theorem 2, there exists a $W_0 \in \mathscr{W}$ such that $L = X \setminus W_0$ is Lindelöf. Clearly $L \supset K$ and $X \setminus L$ is connected.

(2) Not productive: by Proposition 3 (2). Not additive: by Remark 1 (1). Not preserved by the quotient mapping: by Example 4.

Not hereditary with respect to closed subspaces.

For (a), (b) and (c): the strong J-space $\mathbb{R}^+$ has a closed discrete subspace $\mathbb{Z}^+$ which is not a J-space.

For (A): the long ling Z is a strong LJ-space having a closed subspace $[0,\omega_1)\times\{0\}$ homeomorphic to $[0,\omega_1)$ which is not a strong LJ-space by Proposition 4.

For (B) and (C): in Example 5, the semi-strong LJ-space Y has a discrete closed subspace F which is uncountable, so F is not an LJ-space.

(3) is obvious since in a countably compact space Lindelöfness$\Leftrightarrow$ compactness.

(4) Clearly, "X is Lindelöf" $\Rightarrow (A) \Rightarrow (B)$ and "X is compact" $\Rightarrow (a) \Rightarrow (b)$. To show that $(B) \Rightarrow$ "X is Lindelöf" ($(b) \Rightarrow$ "X is compact"), let $K \subset X$ be compact. By (B) ((b)) X has a closed Lindelöf (a compact) $L \supset K$ and a connected $C \subset X \setminus K$ such that $L \cup C = X$. Since X is hereditarily disconnected, $C = \emptyset$ or C is a one-point set. So X is Lindelöf (compact). □

6. Examples

Example 1. An L-map which is not boundary-perfect (so not perfect).

Let $I_i = [o_i, 1_i]$ $(i \in \omega)$ be the copy of the unit closed interval $I = [0,1]$ and let $X = \oplus\{I_i\colon i \in \omega\}$ be the topological sum. Define an equivalence relation $\mathscr{R}$ on X as follows: for each $x_i \in I_i$, if $x_i \neq o_i$, then $x_i \mathscr{R} x_i$; if $x_i = o_i$, then $o_i \mathscr{R} o_j$, $j \in \omega$. Then the natural map $f\colon X \to Y = X/\mathscr{R}$ is an L-map, but not a boundary-perfect map.

Example 2. A closed and open map $f\colon X \to Y$ from a locally compact strong J-space (so a strong LJ-space) X onto a non-Lindelöf q-space Y which is not an L-map.

P r o o f. Let Z be the long line. Then Z is non-Lindelöf and first countable (so a q-space). Let $X = Z \times Z$, $Y = Z$ and let $f\colon X \to Y$ be the projection onto the first coordinate. Then f is open. Let us show that f is also closed. Note that X is countably compact since Z is countably compact and first countable (see Theorem 3.10.36 of [1]). Let $F \subset X$ be closed, then F is countably compact and therefore $f(F)$ is countably compact in Z and thus closed in Z. Since Z is connected non-compact, $X = Z \times Z$ is a strong J-space by Proposition 2.5 of [3]. Clearly f is not an L-map. □

Example 3. The Niemytzki plane X is not an LJ-space.

P r o o f. Let $A = [0,1] \times [0,1], B = X \setminus (0,1) \times [0,1)$. Then $\{A, B\}$ is a closed cover with $A \cap B$ compact, but neither A nor B is Lindelöf. $\square$

Example 4. A strong J-space X whose quotient space Q is not an LJ-space.

P r o o f. Let Q be the Niemytzki plane. Put $X = Q \times \mathbb{R}$, where $\mathbb{R}$ is the real line. By Proposition 2.5 of [3], the product space X is a strong J-space. Clearly Q is a quotient space and the projection $p\colon Q \times \mathbb{R} \to Q$ is the quotient map. $\square$

The following ω_1-broom space $Y(\omega_1)$ is an interesting space. From Theorem 1, Theorem 6 (2), Remarks 2 and 5, we have seen that it plays an important role in this note.

Let Z be the long line and $X = Z \times \mathbb{R}^+$ with the product topology, where $\mathbb{R}^+$ is with the usual topology. For $\alpha \in [0, \omega_1)$ and integer $i \geqslant 1$, let $E_{\alpha,i}$ be the closed segment joining $\langle\langle\alpha, 0\rangle, 0\rangle)$ to $\langle\langle\alpha+1, 0\rangle, \frac{1}{i}\rangle$, where $\langle\alpha, 0\rangle$ and $\langle\alpha+1, 0\rangle$ are points of Z. Put

$$E_\alpha = \left(\bigcup_{i=1}^{\infty} E_{\alpha,i}\right) \cup ([\langle\alpha,0\rangle, \langle\alpha+1,0\rangle] \times \{0\}),$$

where $[\langle\alpha,0\rangle, \langle\alpha+1,0\rangle]$ is a closed interval of Z.

We define $Y(\omega_1) = \cup\{E_\alpha\colon \alpha \in [0,\omega_1)\}$ to be a subspace of X and call $Y(\omega_1)$ *the* ω_1*-broom space*; we also write Y instead of $Y(\omega_1)$.

Example 5. The ω_1-broom space Y is a semi-strong LJ-space such that

(1) Y is not a strong LJ-space;

(2) Y has a closed cover $\{A, B\}$ with $A \cap B$ non-Lindelöf and both A and B are strong LJ-spaces;

(3) Y has a closed discrete subspace F which is uncountable;

(4) there is a perfect map $f\colon M \to Y$ from a strong LJ-space M onto Y.

P r o o f. For any $\alpha \in [0,\omega_1)$, let $L_\alpha = \{\langle y_1, y_2\rangle \in Y\colon y_1 \leqslant \langle\alpha, 0\rangle\}$ and $C_\alpha = \overline{Y \setminus L_\alpha}$. Then L_α is Lindelöf, C_α is connected and $L_\alpha \cup C_\alpha = Y$. Now for any compact $K \subset Y$, pick α such that $K \subset L_\alpha$. Then $K \subset L_{\alpha+1}$, $C_{\alpha+1} \subset Y \setminus K$ and $L_{\alpha+1} \cup C_{\alpha+1} = Y$. So Y is a semi-strong LJ-space.

(1) Y is not a strong LJ-space. In fact, for the "beginning point" $\langle\langle 0,0\rangle, 0\rangle$ of Y, let the compact subset H be the one-point set $\{\langle\langle 0,0\rangle,0\rangle\}$. If $L \subset Y$ is closed, Lindelöf and $H \subset L$, then we can see that $Y \setminus L$ is not connected.

(2) Put $A = (Z \times \{0\}) \cup \left(\bigcup\{E_\alpha\colon \alpha \in [0,\omega_1\right)$, α is a successor ordinal$\})$, $B = (Z \times \{0\}) \cup (\bigcup\{E_\alpha\colon \alpha \in [0,\omega_1)$, α is a limit ordinal$\}$.

Then $\{A, B\}$ is a closed cover of Y with $A \cap B = Z \times \{0\}$ non-Lindelöf.

Let us show that A is a strong LJ-space. For a limit ordinal α, put $L_\alpha^A = \{\langle z, y\rangle \in A\colon z \leqslant \langle \alpha, 0\rangle\}$, then L_α^A is closed Lindelöf, $A \setminus L_\alpha^A$ is connected and each compact $K \subset A$ is a subset of some L_α^A.

Similarly, B is also a strong LJ-space.

(3) Put $F = \{\langle\langle \alpha+1, 0\rangle, 1\rangle\colon \alpha \in [0, \omega_1)\}$, then the uncountable F is a closed discrete subspace of Y.

(4) Let $M = B$ be a subspace of Y and $D = \{\alpha \in [0, \omega_1)\colon \alpha$ is a limit ordinal$\}$. Then D with the order topology is homeomorphic to $[0, \omega_1)$. So there exists an order preserving homeomorphic map $\varphi\colon D \to [0, \omega_1)$. For any $\alpha \in D$, let $f_\alpha\colon E_\alpha \to E_{\varphi(\alpha)}$ be a homeomorphic map. Now we define $f\colon M \to Y$ as follows.

For any $\langle z, y\rangle \in M$,

$$f(\langle z, y\rangle) = \begin{cases} f(\langle z, y\rangle) = f_\alpha(\langle z, y\rangle), & \langle z, y\rangle \in E_\alpha,\ \alpha \in D, \\ f(\langle z, 0\rangle) = \langle\langle \alpha+1, 0\rangle, 0\rangle, & \langle z, 0\rangle \in [\langle \alpha+1, 0\rangle, \langle \alpha^+, 0\rangle] \times \{0\}, \end{cases}$$

where α^+ is the smallest of the limit ordinals greater than α. Then f is a perfect map. □

Corollary 5.

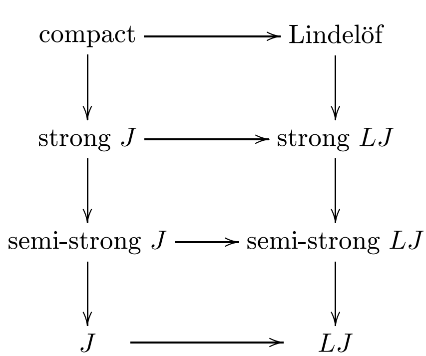
(1) *The ω_1-broom space $Y(\omega_1)$ cannot be the image under an open perfect map of the long line Z.*

(2) *Under CH, the Niemytzki plane cannot be the the image under an perfect map of the long line Z or the ω_1-broom space $Y(\omega_1)$.*

Proof. (1) The long line Z is a strong LJ-space, thus by Proposition 8 (2), so is its open perfect image. But by Example 5, the ω_1-broom space $Y(\omega_1)$ is not a strong LJ-space.

(2) The long line Z and the ω_1-broom space $Y(\omega_1)$ are LJ-spaces by Proposition 8 (1), so their perfect images, but the Niemytzki plane is not an LJ-spaces. □

Now we illustrate the harmonious relationships with a diagram.



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