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DIRECT SUMMANDS AND RETRACT MAPPINGS OF  
GENERALIZED  $MV$ -ALGEBRAS

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*Abstract.* In the present paper we deal with generalized  $MV$ -algebras ( $GMV$ -algebras, in short) in the sense of Galatos and Tsinakis. According to a result of the mentioned authors,  $GMV$ -algebras can be obtained by a truncation construction from lattice ordered groups. We investigate direct summands and retract mappings of  $GMV$ -algebras. The relations between  $GMV$ -algebras and lattice ordered groups are essential for this investigation.

*Keywords:* residuated lattice, lattice ordered group, generalized  $MV$ -algebra, direct summand

*MSC 2000:* 06D35, 06F15

1. INTRODUCTION

In [5], the notion of generalized  $MV$ -algebra ( $GMV$ -algebra, in short) has been introduced; it has been studied in the context of residuated lattices.

The fundamental result of [5] is Theorem (A). From this it follows that each  $GMV$ -algebra can be represented by using lattice ordered groups. For a detailed formulation of this result, cf. Section 2 below.

In the present paper we apply the mentioned representation for investigating direct summands and retract mappings of  $GMV$ -algebras.

Let  $\mathbf{M}$  be a  $GMV$ -algebra and let  $\ell(\mathbf{M})$  be the underlying lattice of  $\mathbf{M}$ . Further, let  $\mathbf{A}$  be a subalgebra of  $\mathbf{M}$ . We prove that  $\mathbf{A}$  is a direct summand of  $\mathbf{M}$  iff the underlying lattice  $\ell(\mathbf{A})$  of  $\mathbf{A}$  is an internal direct factor of the lattice  $\ell(\mathbf{A})$ .

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The main result concerning retract mappings of *GMV*-algebras is Theorem (C) presented in Section 7 below.

We recall that the investigation of direct summands of some types of algebraic structures is frequent in the literature. E.g., a rather large series of papers has dealt with direct summands of abelian groups; cf. the references given in [4].

The related notion of direct product decomposition of *MV*-algebras was dealt with in [9]; for the case of pseudo *MV*-algebras cf. [10] and [20] (under a different terminology).

Retract mappings and retracts of lattice ordered groups were investigated in [13], [14], [15], [16]. Retract mappings of *MV*-algebras were studied in [17].

An important tool in the investigation of the relation between *GMV*-algebras and lattice ordered groups that is applied in [5] is the negative cone of a lattice ordered group. In the introduction of [5], the authors mention the papers of Chang [1], Mundici [18] and Dvurečenskij [3] on *MV*-algebras and pseudo *MV*-algebras; here the authors write: ‘It should be noted that all the three authors have expressed their results in terms of the positive cone rather than the negative cone.’ Hence in this respect, the method of [5] differs from that of [1], [3], [18].

We also remark that the term ‘generalized *MV*-algebra’ was applied in a different sense in [17]; in the sense of [17], this term is equivalent to the notion of pseudo *MV*-algebra (cf. [3], [6], [7], and also [10], [11], [12], [19] and [20]).

In what follows, the term ‘*GMV*-algebra’ will be used in the sense of [5].

## 2. PRELIMINARIES

For the sake of completeness, we recall some basic definitions. We also quote some results of [5].

A *residuated lattice* is an algebra  $\mathbf{L} = (L; \wedge, \vee, \cdot, \backslash, /, e)$  of type  $(2, 2, 2, 2, 2, 0)$  such that  $(L; \wedge, \vee)$  is a lattice,  $(L; \cdot, e)$  is a monoid and for each  $x, y, z \in L$ ,

$$x \cdot y \leq \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x \backslash z.$$

A residuated lattice is *commutative* if  $xy = yx$  for each  $x, y \in L$ ; it is *integral* if  $x \wedge e = x$  for each  $x \in L$ .

The *negative cone* of a residuated lattice  $\mathbf{L}$  is an algebra  $\mathbf{L}^- = (L^-; \wedge, \vee, \cdot, \backslash_{L^-}, /_{L^-}, e)$  where

$$\begin{aligned} L^- &= \{x \in L: x \leq e\}, \\ x \backslash_{L^-} y &= (x \backslash y) \wedge e, \quad x /_{L^-} y = (x/y) \wedge e. \end{aligned}$$

Then  $\mathbf{L}^-$  is a residuated lattice as well.

A *generalized MV-algebra* (*GMV-algebra*, in short) is a residuated lattice satisfying the identities

$$x / ((x \vee y) \setminus x) = x \vee y = (x / (x \vee y)) \setminus x.$$

If  $\mathbf{L}$  is a *GMV-algebra*, then its negative cone  $\mathbf{L}^-$  is a *GMV-algebra* as well.

Let  $P$  be a partially ordered set. A mapping  $\gamma: P \rightarrow P$  is a *closure operator* on  $P$  if  $\gamma(x) \leq \gamma(y)$  whenever  $x \leq y$ ,  $x \leq \gamma(x)$  and  $\gamma(\gamma(x)) = x$ . Put  $\gamma(P) = P_\gamma$ . Then

$$\gamma(x) = \min\{t \in P_\gamma: x \leq t\}$$

for each  $x \in P$ ; hence the mapping  $\gamma$  is uniquely determined by the set  $P_\gamma$ .

Let  $\mathbf{L}$  be a residuated lattice. A closure operator  $\gamma$  on  $L$  satisfying  $\gamma(a)\gamma(b) \leq \gamma(a, b)$  for each  $a, b \in L$  is a *nucleus* on  $L$ . If  $L_\gamma$  is the image of a nucleus  $\gamma$  on  $L$ , then the set  $L_\gamma$  is endowed with a residuated lattice structure in the following way:

$$L_\gamma = (L_\gamma; \wedge, \vee, \circ_\gamma, \setminus, /, \gamma(e)),$$

where

$$\gamma(a) \vee_\gamma \gamma(b) = \gamma(a \vee b), \quad \gamma(a) \circ_\gamma \gamma(b) = \gamma(ab).$$

A residuated lattice  $\mathbf{A}$  is a *direct sum* of its subalgebras  $\mathbf{B}$  and  $\mathbf{C}$ , in symbols  $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$ , if the map  $B \times C \rightarrow A$  defined by  $f(x, y) = xy$  is an isomorphism. In such case  $\mathbf{B}$  and  $\mathbf{C}$  are *direct summands* of  $\mathbf{A}$ . Under the above notation, put  $z = xy$ ; we denote  $x = z(\mathbf{B})$  and  $y = z(\mathbf{C})$ . We say that  $x$  and  $y$  is the *component* of  $z$  in  $\mathbf{B}$  or in  $\mathbf{C}$ , respectively.

For lattice ordered groups we use the terminology and the notation as in [8].

Let  $\mathbf{G} = (G; \wedge, \vee, \cdot, ^{-1}, e)$  be a lattice ordered group. The algebra

$$\mathbf{G}^* = (G; \wedge, \vee, \cdot, \setminus, /, e)$$

where  $x \setminus y = x^{-1}y$  and  $y/x = yx^{-1}$ , is a *GMV-algebra*.

The following theorem is one of the main results of [5]; we use a slightly modified notation.

**Theorem 2.1** (cf. [5], Theorem (A)). *A residuated lattice  $\mathbf{M}$  is a GMV-algebra if and only if there are lattice ordered groups  $\mathbf{G}$  and  $\mathbf{G}_1$  and a nucleus  $\gamma$  on  $(\mathbf{G}_1^*)^-$  such that*

$$\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma,$$

where  $\mathbf{L} = (\mathbf{G}_1^*)^-$ .

**Theorem 2.2** (cf. [5], Theorem 3.4). *If  $\mathbf{L} = (L; \wedge, \vee, \cdot, \setminus, /, e)$  is a GMV-algebra and  $\gamma$  is a nucleus on  $\gamma$ , then*

- (i)  $\vee_\gamma = \vee$ ;
- (ii)  $\gamma$  preserves binary joins;
- (iii)  $\gamma(e) = e$ ;
- (iv)  $\mathbf{L}_\gamma = (L_\gamma; \wedge, \vee, \circ_\gamma, \setminus, /, e)$  is a GMV-algebra;
- (v)  $L_\gamma$  is a filter of the lattice  $(L; \wedge, \vee)$ .

### 3. INTERNAL DIRECT FACTORS OF PARTIALLY ORDERED SETS

Assume that  $P$  is a partially ordered set and that  $(P_i)_{i \in I}$  is an indexed system of partially ordered sets. The *direct product*  $\prod_{i \in I} P_i$  is defined in the usual way. The elements of  $\prod_{i \in I} P_i$  are written in the form  $t = (t_i)_{i \in I}$ . If

$$\varphi: P \rightarrow \prod_{i \in I} P_i$$

is an isomorphism, then we say that  $\varphi$  is a *direct product decomposition* of  $P$ . In such case, for each  $i \in I$  and each  $a \in P$  we put

$$P(i, a) = \{x \in P: \varphi(x)_j = \varphi(a)_j \text{ for each } j \in I \setminus \{i\}\}.$$

The set  $P(i, a)$  endowed with the partial order induced from  $P$  is an *internal direct factor of  $P$  with respect to the element  $a$* . Obviously,  $P(i, a)$  is isomorphic to  $P_i$ .

For each  $y \in P$ , we denote by  $\varphi_i^a(y)$  the element of  $P(i, a)$  such that

$$(\varphi(\varphi_i^a(y)))_i = (\varphi(y))_i.$$

Then the mapping

$$(1) \quad \varphi^a: P \rightarrow \prod_{i \in I} P(i, a)$$

where  $\varphi^a(y) = (\varphi_i^a(y))_{i \in I}$  for each  $y \in P$ , is an isomorphism. We say that  $\varphi^a$  defines an *internal direct product decomposition of  $P$  with respect to the element  $a$* .

For each  $x \in P$  we now put

$$x_i = (\varphi^a(x))_i;$$

$x_i$  is the  $i$ -th component of  $x$  with respect to (1). We also say that  $x_i$  is the component of  $x$  in  $P(i, a)$  and we write  $x_i = x(P(i, a))$ . Then

$$(2) \quad a_i = a \text{ for each } i \in I,$$

$$(3) \quad (x_i)_i = x_i \text{ and } (x_i)_j = a \text{ if } j \in I, j \neq i.$$

Now let  $I_1$  and  $I_2$  be nonempty subsets of  $I$  such that  $I_1 \cap I_2 = \emptyset$  and  $I_1 \cup I_2 = I$ . Put

$$P(I_1, a) = \{x \in P: x_i = a_i \text{ for each } i \in I_1\},$$

$$P(I_2, a) = \{x \in P: x_i = a_i \text{ for each } i \in I_2\}.$$

Let  $x \in P$ . The element  $y \in P$  such that

$$y_i = \begin{cases} x_i & \text{if } i \in I_1, \\ a_i & \text{if } i \in I_2 \end{cases}$$

will be denoted by  $x_{I_1}$ . Analogously we define  $x_{I_2}$ . Then the mapping

$$x \rightarrow (x_{I_1}, x_{I_2})$$

defines an internal direct product decomposition

$$(*_1) \quad P \rightarrow P(I_1, a) \times P(I_2, a).$$

Further, we have internal direct product decompositions

$$(*_2) \quad P(I_1, a) \rightarrow \prod_{i \in I_1} P(i, a),$$

$$(*_3) \quad P(I_2, a) \rightarrow \prod_{i \in I_2} P(i, a).$$

All the internal direct product decompositions  $(*_1)$ ,  $(*_2)$  and  $(*_3)$  are taken with respect to the element  $a$ .

**Lemma 3.1.** *Assume that  $P$  is a partially ordered set and that  $a$  is the greatest element of  $P$ . Let (1) be valid. Then*

$$x = \bigwedge_{i \in I} x_i$$

for each  $x \in P$ .

*Proof.* This is a consequence of the relations (2) and (3). □

If (1) holds and  $i \in I$ , then we put

$$P'(i, a) = \{x \in P: x_i = a\}.$$

Then in view of  $(*_2)$  we have an internal product decomposition

$$P'(i, a) \rightarrow \prod_{j \in I \setminus \{i\}} P(j, a).$$

Moreover, according to  $(*_1)$  we obtain a two-factor internal direct product decomposition

$$(4) \quad P \rightarrow P(i, a) \times P'(i, a).$$

For  $x \in P$  we put  $x(P'(i, a)) = x'_i$ . □

**Lemma 3.2.** *Let  $P$  be as in Lemma 3.1. Further, let  $x$  and  $y$  be elements of  $P$ . Then*

$$x_i \wedge x'_i = x, \quad x_i \vee y'_i = a.$$

*Proof.* The validity of the first relation is a consequence of Lemma 3.1 and of (4). In view of (2) and (3), the second relation holds. □

#### 4. THE NEGATIVE CONE

Let  $\mathbf{G}$  be a lattice ordered group. The algebra

$$\mathbf{G}^- = \{G^-; \wedge, \vee, \cdot, e\},$$

where  $\mathbf{G}^- = \{g \in G: g \leq e\}$  is the *negative cone* of  $G$ . For  $x, y \in G^-$  we put  $x \setminus y = (x^{-1}y) \wedge e$  and  $y/x = (yx^{-1}) \wedge e$ . An elementary calculation shows that the algebra

$$(\mathbf{G}^-)^* = (G^-; \wedge, \vee, \cdot, \setminus, /, e)$$

is a *GMV*-algebra; moreover, under the notation as in Section 2 we have  $(\mathbf{G}^-)^* = (\mathbf{G}^*)^-$ .

We denote by  $\ell(\mathbf{G})$  and  $\ell(\mathbf{G}^-)$  the underlying lattice of  $\mathbf{G}$  or of  $\mathbf{G}^-$ , respectively.

A filter  $C$  of  $\ell(G^-)$  will be called *regular* if for each  $x \in G^-$ , the set  $\{c \in C: x \leq c\}$  has a minimal element; in such case, this minimal element will be denoted by  $\gamma_C(x)$ . Clearly,

$$\gamma_C(G^-) = C.$$

**Lemma 4.1.** *Let  $C$  be a regular filter of the lattice  $\ell(\mathbf{G}^-)$ . Then  $\gamma_C$  is a nucleus on  $G^-$  (with respect to the *GMV*-algebra  $(\mathbf{G}^-)^*$ ).*

*Proof.* It is obvious that  $\gamma_C$  is a closure operator on the lattice  $\ell(\mathbf{G}^-)$ . Let  $a, b \in G^-$ . In view of the definition of the operation  $/_{L-}$  we have

$$\gamma(a)/_{L-}b = (\gamma(a)b^{-1}) \wedge e.$$

From  $b \in G^-$  we obtain  $b^{-1} \geq e$ , thus  $\gamma(a)b^{-1} \geq \gamma(a)$ , whence

$$\gamma(a) \leq (\gamma(a)b^{-1}) \wedge e \leq e$$

and thus  $\gamma(a)/_{L-}b \in \gamma_C(G^-)$ . Analogously,  $b \setminus_{L-} \gamma(a) \in \gamma_C(G^-)$ . Hence in view of [5], Lemma 1.3,  $\gamma_C$  is a nucleus. □

Let  $C$  be as in Lemma 4.1. Denote

$$(5) \quad \gamma_C = \gamma, \quad \mathbf{L} = (\mathbf{G}^-)^*, \quad P = \ell(\mathbf{L}_\gamma), \quad a = e.$$



**Lemma 4.2.** Assume that (1) is valid. Let  $x, y \in P$  and  $i \in I$ . Then

$$x \circ_\gamma y = (x_i \circ_\gamma y_i) \circ_\gamma (x'_i \circ_\gamma y'_i).$$

*Proof.* We use the relation (4). In view of Lemma 3.2,

$$x = x_i \wedge x'_i, \quad y = y_i \wedge y'_i.$$

Also,  $x_i \vee x'_i = e$ . From this and from Lemma 2.10 in [5] we obtain  $x = x_i \circ_\gamma x'_i$ . Similarly,  $y = y_i \circ_\gamma y'_i$ . Thus

$$x \circ_\gamma y = (x_i \circ_\gamma x'_i) \circ_\gamma (y_i \circ_\gamma y'_i) = x_i \circ_\gamma (x'_i \circ_\gamma y_i) \circ_\gamma y'_i.$$

Using Lemma 3.2 again we get

$$x'_i \circ_\gamma y_i = x'_i \wedge y_i = y_i \wedge x'_i = y_i \circ_\gamma x'_i,$$

whence

$$x \circ_\gamma y = (x_i \circ_\gamma y_i) \circ_\gamma (x'_i \circ_\gamma y'_i).$$

□

**Lemma 4.3.** Assume that (1) is valid. Let  $i \in I$  and  $x, y \in P_i$ . Then  $x \circ_\gamma y \in P_i$ .

*Proof.* Put  $x \circ_\gamma y = z$ . In view of (1),

$$z = z_i \wedge z'_i = z_i \circ_\gamma z'_i.$$

From the relations  $x \in P_i, z'_i \in P'_i$  we get

$$x \vee z'_i = e.$$

Hence

$$(x \vee z_i) \circ_\gamma y = (x \circ_\gamma y) \vee (z'_i \circ_\gamma y) = e \circ_\gamma y = y.$$

Further,  $z'_i \circ_\gamma y = z'_i \wedge y$ , thus

$$z \vee (z'_i \wedge y) = y.$$

By the distributivity of  $\ell$ -groups, we have

$$z \vee (z'_i \wedge y) = (z \vee z'_i) \wedge (z \vee y) = z'_i \wedge y.$$

Therefore  $z'_i \geq y$ . Since  $z'_i \vee y = e$  we get  $z'_i = e$ . This yields  $z = z_i$ , whence  $z \in P_i$ . □

**Lemma 4.4.** *Let (1) be valid. We use the notation as in (5). Let  $a \in P$ ,  $i \in I$ ,  $a_1 \in P_i$ ,  $a_2 \in P'_i$  and  $a = a_1 \wedge a_2$ . Then  $a_1 = a_i$  and  $a_2 = a'_i$ .*

*Proof.* We have

$$a_1 = a_1 \vee a = a_1 \vee (a_i \wedge a'_i) = (a_1 \vee a_i) \wedge (a_1 \vee a'_i).$$

Since  $a_1 \vee a'_i = e$  we get  $a_1 = a_1 \vee a_i$ , whence  $a_1 \geq a_i$ . By an analogous argument we obtain  $a_i \geq a_1$ , thus  $a_i = a_1$ . Similarly,  $a_2 = a'_i$ .  $\square$

**Lemma 4.5.** *Let (1) and (5) be valid. Let  $i \in I$  and  $x, y \in P$ . Then*

$$(x \circ_\gamma y)_i = x_i \circ_\gamma y_i, \quad (x \circ_\gamma y)'_i = x'_i \circ_\gamma y'_i.$$

*Proof.* In view of Lemma 4.3 we have  $x_i \circ_\gamma y_i \in P_i$ . Analogously,  $x'_i \circ_\gamma y'_i \in P'_i$ . Now it suffices to apply Lemma 4.2 and Lemma 4.4.  $\square$

Again, let us suppose that (1) and (5) are valid.

Let  $y, z \in P$ . Put  $z/y = t$ . In view of the definition of residuated lattice we have

$$x \circ_\gamma y \leq z \Leftrightarrow x \leq t.$$

This yields

$$t = \max P(y, z),$$

where

$$P(y, z) = \{x \in P: x \circ_\gamma y \leq z\}.$$

According to Lemma 4.5 the relation  $x \circ_\gamma y \leq z$  is equivalent with the validity of the two following conditions:

$$(6a) \quad x_i \circ_\gamma y_i \leq z_i$$

$$(6b) \quad x'_i \circ_\gamma y'_i \leq z'_i.$$

**Lemma 4.6.** *Let (1) and (5) be valid. Let  $i \in I$  and  $y, z \in P_i$ . Then  $z/y \in P_i$ .*

*Proof.* Let us apply the notation as above. From  $y, z \in P_i$  we obtain  $y_i = y$  and  $z_i = z$ ; thus in view of (6a)

$$x_i \circ_\gamma y \leq z$$

for each  $x \in P(y, z)$ . Since  $t \in P(y, z)$ , we have  $t_i \circ_\gamma y \leq z$ , hence  $t_i \in P(y, z)$ . Clearly  $t_i \geq t$ . Therefore we must have  $t_i = t$ . This yields  $t \in P_i$ .  $\square$

Analogously, we have (by applying (6b))

**Lemma 4.6.1.** *Let (1) and (5) be valid. Let  $i \in I$  and  $y, z \in P'_i$ . Then  $z/y = P'_i$ .*

**Lemma 4.7.** *Let (1) and (5) be valid. Let  $i \in I$  and  $y, z \in P$ . Then  $(z/y)_i = z_i/y_i$ .*

*Proof.* As above, let  $z/y = t$ . Further, put  $z_i/y_i = q$ . In view of Lemma 4.6,  $q \in P_i$ .

Since  $t \in P(y, z)$ , (6b) yields

$$t_i \circ_\gamma y_i \leq z_i,$$

hence  $t_i \in P(y_i, z_i)$ . Since  $q = \max P(y_i, z_i)$ , we obtain  $q \geq t_i$ .

Denote  $q \wedge t'_i = q_1$ . According to Lemma 4.4,

$$(q_1)_i = q, \quad (q_1)'_i = t'_i.$$

From this and from (6a), (6b) we conclude that  $q_1$  belongs to  $P(y, z)$ . Therefore  $q_1 \leq t$ . Hence  $(q_1)_i \leq t_i$ . Thus  $q = t_i$ . This completes the proof.  $\square$

Similarly, we have

**Lemma 4.7.1.** *Under the assumptions as in Lemma 4.7,  $(z/y)'_i = z'_i/y'_i$ .*

The results analogous to Lemma 4.7 and Lemma 4.7.1 are valid for the operation  $y \setminus z$ .

Summarizing, from the previous lemmas of the present section we obtain

**Proposition 4.8.** *Let  $\mathbf{G}$  be a lattice ordered group and let us use the notation as in (5). Suppose that*

$$\varphi: P \rightarrow P_i \times P'_i$$

*is an internal direct product decomposition of the lattice  $P$  with respect to the element  $e$ .*

- (i) *Both  $P_i$  and  $P'_i$  are closed with respect to the operations  $\wedge, \vee_\gamma, \circ_\gamma, \setminus$  and  $/$ ; also,  $e \in P_i \cap P'_i$ . Thus the algebras  $\mathbf{P}_i = (P_i, \vee, \wedge_\gamma, \circ_\gamma, \setminus, /, e)$  and  $\mathbf{P}'_i = (P'_i, \wedge, \vee_\gamma, \circ_\gamma, \setminus, /, e)$  are subalgebras of the GMV-algebra  $\mathbf{L}_\gamma$ .*
- (ii) *The mapping  $\varphi$  determines a direct sum decomposition*

$$\varphi: \mathbf{L}_\gamma = \mathbf{P}_i \oplus \mathbf{P}'_i.$$

It is obvious that if  $\mathbf{L}_\gamma$  is represented as a direct sum

$$\mathbf{L}_\gamma = \mathbf{X} \oplus \mathbf{Y}$$

and if for  $z \in L_\gamma$  we have  $z = x \cdot y$  with  $x \in X$  and  $y \in Y$ , then the mapping  $\varphi(z) = (x, y)$  determines an internal direct product of the corresponding lattices

$$\varphi: \ell(\mathbf{L}_\gamma) \rightarrow \ell(\mathbf{X}) \times \ell(\mathbf{Y}).$$

From this and from Proposition 2.10 we obtain the following.

**Corollary 4.9.** *Let us use the notation as in Proposition 4.8. Let  $F(P)$  be the set of all internal direct factors of  $P$  taken with respect to the element  $e$ . Further, let  $S(\mathbf{L}_\gamma)$  be the set of all direct summands of  $\mathbf{L}_\gamma$ . For each  $\mathbf{A} \in S(\mathbf{L}_\gamma)$  put  $\psi(\mathbf{A}) = \ell(\mathbf{A})$ . Then  $\psi$  is a one-to-one mapping of the set  $S(\mathbf{L}_\gamma)$  onto the set  $F(P)$ .*

From the mentioned relations between elements of  $F(P)$  and  $S(\mathbf{L}_\gamma)$  and from the well-known properties of internal direct factors of partially ordered sets we immediately obtain the following facts:

**4.10.1.** Let  $\mathbf{P}_i \in S(\mathbf{L}_\gamma)$  and  $x \in L_\gamma$ . Then the component  $x_L$  of  $x$  in  $\mathbf{P}_i$  is uniquely determined; namely

$$x_i = \min\{t \in P_i : t \geq x\}.$$

**4.10.2.** Let  $\mathbf{P}_i \in S(\mathbf{L}_\gamma)$ . Then the corresponding  $\mathbf{P}'_i$  (under the notation as above) is uniquely determined; namely,

$$P'_i = \{t \in L_\gamma : t \vee p = e \text{ for each } p \in P_i\}.$$

**4.10.3.** The system  $S(\mathbf{L}_\gamma)$  partially ordered by the set-theoretical inclusion is a Boolean algebra. If  $\mathbf{X}, \mathbf{Y} \in S(\mathbf{L}_\gamma)$ , then the underlying set of  $\mathbf{X} \wedge \mathbf{Y}$  is  $X \cap Y$ . Under the notation as above,  $\mathbf{P}'_i$  is the complement of  $\mathbf{P}_i$  in the Boolean algebra  $S(\mathbf{L}_\gamma)$ .

## 5. ON THE *GMV*-ALGEBRA $\mathbf{G}^*$

Assume that  $\mathbf{M}$  is a *GMV*-algebra and that

$$\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma,$$

where  $\mathbf{G}^*$  and  $\mathbf{L}_\gamma$  are as in Theorem 2.1.

In this section we investigate the *GMV*-algebra  $\mathbf{G}^*$ . Put  $\ell(\mathbf{G}^*) = Q$ . We have  $\ell(\mathbf{G}^*) = \ell(\mathbf{G})$ .

Suppose that

$$\psi: Q \rightarrow \prod_{j \in J} Q_j$$

is an internal direct product decomposition of the lattice  $Q$  with respect to the element  $e$ .

Let  $j \in J, x \in Q$ . We denote by  $x_j$  or  $x(Q_j)$  the component of  $x$  in  $Q_j$ . Further, let  $Q'_j$  be defined analogously as  $P'_i$  in Section 3. Then we have an internal direct product decomposition

$$\psi^j: Q \rightarrow Q_j \times Q'_j$$

where  $\psi^j(x) = (x_j, x'_j)$  for each  $x \in Q$ . Then in view of Proposition 4.8 (applied for  $\psi^j$ ) we conclude that  $Q_j$  is the underlying sublattice of an  $\ell$ -subgroup  $\mathbf{Q}_j$  of  $\mathbf{G}$  (the meaning of  $\mathbf{Q}'_j$  is analogous) and that  $\psi^j$  yields also a direct product decomposition of the lattice ordered group  $\mathbf{G}$ , i.e.,

$$(7) \quad \psi^j: \mathbf{G} \rightarrow \mathbf{Q}_j \times \mathbf{Q}'_j$$

is a direct product decomposition of the lattice ordered group  $\mathbf{G}$ .

Let us consider the *GMV*-algebras  $\mathbf{G}^*$ ,  $\mathbf{Q}_j^*$  and  $(\mathbf{Q}'_j)^*$ . For  $x, y \in G$  we have

$$x/y = xy^{-1}, \quad y \setminus x = y^{-1}x.$$

Thus in view of (7) we obtain that  $Q_j$  and  $Q'_j$  are closed with respect to the operations  $/$  and  $\setminus$ ; therefore

$$\begin{aligned} (x/y)_j &= x_j y_j^{-1} = x_j / y_j^{-1}, \\ (x/y)'_j &= x'_j (y^{-1})'_j = x'_j / (y^{-1})'_j \end{aligned}$$

and analogously for the operation  $\setminus$ . Hence

$$\mathbf{G}^* = \mathbf{Q}_j^* \oplus \mathbf{Q}'_j^*.$$

We verified that if  $Q_j$  is an internal direct factor of the lattice  $Q$  with respect to the element  $e$ , then  $\mathbf{Q}_j^*$  is a direct summand of the *GMV*-algebra  $\mathbf{G}^*$ . Clearly,  $\ell(\mathbf{Q}_j^*) = Q_j$ .

Conversely, it is obvious that if  $\mathbf{X}$  is a direct summand of the *GMV*-algebra  $\mathbf{G}^*$ , then the lattice  $\ell(\mathbf{X})$  is a direct summand of the lattice  $\ell(\mathbf{G}^*)$  with respect to the element  $e$ .

We denote by  $F(Q)$  the system of all internal direct factors of the lattice  $Q$  taken with respect to the element  $e$ . Further, let  $S(\mathbf{G}^*)$  be the system of all direct summands of the *GMV*-algebra  $\mathbf{G}^*$ . In view of the above argument we have proved

**Lemma 5.1.** For each  $\mathbf{X} \in S(\mathbf{G}^*)$  let  $\chi(\mathbf{X}) = \ell(\mathbf{X})$ . Then  $\chi$  is a one-to-one mapping of the set  $S(\mathbf{G}^*)$  onto the set  $F(\ell(\mathbf{G}^*))$ .

Now, let us assume that  $\mathbf{M}$  is any *GMV*-algebra. Further, suppose that  $I$  is a nonempty set and that for each  $i \in I$ ,  $\mathbf{M}_i$  is a direct summand of  $M$ . If  $x \in M$ , then, as above,  $x(\mathbf{M}_i)$  will denote the component of  $x$  in  $\mathbf{M}_i$ . Consider the mapping

$$\alpha: M \rightarrow \prod_{i \in I} M_i$$

defined by  $\alpha(x) = (x(\mathbf{M}_i))_{i \in I}$  for each  $x \in M$ . If  $\alpha$  is bijective then we say that  $\mathbf{M}$  is a *complete direct sum* of the system  $(\mathbf{M}_i)_{i \in I}$  and we express this fact by writing

$$\mathbf{M} = \sum_{i \in I}^* \mathbf{M}_i.$$

**Proposition 5.2.** Let  $\mathbf{G}$  be a lattice ordered group. Assume that

$$\varphi: \ell(\mathbf{G}) \rightarrow \prod_{i \in I} P_i$$

is an internal direct product decomposition of the lattice  $\ell(\mathbf{G})$ . Let  $\chi$  be as in Lemma 5.1; for each  $i \in I$  put  $\mathbf{T}_i = \chi^{-1}(P_i)$ . Then

$$\mathbf{G}^* = \sum_{i \in I}^* \mathbf{T}_i.$$

*Proof.* Since  $\ell(\mathbf{G}) = \ell(\mathbf{G}^*)$ , the assertion follows from Lemma 5.1. □

Analogously, from Corollary 4.9 we obtain

**Proposition 5.3.** Let  $\mathbf{L}_\gamma$  be as in Corollary 4.9. Assume that

$$\varphi: \ell(\mathbf{L}_\gamma) \rightarrow \prod_{i \in I} P_i$$

is an internal direct product decomposition of the lattice  $\ell(\mathbf{L}_\gamma)$  with respect to the element  $e$ . Put  $\psi^{-1}(P_i) = \mathbf{Q}_i$  for each  $i \in I$ . Then

$$\mathbf{L}_\gamma = \sum_{i \in I}^* \mathbf{Q}_i.$$

## 6. DIRECT SUMMANDS OF $\mathbf{M}$

Again, let  $\mathbf{M}$  be a *GMV*-algebra and let  $S(\mathbf{M})$  be the system of all direct summands of  $\mathbf{M}$ . Further, we denote by  $F(\ell(\mathbf{M}))$  the system of all internal direct factors of the lattice  $\ell(\mathbf{M})$  with respect to the element  $e$ .

If  $\mathbf{X} \in S(\mathbf{M})$ , then, obviously, the lattice  $\ell(\mathbf{X})$  belongs to  $F(\ell(\mathbf{M}))$ .

Conversely, assume that  $X$  is an element of  $F(\ell(\mathbf{M}))$ . Then there exists  $Y \in F(\ell(\mathbf{M}))$  and an internal direct product decomposition

$$\varphi_0: \ell(\mathbf{M}) \rightarrow X \times Y.$$

At the same time, in view of Theorem 2.1, we have an internal direct product decomposition with respect to the element  $e$

$$\varphi_1: \ell(M) \rightarrow \ell(\mathbf{G}^*) \times \ell(\mathbf{L}_\gamma).$$

It is well-known that any two internal direct product decompositions of a lattice (taken with respect to the same element) have a common refinement; hence from  $\varphi_0$  and  $\varphi_1$  we can construct a new internal direct product decomposition

$$\varphi_2: \ell(\mathbf{M}) \rightarrow (X \cap \ell(\mathbf{G}^*)) \times (X \cap \ell(\mathbf{L}_\gamma)) \times (Y \cap \ell(\mathbf{G}^*)) \times (Y \cap \ell(\mathbf{L}_\gamma)).$$

At the same time, we have internal direct product decompositions with respect to the element  $e$

$$\begin{aligned} \varphi_{21}: \ell(\mathbf{G}^*) &\rightarrow (X \cap \ell(\mathbf{G}^*)) \times (Y \cap \ell(\mathbf{G}^*)), \\ \varphi_{22}: \ell(\mathbf{L}_\gamma) &\rightarrow (X \cap \ell(\mathbf{L}_\gamma)) \times (Y \cap \ell(\mathbf{L}_\gamma)), \\ \varphi_{23}: X &\rightarrow (X \cap \ell(\mathbf{G}^*)) \times (X \cap \ell(\mathbf{L}_\gamma)), \\ \varphi_{24}: Y &\rightarrow (X \cap \ell(\mathbf{G}^*)) \times (Y \cap \ell(\mathbf{L}_\gamma)). \end{aligned}$$

In view of  $\varphi_{21}$  and of Proposition 5.2 we conclude that there are *GMV*-algebras  $\mathbf{M}_1$  and  $\mathbf{M}_2$  such that

$$\ell(\mathbf{M}_1) = X \cap \ell(\mathbf{G}^*), \quad \ell(\mathbf{M}_2) = Y \cap \ell(\mathbf{G}^*)$$

and

$$\mathbf{G}^* = \mathbf{M}_1 \oplus \mathbf{M}_2.$$

Analogously, according to the relation  $\varphi_{22}$  and Proposition 5.3, there are *GMV*-algebras  $\mathbf{M}_3$  and  $\mathbf{M}_4$  such that

$$\ell(\mathbf{M}_3) = X \cap \ell(\mathbf{L}_\gamma), \quad \ell(\mathbf{M}_4) = Y \cap \ell(\mathbf{L}_\gamma)$$

and

$$\mathbf{L}_\gamma = \mathbf{M}_3 \oplus \mathbf{M}_4.$$

Therefore, in view of Theorem 2.1, we have

$$\mathbf{M} = (M_1 \oplus M_2) \oplus (\mathbf{M}_3 \oplus \mathbf{M}_4).$$

It is obvious that the operation  $\oplus$  is associative and commutative; hence

$$\mathbf{M} = (\mathbf{M}_1 \oplus \mathbf{M}_3) \oplus (\mathbf{M}_2 \oplus \mathbf{M}_4).$$

Thus  $\mathbf{M}_1 \oplus \mathbf{M}_3$  is a direct summand of  $\mathbf{M}$ . Further, in view of  $\varphi_{23}$  we conclude that

$$\ell(M_1 \oplus M_3) = X.$$

Summarizing, we have

**Theorem 6.1.** *Let  $\mathbf{M}$  be a GMV-algebra. For each  $\mathbf{M}_1 \in S(\mathbf{M})$  put  $\varphi(\mathbf{M}_1) = \ell(\mathbf{M}_1)$ . Then  $\varphi$  is a bijection of  $S(\mathbf{M})$  onto  $\mathbf{F}(\ell(\mathbf{M}))$ .*

**Theorem 6.2.** *Let  $\mathbf{M}$  be a GMV-algebra. Assume that*

$$\varphi_1: \ell(\mathbf{M}) \rightarrow \prod_{i \in I} P_i$$

*is an internal direct product decomposition of the lattice  $\ell(\mathbf{M})$  with respect to the element  $e$ . Put  $\varphi^{-1}(P_i) = \mathbf{Q}_i$  for each  $i \in I$ , where  $\varphi$  is as in Theorem 6.1. Then*

$$(8) \quad \mathbf{M} = \sum_{i \in I}^* \mathbf{Q}_i.$$

**P r o o f.** In view of Theorem 6.1, each  $\mathbf{Q}_i$  is a direct summand of  $\mathbf{M}$ . Moreover, for  $x \in M$ , the component of  $x$  in  $\mathbf{Q}_i$  coincides with the component of  $x$  in  $P_i$ . Hence the mapping  $x \mapsto x(P_i)$  is a homomorphism of  $\mathbf{M}$  into  $\mathbf{Q}_i$ . From this and from the direct product decomposition  $\varphi_1$  we infer that (8) holds.  $\square$

Since any two internal direct product decompositions of a lattice have a common refinement, we obtain



**Corollary 6.3.** *Any two complete direct sum decompositions of a GMV-algebra have a common refinement. Namely, if (8) is valid, and, at the same time,*

$$(9) \quad \mathbf{M} = \sum_{j \in J}^* \mathbf{T}_j,$$

then

$$\mathbf{M} = \sum_{i \in I, j \in J}^* \mathbf{V}_{i,j},$$

where  $V_{i,j} = Q_i \cap T_j$  for each  $i \in I$  and  $j \in J$ , and  $\mathbf{V}_{i,j}$  is a subalgebra of  $\mathbf{Q}_i$  and of  $\mathbf{T}_j$ .

A GMV-algebra is *directly irreducible* if, whenever  $\mathbf{M} = \mathbf{M}_1 \oplus \mathbf{M}_2$ , then either  $M_1$  or  $M_2$  is a one-element set. In the opposite case,  $\mathbf{M}$  is *directly irreducible*.

For monoids, we define the notation of direct sum, direct summand, direct irreducibility and direct reducibility in the same way as for GMV-algebras.

Let  $\mathbf{M} = (M; \wedge, \vee, \cdot, \backslash, /, e)$  be an GMV-algebra; we consider the monoid  $\text{mon } \mathbf{M} = (M; \cdot, e)$ .

If  $\mathbf{M} = \mathbf{M}_1 \oplus \mathbf{M}_2$ , then we obviously have

$$\text{mon } \mathbf{M} = \text{mon } \mathbf{M}_1 \oplus \text{mon } \mathbf{M}_2.$$

The natural question arises whether the situation here is analogous to the situation when we consider direct summands of  $\mathbf{M}$  and internal direct factors of  $\ell(\mathbf{M})$ ; i.e., we ask whether there exists a one-to-one correspondence between direct summands of  $\mathbf{M}$  and direct summands of  $\text{mon } \mathbf{M}$ .

The answer is ‘No’. Moreover, it can happen that  $\mathbf{M}$  is directly irreducible and  $\text{mon } \mathbf{M}$  is directly reducible.

**Example.** Let  $R$  be the additive group of all reals with the natural linear order and  $\mathbf{G} = R \circ R$ , where  $\circ$  denotes the operation of lexicographic product. Put  $\mathbf{M} = \mathbf{G}^*$ . Since  $\ell(G^*)$  is a chain, it is directly irreducible and hence  $\mathbf{M}$  is directly irreducible as well. On the other hand, the monoid  $\text{mon } \mathbf{M}$  is directly reducible.

## 7. RETRACT MAPPINGS OF $GMV$ -ALGEBRAS

A *retract mapping* of an algebra  $\mathcal{A}$  is an endomorphism of  $\mathcal{A}$  such that  $f^2 = f$ .

Let  $\mathbf{M}$  be a  $GMV$ -algebra; we apply the notation as in Theorem 2.1. Thus  $\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma$ .

Let  $z \in M$ . As above, we denote by  $z(\mathbf{G}^*)$  the component of  $z$  in the direct summand  $\mathbf{G}^*$ . The meaning of  $z(\mathbf{L}_\gamma)$  is analogous. If  $x = z(\mathbf{G}^*)$  and  $y = z(\mathbf{L}_\gamma)$ , then  $z = xy$ . Conversely, if  $z = x_1y_1$  and  $x_1 \in G$ ,  $y_1 \in L_\gamma$ , then  $x_1 = z(\mathbf{G}^*)$  and  $y_1 = z(\mathbf{L}_\gamma)$ .

In the present section we prove that each retract mapping  $f$  of  $\mathbf{M}$  is determined by a pair  $(f_1, f_2,)$  of mappings such that

- (i<sub>0</sub>)  $f_1$  is a retract mapping of  $\mathbf{G}^*$ ;
- (ii<sub>0</sub>)  $f_2$  is a retract mapping of  $\mathbf{L}_\gamma$ .

We denote by  $\mathcal{R}(\mathbf{M})$  the set of all retract mappings of  $\mathbf{M}$ . Further, let  $\mathcal{T}(\mathbf{M})$  be the system of all pairs of mappings  $(f_1, f_2,)$  such that the conditions (i<sub>0</sub>) and (ii<sub>0</sub>) are valid.

Our aim is to construct a bijection

$$\psi: \mathcal{R}(\mathbf{M}) \rightarrow \mathcal{T}(\mathbf{M}).$$

**Lemma 7.1.** *Let  $z \in M$ . The following conditions are equivalent:*

- (i)  $z \in G$ ;
- (ii) there exists  $z_1 \in M$  such that  $zz_1 = e$ ;
- (iii) there exists  $z_2 \in M$  such that  $z_2z = e$ .

*Proof.* In view of Theorem 2.1,  $(G; \cdot, e)$  is a group with neutral element  $e$ . Thus (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

Assume that (ii) holds. We express  $z$  in the form  $z = xy$ , where  $x = z(\mathbf{G}^*)$  and  $y = z(\mathbf{L}_\gamma)$ . Under analogous notation, let  $z_1 = x_1y_1$ . By way of contradiction, suppose that  $z$  does not belong to  $G$ . Hence  $y \neq e$ . Thus  $e > y$  and  $y \geq yy_1$ . We obtain  $yy_1 \neq e$ , whence  $zz_1$  does not belong to  $G$ , which is a contradiction. Therefore (ii) $\Rightarrow$ (i). Analogously, (iii) $\Rightarrow$ (i). □

**Corollary 7.2.** *Let  $z, z_1 \in M$ ,  $zz_1 = e$ . Then both  $z$  and  $z_1$  belong to  $G$ .*

**Lemma 7.3.** *Let  $f$  be a retract mapping of  $\mathbf{M}$ . Let  $z \in G$ . Then  $f(z) \in G$ .*

*Proof.* In view of Lemma 7.1, there exists  $z_1 \in M$  with  $zz_1 = e$ . We have  $f(e) = e$  and  $f(z_1) = f(z)f(z_1)$ , hence  $f(z)f(z_1) = e$ . Then Corollary 7.2 yields that  $f(z)$  belongs to  $G$ . □

Under the notation as in Lemma 7.3, we put  $f|G = f_1$ . In view of Lemma 7.3, we get

**Lemma 7.4.**  $f_1$  is a retract mapping of  $\mathbf{G}^*$ .

Let  $y \in L_\gamma$ . We denote

$$x = f(y)(\mathbf{G}^*), \quad y_1 = f(y)(\mathbf{L}_\gamma).$$

Further, we put

$$f_2(y) = y_1, \quad f_3(y) = x.$$

We obtain mappings

$$f_2: L_\gamma \rightarrow L_\gamma, \quad f_3: L_\gamma \rightarrow G.$$

**Lemma 7.5.**  $f_2$  is a retract mapping of  $\mathbf{L}_\gamma$ .

**Proof.** It is obvious that  $f_2$  is an endomorphism of  $\mathbf{L}_\gamma$ . It remains to verify that  $f_2(f_2(y)) = f_2(y)$  for each  $y \in L_\gamma$ .

Under the notation as above, we have  $f_2(y) = y_1$  and  $f(y) = xy_1$ . Denote

$$x_1 = f(y_1)(\mathbf{G}^*), \quad y_2 = f(y_1)(\mathbf{L}_\gamma).$$

Thus, in view of the definition of  $f_2$ , we get  $f_2(y_1) = y_2$ . Further, we obtain

$$\begin{aligned} f(f(y)) &= f(y) = xy_1, \\ f(f(y)) &= f(xy_1) = f(x)f(y_1) = f(x)x_1y_2. \end{aligned}$$

Since  $x \in G$ , in view of 7.3 we have  $f(x)x_1 \in G$ . Thus from

$$xy_1 = f(x)x_1y_2$$

we obtain  $x = f(x)x_1$  and  $y_1 = y_2$ . We have verified that  $f_2(y_1) = y_1$ . □

Under the above notation we put

$$\psi(f) = (f_1, f_2).$$

From 7.4 and 7.5 we obtain

**Theorem (A).**  $\psi$  is a mapping of the set  $\mathcal{R}(\mathbf{M})$  into the set  $\mathcal{T}(\mathbf{M})$ .

**Lemma 7.6.**  $f_3 = e$  for each  $y \in L_\gamma$ .

*P r o o f.* Let  $y \in L_\gamma$ . If we view  $\mathbf{M}$  as a direct product, we have

$$\begin{aligned} e = e_{\mathbf{M}} &= f(e_{\mathbf{M}}) = f(e_{\mathbf{M}}/y) = f(e_{\mathbf{M}})/f(y) \\ &= (e_{\mathbf{G}}, e_{L_\gamma}/(x, y_1)) = (e_{\mathbf{G}}/x, e_{L_\gamma}) = (x^{-1}, e_{L_\gamma}), \end{aligned}$$

so  $x^{-1} = e_{\mathbf{G}}$  and  $x = e_{\mathbf{G}}$ . Therefore  $f_3(y) = e_{\mathbf{G}}$ . □

**Corollary 7.7.**  $f(y) = f_2(y)$  for each  $y \in L_\gamma$ .

In view of Corollary 7.7 we conclude that the mapping  $\psi$  is a monomorphism.

Now let us suppose that  $f_1$  and  $f_2$  are as in conditions (i<sub>0</sub>) and (ii<sub>0</sub>). Further, let  $z \in M$ ,  $z = xy$ , where  $x \in G$  and  $y \in L_\gamma$ . We put

$$f_0(z) = f_1(x)f_2(y).$$

Then in view of Theorem 2.1 we obtain

**Lemma 7.8.**  $f_0$  is a retract mapping of  $\mathbf{M}$ .

For a pair  $(f_1, f_2)$  belonging to the system  $\mathcal{T}(\mathbf{M})$  we put

$$\chi((f_1, f_2)) = f_0,$$

where  $f_0$  is as above.

According to Lemma 7.8 we get

**Theorem (B).**  $\chi$  is a mapping of the system  $\mathcal{T}(\mathbf{M})$  into the set  $\mathcal{R}(\mathbf{M})$ .

We have already noticed above that the mapping  $\psi$  is a monomorphism. Now from the definitions of  $\psi$  and  $\chi$  we immediately obtain that  $\chi = \psi^{-1}$ . Hence we have

**Theorem (C).** Let  $\mathbf{M}$  be a GMV-algebra. The mapping  $\psi$  is a bijection of the set  $\mathcal{R}(\mathbf{M})$  onto the system  $\mathcal{T}(\mathbf{M})$ .

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