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INTERIOR AND CLOSURE OPERATORS ON BOUNDED
RESIDUATED LATTICE ORDERED MONOIDS

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Abstract. *GMV*-algebras endowed with additive closure operators or with its duals-multiplicative interior operators (closure or interior *GMV*-algebras) were introduced as a non-commutative generalization of topological Boolean algebras. In the paper, the multiplicative interior and additive closure operators on *DRL*-monoids are introduced as natural generalizations of the multiplicative interior and additive closure operators on *GMV*-algebras.

Keywords: *GMV*-algebra, *DRL*-monoid, filter

MSC 2000: 06D35, 06F05, 03G25

1. INTRODUCTION

In 1965, the commutative dually residuated lattice-ordered semigroups (*DRL*-semigroups) were introduced by K. L. N. Swamy [28] as a common generalization of abelian *l*-groups and Brouwerian algebras. Bounded commutative *DRL*-monoids are in a close connection with algebras of fuzzy logic. For example, each *BL*-algebra (or more precisely the dual to each *BL*-algebra) and each *MV*-algebra can be considered as a special case of a bounded commutative *DRL*-semigroup.

The non-commutative extension of *DRL*-semigroups was introduced first by K. Swamy and later in 1996 T. Kovář dealt with them in his Ph.D. Thesis [15].

Definition 1.1. An algebra $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ is called a *bounded residuated lattice ordered monoid (bounded RL-monoid)* iff for each $x, y, z \in M$

- (i) $(M; \odot, 1)$ is a monoid,
- (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

- (iii) $x \odot y \leq z \iff x \leq y \rightarrow z \iff y \leq x \rightsquigarrow z$,
- (iv) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$.

Bounded *Rl*-monoids are in fact the so called bounded integral generalized *BL*-algebras—a special class of residuated lattices studied in [1] and [2]. One can show that the lattice $(M; \vee, \wedge)$ is distributive and also that the binary operation \odot distributes over \vee and \wedge —see [9].

Bounded *Rl*-monoids form a variety of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$. For example, every *GMV*-algebra (pseudo *MV*-algebra) and every pseudo *BL*-algebra are special cases of *Rl*-monoids. In the sequel, an *Rl-monoid* will mean a bounded *Rl*-monoid.

The aim of the paper is to generalize the results of the paper [26] (where one works with additive closure and multiplicative interior operators on *GMV*-algebras) to the wider class of algebras, to the class of *Rl*-monoids. In the second section of the paper we will introduce multiplicative interior and additive closure operators on *Rl*-monoids as natural generalization of the same operators on *GMV*-algebras and we will describe their mutual relation. In the final third section we will study operators of interior on algebras derived from *Rl*-monoids, for example derived by factorization by their filters.

In the next lemma we will show some of the basic properties of *Rl*-monoids.

Lemma 1.1 ([15], [23]). *Let $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be an *Rl-monoid*. Then the following assertions hold for every $x, y, z \in M$:*

- (i) $x \odot y \leq x \wedge y \leq x, y$;
- (ii) if $x \leq y$ then $x \odot z \leq y \odot z$ and $z \odot x \leq z \odot y$;
- (iii) if $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$;
- (iv) if $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$;
- (v) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$;
- (vi) $x \rightarrow x = x \rightsquigarrow x = 1$;
- (vii) $1 \rightarrow x = 1 \rightsquigarrow x = x$;
- (viii) $y \leq x \rightarrow y$ and $y \leq x \rightsquigarrow y$;
- (ix) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
- (x) if $x \leq y$ then $y \rightarrow 0 \leq x \rightarrow 0$ and $y \rightsquigarrow 0 \leq x \rightsquigarrow 0$.

2. MULTIPLICATIVE INTERIOR AND ADDITIVE CLOSURE
OPERATORS ON *Rl*-MONOIDS

Definition 2.1. Let $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be an *Rl*-monoid and $f: M \rightarrow M$ a mapping. Then f is called a *multiplicative interior operator* (or *mi-operator*) on \mathcal{M} iff for each $x, y \in M$

1. $f(x \odot y) = f(x) \odot f(y)$,
2. $f(x) \leq x$,
3. $f(f(x)) = f(x)$,
4. $f(1) = 1$.

Lemma 2.1. *Each mi-operator on an *Rl*-monoid \mathcal{M} is isotone.*

Proof. Let us consider an *mi-operator* f on \mathcal{M} and $x, y \in M$ such that $x \leq y$. Then

$$f(x) = f(y \wedge x) = f((y \rightarrow x) \odot y) = f(y \rightarrow x) \odot f(y).$$

Since $f(y) \odot f(y \rightarrow x) \leq f(y)$, we have also $f(x) \leq f(y)$. That means, f is isotone on \mathcal{M} . □

Hence (by 2 and 3 from Definition 2.1) f is an interior operator on the lattice $(M; \vee, \wedge)$ of the *Rl*-monoid \mathcal{M} .

Lemma 2.2. *For an mi-operator f on an *Rl*-monoid \mathcal{M} and for each $x, y \in M$,*

$$\begin{aligned} f(x \rightarrow y) &\leq f(x) \rightarrow f(y), \\ f(x \rightsquigarrow y) &\leq f(x) \rightsquigarrow f(y). \end{aligned}$$

Proof. Let $x, y \in M$. Then

$$(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y \leq y$$

and by Lemma 2.1

$$f(x \rightarrow y) \odot f(x) = f(x) \odot f(x \rightsquigarrow y) \leq f(y).$$

So, by Definition 1.1, the inequalities we are proving, hold on \mathcal{M} . □

On an arbitrary *Rl*-monoid \mathcal{M} we define two unary operations, *negations* $^-: M \rightarrow M$ and $\rightsquigarrow: M \rightarrow M$ by

$$\begin{aligned} x^- &:= x \rightarrow 0, \\ x^\rightsquigarrow &:= x \rightsquigarrow 0 \end{aligned}$$

for each element $x \in M$.

We can characterize *GMV*-algebras by means of the negations, because by [22], the class of *GMV*-algebras is a subvariety of the variety of *Rl*-monoids determined by the identities $x^{-\sim} = x = x^{\sim-}$ and $(x^- \odot y^-)^{\sim} = (x^{\sim} \odot y^{\sim})^-$.

Let us show now some properties of the two operations of negation.

Lemma 2.3. *In every Rl-monoid \mathcal{M} the following assertions hold for each elements $x, y \in M$:*

- (i) $0^{\sim-} = 0 = 0^{-\sim}$, $1^{\sim-} = 1 = 1^{-\sim}$;
- (ii) $x \leq x^{-\sim}, x^{\sim-}$;
- (iii) $x^- = x^{-\sim-}$, $x^{\sim} = x^{\sim-\sim}$;
- (iv) $x \leq y \implies x^- \geq y^-, x^{\sim} \geq y^{\sim}$.

Proof. See [23]. □

Let us consider a mapping $f: M \rightarrow M$ and two new mappings

$$f_{\sim}^-: M \rightarrow M, \quad f_{\sim}^-: M \rightarrow M$$

such that for each $x \in M$

$$f_{\sim}^-(x) := (f(x^-))^{\sim}$$

and

$$f_{\sim}^-(x) := (f(x^{\sim}))^-.$$

Proposition 2.4. *If f is an *mi*-operator on an *Rl*-monoid \mathcal{M} then both the mappings f_{\sim}^-, f_{\sim}^- are isotone.*

Proof. Let us consider elements $x, y \in M$ such that $x \leq y$. Then $y^- \leq x^-$ (see Lemma 3.3(iv)), so $f(y^-) \leq f(x^-)$. Therefore $(f(x^-))^{\sim} \leq (f(y^-))^{\sim}$, or equivalently $f_{\sim}^-(x) \leq f_{\sim}^-(y)$. Analogously for f_{\sim}^- . □

Proposition 2.5. *If f is an *mi*-operator on an *Rl*-monoid \mathcal{M} then for each element $x \in M$ we have*

- 2'. $x \leq f_{\sim}^-(x)$,
- 3'. $f_{\sim}^-(f_{\sim}^-(x)) = f_{\sim}^-(x)$,
- 4'. $f_{\sim}^-(0) = 0$.

Proof. Let us consider an arbitrary element $x \in M$.

2': $f_{\sim}^-(x) = (f(x^-))^{\sim} \geq x^{-\sim} \geq x$.

3': By 2' we have

$$f_{\sim}^-(x) \leq f_{\sim}^-(f_{\sim}^-(x)).$$

Further we know that

$$f(x^-) \leq (f(x^-))^{\sim-}$$

and so

$$f(x^-) = f(f(x^-)) \leq f((f(x^-))^{\sim-}).$$

Therefore

$$f^{\sim}(f^{\sim}(x)) = f((f(x^-))^{\sim-})^{\sim} \leq f(x^-)^{\sim} = f^{\sim}(x).$$

$$4': f^{\sim}(0) = (f(0^-))^{\sim} = (f(1))^{\sim} = 1^{\sim} = 0. \quad \square$$

Remark 2.6.

- a) Of course, relations 2'-4' from the preceding proposition are satisfied also for the operator $f^{\sim-}$.
- b) By Propositions 2.4 and 2.5 and Remark 2.6 a), $f^{\sim-}$ and f^{\sim} are closure operators on the lattice $(M; \vee, \wedge)$.
- c) By the proof of part 2' of Proposition 2.5 the stronger inequality $x^{\sim-} \leq f^{\sim}(x)$ is satisfied in \mathcal{M} .

Definition 2.2. An *Rl*-monoid \mathcal{M} is said to be *good* if and only if

$$(G) \quad x^{\sim-} = x^{\sim-}$$

holds for each element $x \in M$.

Remark 2.7. The identity (G) holds for example in every *GMV*-algebra. On the other hand, the situation is not so clear for the case of pseudo *BL*-algebras. It was proved [5], [6] that every linearly ordered pseudo *BL*-algebra, hence every representable pseudo *BL*-algebra is good. Anyway, the general problem is still open-see [12], Open problem 3.21.

Moreover, we can show (see [23]) that every good *BL*-algebra and every Heyting algebra satisfy the identity

$$(N1) \quad (x \odot y)^{\sim-} = x^{\sim-} \odot y^{\sim-}$$

or the equivalent form

$$(N2) \quad (x \odot y)^{\sim-} = x^{\sim-} \odot y^{\sim-}.$$

Therefore, it is clear that the class of *Rl*-monoids satisfying (N1) and (N2) is really wide, which leads us to the following definition.

Definition 2.3. An *Rl-monoid* \mathcal{M} is said to be *normal* if and only if it satisfies both the identities (N1) and (N2).

We can define a new binary operation “ \oplus ” on every *Rl-monoid* $\mathcal{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$. For arbitrary elements $x, y \in M$ we put

$$x \oplus y := (x^- \odot y^-)^\sim.$$

Then this new binary operation has the following properties.

Lemma 2.8. *If \mathcal{M} is a good Rl-monoid and $x, y, z \in M$, then*

- (a) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- (b) $x \oplus y = (x^\sim \odot y^\sim)^-$,
- (c) $x, y \leq x \vee y \leq x \oplus y$,
- (d) $x \oplus 0 = x^{-\sim} = 0 \oplus x$,
- (e) $x \oplus 1 = 1 = 1 \oplus x$.

Proof. See [8]. □

Lemma 2.9. *The following equalities hold in every good normal Rl-monoid \mathcal{M} :*

- (i) $(x \oplus y)^- = x^- \odot y^-$;
- (ii) $(x \oplus y)^\sim = x^\sim \odot y^\sim$;
- (iii) $(x \odot y)^- = x^- \oplus y^-$;
- (iv) $(x \odot y)^\sim = x^\sim \oplus y^\sim$.

Proof. Let us choose arbitrary $x, y \in M$. Then we have

- (i): $(x \oplus y)^- = (x^- \odot y^-)^{\sim-} = x^{-\sim-} \odot y^{-\sim-} = x^- \odot y^-$ (we have used (N2) and Lemma 2.3 (iii));
- (ii): $(x \oplus y)^\sim = (x^\sim \odot y^\sim)^{-\sim} = x^{\sim-\sim} \odot y^{\sim-\sim} = x^\sim \odot y^\sim$ (we have used Lemma 2.8(b), (N1) and Lemma 3.3(iii));
- (iii): $(x \odot y)^- = (x \odot y)^{-\sim-} = (x^{-\sim} \odot y^{-\sim})^- = x^- \oplus y^-$ (we have used Lemma 2.3(iii), (N1) and Lemma 3.8(b));
- (iv): $(x \odot y)^\sim = (x \odot y)^{\sim-\sim} = (x^{\sim-} \odot y^{\sim-})^\sim = x^\sim \oplus y^\sim$ (we have used Lemma 2.3(iii) and (N2)). □

Definition 2.4. If \mathcal{M} is an *Rl-monoid* and $g: M \rightarrow M$ a mapping then g is called an *additive closure operator* (*ac-operator*) on \mathcal{M} iff for each $x, y \in M$

- 1'. $g(x \oplus y) = g(x) \oplus g(y)$,
- 2'. $x \leq g(x)$,
- 3'. $g(g(x)) = g(x)$,
- 4'. $g(0) = 0$.

Theorem 2.10. *If \mathcal{M} is a good normal Rl-monoid and f is an mi-operator on \mathcal{M} then the mappings f_{\sim}^{-} and f_{\sim}^{-} are ac-operators on \mathcal{M} , which are moreover isotone.*

Proof. Thanks to Proposition 2.6 it is enough to check the identity 1' from the definition of an ac-operator. Let us do it for f_{\sim}^{-} , for f_{\sim}^{-} it is analogous. Let $x, y \in M$. Then by Lemma 2.9(i)

$$f_{\sim}^{-}(x \oplus y) = (f((x \oplus y)^{-}))^{\sim} = (f(x^{-} \odot y^{-}))^{\sim} = (f(x^{-}) \odot f(y^{-}))^{\sim}.$$

By Lemma 2.9(iv) we further get

$$(f(x^{-}) \odot f(y^{-}))^{\sim} = (f(x^{-}))^{\sim} \oplus (f(y^{-}))^{\sim} = f_{\sim}^{-}(x) \oplus f_{\sim}^{-}(y).$$

The isotony of the operators f_{\sim}^{-} and f_{\sim}^{-} is a direct consequence of the isotony of f and Lemma 2.3(iv). \square

Now, let us consider the converse situation. We choose an ac-operator g on an Rl-monoid \mathcal{M} and we will study properties of the mappings g_{\sim}^{-} and g_{\sim}^{-} .

Lemma 2.11. *Every ac-operator g on a good Rl-monoid \mathcal{M} satisfies the equality*

$$g(x^{-\sim}) = (g(x))^{-\sim}.$$

Proof. We have

$$g(x^{-\sim}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{-\sim}.$$

\square

Theorem 2.12. *Let g be an ac-operator on a good normal Rl-monoid \mathcal{M} . Then the mappings g_{\sim}^{-} and g_{\sim}^{-} satisfy identities 1, 3, 4 from Definition 2.1. Moreover, if g is isotone then both g_{\sim}^{-} and g_{\sim}^{-} are also isotone.*

Proof. Let us choose arbitrary elements $x, y \in M$. Then for example for g_{\sim}^{-} we have

$$1: g_{\sim}^{-}(x \odot y) = (g((x \odot y)^{-}))^{\sim} = (g(x^{-} \oplus y^{-}))^{\sim} = (g(x^{-}) \oplus g(y^{-}))^{\sim} = (g(x^{-}))^{\sim} \odot (g(y^{-}))^{\sim} = g_{\sim}^{-}(x) \odot g_{\sim}^{-}(y);$$

$$3: g_{\sim}^{-}(g_{\sim}^{-}(x)) = (g((g(x^{-}))^{\sim-}))^{\sim} = (g(g(x^{-\sim-}))^{\sim})^{\sim} = (g(g(x^{-}))^{\sim})^{\sim} = (g(x^{-}))^{\sim} = g_{\sim}^{-}(x) \text{ (we have used Lemma 3.10);}$$

$$4: g_{\sim}^{-}(1) = (g(1^{-}))^{\sim} = (g(0))^{\sim} = 0^{\sim} = 1.$$

The second part is obvious. \square

Remark 2.13. Axiom 2. from Definition 2.1 need not be satisfied by g_{\sim}^{-} (or g_{\sim}^{-}) in general. Only the following weaker inequality holds for arbitrary $x \in M$:

$$g_{\sim}^{-}(x) = (g(x^{-}))^{\sim} \leq x^{-\sim}.$$

Theorem 2.14. *Let us consider a good normal Rl -monoid \mathcal{M} and an operator h on \mathcal{M} which satisfies identities 1, 3, 4 from Definition 2.1 and the inequality $h(x) \leq x^{-\sim}$ for arbitrary $x \in M$. Then the mappings h_{\sim}^- and h_{\sim}^- are ac -operators on \mathcal{M} .*

Proof. We must check axioms 1'–4' from Definition 2.4 for our mappings h_{\sim}^- and h_{\sim}^- and the Rl -monoid \mathcal{M} . So, for an arbitrary element $x \in M$ we have

$$2': h_{\sim}^-(x) = (h(x^-))^{\sim} \geq ((x^-)^{\sim})^{\sim} = x^{-\sim} \geq x.$$

For the other three identities 1', 3' and 4' we have now the same situation as in Proposition 2.5 and Theorem 2.10. \square

3. OPERATORS ON ALGEBRAS DERIVED FROM Rl -MONOIDS

Let us have an Rl -monoid \mathcal{M} and its mi -operator f . In this chapter, the algebra $(\mathcal{M}, f) = (M; \odot, \vee, \wedge, \rightarrow, 0, 1, f)$ will be called an *interior Rl -monoid* (analogously to the GMV -algebras in [26]).

Definition 3.1. If \mathcal{M} is an Rl -monoid then a non-empty subset F of M is called a *filter* in \mathcal{M} iff

- (F1) $x, y \in F \implies x \odot y \in F$,
- (F2) $x \in F, y \in M, x \leq y \implies y \in F$.

A filter F is called *normal* iff

- (F3) $x \rightarrow y \in F \iff x \rightsquigarrow y \in M$ for each $x, y \in M$.

It is known (see [2] or [18]) that normal filters of Rl -monoids coincide with kernels of their congruences. If F is a normal filter of an Rl -monoid \mathcal{M} then F is the kernel of the unique congruence $\Theta(F)$ such that

$$\langle x, y \rangle \in \Theta(F) \iff (x \rightarrow y), (y \rightarrow x) \in F$$

for each $x, y \in M$. Therefore, for each Rl -monoid M we can consider the quotient Rl -monoid \mathcal{M}/F by its filter F .

Definition 3.2. Let F be a filter in an interior Rl -monoid (\mathcal{M}, f) . Then F is called an *i -filter* (or *interior filter*) iff

- (F4) $x \in F \implies f(x) \in F$.

Theorem 3.1. Let (\mathcal{M}, f) be an interior *Rl*-monoid and let F be its normal *i*-filter. Further, let us consider the mapping $\tilde{f}: \mathcal{M}/F \rightarrow \mathcal{M}/F$ such that for each $x \in M$,

$$\tilde{f}(x/F) := f(x)/F.$$

Then the *Rl*-monoid \mathcal{M}/F endowed with \tilde{f} is an interior *Rl*-monoid.

Proof. Let us consider $x, y \in M$ such that $x/F = y/F$. So we have $\langle x, y \rangle \in \Theta(F)$ or equivalently $(x \rightarrow y), (y \rightarrow x) \in F$ and further $f(x \rightarrow y), f(y \rightarrow x) \in F$ with regard to (F4). According to Lemma 2.2,

$$f(x \rightarrow y) \leq f(x) \rightarrow f(y), \quad f(y \rightarrow x) \leq f(y) \rightarrow f(x),$$

therefore also $f(x) \rightarrow f(y), f(y) \rightarrow f(x) \in F$ and $\langle f(x), f(y) \rangle \in \Theta(F)$. This means that the unary operation \tilde{f} is correctly defined on \mathcal{M}/F . We have to check conditions 1–4 from the definition of the *mi*-operator on the *Rl*-monoid for \tilde{f} and the proof will be done. Let x, y be arbitrary elements from M .

- 1: $\tilde{f}(x/F) \odot \tilde{f}(y/F) = f(x)/F \odot f(y)/F = (f(x) \odot f(y))/F = f(x \odot y)/F = \tilde{f}((x \odot y)/F) = \tilde{f}((x/F) \odot (y/F));$
- 2: $\tilde{f}(x/F) = f(x)/F \leq x/F;$
- 3: $\tilde{f}(\tilde{f}(x/F)) = \tilde{f}(f(x)/F) = f(f(x))/F = f(x)/F = \tilde{f}(x/F);$
- 4: $\tilde{f}(1/F) = f(1)/F = 1/F.$ □

Corollary 3.2. There is a one-to-one correspondence between the normal *i*-filters and the congruences of the interior *Rl*-monoids.

We will denote by $D(\mathcal{M}) = \{x \in M: x^{-\sim} = 1\}$ the set of all *dense* elements of a good *Rl*-monoid \mathcal{M} .

Proposition 3.3. For every good *Rl*-monoid \mathcal{M} the set $D(\mathcal{M})$ is a normal filter in \mathcal{M} .

Proof. See [23], Theorem 10. □

Similarly to the commutative case, we can show (see [23], Theorems 9, 10) that for a good *Rl*-monoid \mathcal{M} the quotient *Rl*-monoid $M/D(\mathcal{M})$ is a *GMV*-algebra. By Theorem 3.1, Proposition 3.3 and [26] we have

Theorem 3.4. Let us consider an interior Rl -monoid (\mathcal{M}, f) . Further, consider a mapping $\tilde{f}: \mathcal{M}/D(\mathcal{M}) \rightarrow \mathcal{M}/D(\mathcal{M})$ such that for each element $x \in M$,

$$\tilde{f}(x/D(\mathcal{M})) := f(x)/D(\mathcal{M}).$$

Then \tilde{f} is an mi -operator on the GMV -algebra $\mathcal{M}/D(\mathcal{M})$.

Let us consider an Rl -monoid \mathcal{M} and the set $R(\mathcal{M}) = \{x \in M : x^{-\sim} = x = x^{-\sim}\}$. It is known (see [8]) that if an Rl -monoid \mathcal{M} is good then $\mathcal{R}(\mathcal{M}) = (R(\mathcal{M}); \oplus_R, -_R, \sim_R, 0, 1)$, where “ \oplus_R ” is introduced on $\mathcal{R}(\mathcal{M})$ in the same way as on the whole \mathcal{M} , and “ $-_R$ ”, “ \sim_R ” are restrictions of unary operations “ $-$ ”, “ \sim ” of negations from \mathcal{M} to $\mathcal{R}(\mathcal{M})$, is a GMV -algebra.

Theorem 3.5. Let us introduce a mapping $\hat{f}: R(\mathcal{M}) \rightarrow R(\mathcal{M})$ on a good normal interior Rl -monoid $(\mathcal{M}; f)$ by

$$\hat{f}(x) := (f(x))^{-\sim}$$

for each $x \in R(\mathcal{M})$. Then \hat{f} is an mi -operator on the GMV -algebra $\mathcal{R}(\mathcal{M})$.

Proof. Since

$$\hat{f}(x)^{-\sim} = ((f(x))^{-\sim})^{-\sim} = (f(x))^{-\sim} \text{ for each } x \in R(\mathcal{M}),$$

it is clear that \hat{f} is a self-mapping of $\mathcal{R}(\mathcal{M})$. Let us check the conditions from the definition of an mi -operator on a GMV -algebra (see [26]) for \hat{f} and $\mathcal{R}(\mathcal{M})$. For arbitrary $x, y \in R(\mathcal{M})$ we have

- 1: $\hat{f}(x \odot y) = (f(x \odot y))^{-\sim} = (f(x) \odot f(y))^{-\sim} = (f(x))^{-\sim} \odot (f(y))^{-\sim} = \hat{f}(x) \odot \hat{f}(y)$;
- 2: $\hat{f}(x) = (f(x))^{-\sim} \leq x^{-\sim} = x$;
- 3: $\hat{f}(\hat{f}(x)) = \hat{f}(f(x)^{-\sim}) = (f((f(x))^{-\sim}))^{-\sim} \geq (f(f(x)))^{-\sim} = (f(x))^{-\sim} = \hat{f}(x)$.

Conversely, $(f(x))^{-\sim} = \hat{f}(x) \leq x$, so $(f((f(x))^{-\sim}))^{-\sim} \leq (f(x))^{-\sim}$ or equivalently $\hat{f}(\hat{f}(x)) \leq \hat{f}(x)$;

- 4: $\hat{f}(1) = (f(1))^{-\sim} = 1^{-\sim} = 1$. □

It was proved that for every good normal Rl -monoid \mathcal{M} the GMV -algebras $\mathcal{R}(\mathcal{M})$ and $\mathcal{M}/D(\mathcal{M})$ are isomorphic ([23], Th. 10), where the mappings $\varphi: \mathcal{R}(\mathcal{M}) \rightarrow \mathcal{M}/D(\mathcal{M})$ and $\psi: \mathcal{M}/D(\mathcal{M}) \rightarrow \mathcal{R}(\mathcal{M})$ such that

$$\begin{aligned} \varphi(x) &:= x/D(\mathcal{M}), \\ \psi(y/D(\mathcal{M})) &:= y^{-\sim} \end{aligned}$$

are mutually inverse isomorphisms.

Let \mathcal{M} be an *Rl*-monoid. Let us denote by $I(\mathcal{M}) = \{a \in M : a \odot a = a\}$ the set of all *idempotent elements* in M and by $B(\mathcal{M})$ the set of elements from \mathcal{M} which have a complement in the lattice (M, \vee, \wedge) . It is known that if \mathcal{M} is a *GMV*-algebra then $I(\mathcal{M}) = B(\mathcal{M})$ —see [10], Prop. 4.2.

Theorem 3.6. *For each Rl-monoid \mathcal{M} we have $B(I(\mathcal{M})) = B(\mathcal{M})$.*

Proof. Clearly $0, 1 \in I(\mathcal{M})$, so 0 is the least and 1 the greatest element in the lattice $(I(\mathcal{M}); \vee, \wedge)$. Let $x \in B(I(\mathcal{M}))$. Then there exists an element $y \in I(\mathcal{M})$ such that $x \vee y = 1$ and $x \wedge y = 0$ in the lattice $(I(\mathcal{M}); \vee, \wedge)$. Since $I(\mathcal{M}) \subseteq M$ and since the operations “ \vee ”, “ \wedge ” in $(I(\mathcal{M}); \vee, \wedge)$ are restrictions of the “same” operations in $(M; \vee, \wedge)$, y is also a complement of x in the lattice $(M; \vee, \wedge)$. So $x \in B(\mathcal{M})$.

The converse inclusion is proved in [17], Lemma 15. □

Lemma 3.7. *If \mathcal{M} is an Rl-monoid, $a \in I(\mathcal{M})$ and $x \in \mathcal{M}$ then $a \wedge x = a \odot y$.*

Proof. See [17], Lemma 6. □

Lemma 3.8. *If \mathcal{M} is a normal good Rl-monoid and $a \in I(\mathcal{M})$ then $a^{-\sim} \in I(\mathcal{M})$ and $a^{\sim} \oplus a^{\sim} = a^{\sim}$.*

Proof. For an arbitrary element $a \in I(\mathcal{M})$ we have

$$a^{-\sim} \odot a^{-\sim} = (a \odot a)^{-\sim} = a^{-\sim}$$

thanks to normality of \mathcal{M} . Moreover,

$$a^{\sim} \oplus a^{\sim} = (a^{\sim-} \odot a^{\sim-})^{\sim} = (a \odot a)^{\sim-} = a^{\sim}.$$

□

Theorem 3.9. *If \mathcal{M} is an Rl-monoid then $I(\mathcal{M})$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$ of the Rl-monoid \mathcal{M} .*



Proof. Closedness of $I(\mathcal{M})$ with respect to the operation “ \wedge ” follows from Lemma 3.7. It is enough to check that $I(\mathcal{M})$ is closed with respect to the operation “ \vee ”. Let $a, b \in I(\mathcal{M})$. Since “ \odot ” is distributive over the lattice operations join and meet, we conclude

$$(a \vee b) \odot (a \vee b) = (a \odot a) \vee (a \odot b) \vee (b \odot a) \vee (b \odot b) = a \vee b \vee (a \odot b) \vee (b \odot a) = a \vee b.$$

□

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