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ON RAINBOWNESS OF SEMIREGULAR POLYHEDRA

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Dedicated to Professor Miroslav Fiedler on the occasion of his 80th birthday.

Abstract. We introduce the *rainbowness* of a polyhedron as the minimum number k such that any colouring of vertices of the polyhedron using at least k colours involves a face all vertices of which have different colours. We determine the rainbowness of Platonic solids, prisms, antiprisms and ten Archimedean solids. For the remaining three Archimedean solids this parameter is estimated.

Keywords: rainbowness, Platonic solids, prisms, antiprisms, Archimedean solids

MSC 2000: 52B05, 05C15

1. Introduction

A *polyhedron* P in E^3 is a locally finite collection of planar convex polygons, called the *faces*, such that every edge of every polygon is an edge of precisely one other polygon (see [2], [3], [4], [5], [6], [8]). The edges and the vertices of P are defined in the usual way (see [2], [3], [4], [5]).

A polyhedron P is called *semiregular* (see [2], [4], [5], [8]) if all of its faces are regular polygons and there exists a sequence $\sigma = (p_1, p_2, \ldots, p_q)$, called the *cyclic sequence* of P, such that every vertex of P is surrounded by a p_1-gon, a p_2-gon, ..., a p_q-gon, in this order within rotation and reflection. It is a well-known result (see [2], [4], [5]) that if a polyhedron P is the boundary cell complex of a convex polytope (see [5] for definition) and P is semiregular having the cyclic sequence $\sigma = (p_1, \ldots, p_q)$, then $p_i \geqslant 3$, $q \geqslant 3$ and

$$\sum_{i=1}^{q} \frac{1}{p_i} > \frac{q}{2} - 1. \tag{1}$$

A semiregular polyhedron P is called the $(p_1, p_2, \dots, p_q)$-*polyhedron* if it is determined by the cyclic sequence $\sigma = (p_1, p_2, \dots, p_q)$.

The above conditions imply that the set of semiregular polyhedra consists of precisely five Platonic solids, thirteen Archimedean solids (see [3]), a single $(3,4,4,4)$-polyhedron [1] and two infinite families: the prisms, i.e. $(4,4,n)$-polyhedra for every $n \geqslant 3$, $n \neq 4$, and the antiprisms, i.e. $(3,3,3,n)$-polyhedra for every $n \geqslant 4$.

Let P be a polyhedron with the vertex set $V(P)$. Motivated by the paper of Negami [7] on looseness of triangulations we define the *rainbowness* of a polyhedron P, $\mathrm{rb}(P)$, as the minimum number k such that any surjective colour assignment $\varphi\colon V(P) \to \{1, 2, \dots, k\}$ involves a face all vertices of which have different colours.

The main purpose of this paper is to determine the rainbowness of all semiregular polyhedra. Instead of studying convex polyhedra it is enough to study their graphs, i.e. graphs determined by vertices and edges of polyhedra. This is allowed due to a famous theorem by Steinitz (see e.g. [5]) that states that a graph is the graph of a convex polyhedron if and only if it is planar and 3-connected.

We use the standard terminology except for a few notions defined in the sequel. Let φ be a vertex colouring using r colours (or r-colouring) then for any face $\alpha \in F(G)$ the notion $\varphi(\alpha)$ will denote the set of colours used at the vertices of α. If $|\varphi(\alpha)| = \deg(\alpha)$, the size of the face α, then α is called a *rainbow* face. Analogously for any set X of vertices we will denote by $\varphi(X)$ the set of colours used at the vertices of X under a colouring φ. An edge $e = uv$ with $\varphi(u) = \varphi(v)$ will be called a *monochromatic* edge. The set $\{1, 2, \dots, n\}$ will be denoted by $[1, n]$ and the set $\{k, k+1, \dots, m\}$ by $[k, m]$.

The paper is organized as follows: In Chapter 2 we will present our knowledge concerning rainbowness of Platonic solids, Chapter 3 is devoted to rainbowness of prisms and Chapter 4 to antiprisms. In Chapter 5 we consider the Archimedean solids. In short Chapter 6 we investigate the non Archimedean $(3,4,4,4)$-polyhedron.

2. Platonic solids

The set of Platonic solids consists of five members:

(i) the tetrahedron, or equivalently, the $(3,3,3)$-polyhedron,
(ii) the octahedron, or the $(3,3,3,3)$-polyhedron,
(iii) the icosahedron, i.e. the $(3,3,3,3,3)$-polyhedron,
(iv) the cube, i.e. the $(4,4,4)$-polyhedron,
(v) the dodecahedron, i.e. the $(5,5,5)$-polyhedron.

2.1. The tetrahedron.

It is easy to see that the following is true:

Theorem 2.1. *Let T be the tetrahedron. Then*

$$\mathrm{rb}(T) = 3.$$

2.2. The octahedron.

Figure 2.1 shows a 3-colouring of the octahedron O that has no rainbow face. This means that $\mathrm{rb}(O) \geqslant 4$. Next suppose that there exists a 4-colouring of O which has no rainbow face. As O has six vertices, at least one colour, say 1, is used exactly once. Let a vertex x be coloured with 1. Then all neighbours of x must have the same colour. However, for the remaining two colours there is only one vertex, a contradiction. So we have proved

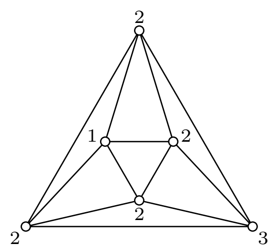

Figure 2.1

Theorem 2.2. *Let O be the octahedron. Then*

$$\mathrm{rb}(O) = 4.$$

2.3. The icosahedron.

Theorem 2.3. *Let I be the icosahedron. Then*

$$\mathrm{rb}(I) = 5.$$

P r o o f. It is easy to find in I two vertex-disjoint wheels as subgraphs. Let the first one have the vertex set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with a central vertex x_6 and spokes $x_i x_6$ for every $1 \leqslant i \leqslant 5$ and rim edges $x_i x_{i+1}$ for every $1 \leqslant i \leqslant 5$; indices here and in the next are taken modulo 5. The second wheel is created by the vertex set $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ with a central vertex y_6, spokes $y_i y_6$, $1 \leqslant i \leqslant 5$, and rim edges $y_i y_{i+1}$, $1 \leqslant i \leqslant 5$. Let the remaining edges of I be $x_i y_i$ for any $1 \leqslant i \leqslant 5$ and $x_i y_{i+1}$, $1 \leqslant i \leqslant 5$.

The following 4-colouring φ that has no rainbow face gives $\mathrm{rb}(I) \geqslant 5$: $\varphi(x_i) = 1$ for any $1 \leqslant i \leqslant 5$, $\varphi(x_6) = 2$, $\varphi(y_i) = 3$ for any $1 \leqslant i \leqslant 5$ and $\varphi(y_6) = 4$.

To prove that $\mathrm{rb}(I) \leqslant 5$ suppose that there is a 5-colouring φ without any rainbow face. Without loss of generality one can suppose that $\varphi(X) = \{1, 2, 3\}$, $\varphi(x_6) = 1$ and $\varphi(x_1) = 1$. Clearly on the rim there is no adjacent pair of colours 2 and 3. One can suppose that $\varphi(x_2) = 2$. Then there are two possibilities for $\varphi(x_3)$, either $\varphi(x_3) = 1$ or $\varphi(x_3) = 2$.

1. Let $\varphi(x_3) = 1$. Then $\varphi(x_4) = 3$ and $\varphi(x_5) \in \{1, 3\}$. Consequently, then $\varphi(y_2) \in \{1, 2\}$, $\varphi(y_3) \in \{1, 2\}$, $\varphi(y_4) \in \{1, 3\}$ and $\{4, 5\} \subseteq \{\varphi(y_1), \varphi(y_5), \varphi(y_6)\}$. Then either the triangle $[y_4 y_5 y_6]$ or the triangle $[y_1 y_2 y_6]$ is a rainbow, a contradiction.

2. Let $\varphi(x_3) = 2$. Then $\varphi(x_4) = 1$, $\varphi(x_5) = 3$, $\varphi(y_1) \in \{1, 3\}$, $\varphi(y_2) \in \{1, 2\}$, $\varphi(y_4) \in \{1, 2\}$, $\varphi(y_5) \in \{1, 3\}$ and $\{\varphi(y_3), \varphi(y_6)\} = \{4, 5\}$.

However, in this case the triangle $[y_2 y_3 y_6]$ is rainbow, a contradiction. □

2.4. The cube.

In Figure 2.2 there is a 5-colouring of the cube Q having no rainbow face. This shows that $\mathrm{rb}(Q) \geqslant 6$. Next suppose that there is a 6-colouring φ of Q having no rainbow face. Observe that on Q there are two vertex-disjoint quadrangles α and β (see Figure 2.2) that cover all eight vertices of Q. Clearly $|\varphi(\alpha)| \leqslant 3$ and $|\varphi(\beta)| \leqslant 3$ but φ is a 6-colouring; so $|\varphi(\alpha)| = 3$, $|\varphi(\beta)| = 3$ and $\varphi(\alpha) \cap \varphi(\beta) = \emptyset$. Two vertices of α that bring α not to be rainbow are either neighbouring or diagonal. In the former case they enforce another face not to be rainbow, but not in the latter case. So the face α can cause that also at most one other neighbouring face of α can be non-rainbow. Analogously the face β can enforce at most one other face not to be rainbow. So at most four faces at Q can be non-rainbow but Q has six faces, and therefore there are at least two rainbow ones, a contradiction. So we have proved

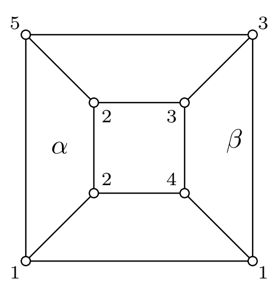

Figure 2.2

Theorem 2.4. *Let Q be the cube. Then*

$$\mathrm{rb}(Q) = 6.$$

2.5. The dodecahedron.

Theorem 2.5. *Let D be a dodecahedron. Then*

$$\mathrm{rb}(D) = 15.$$

P r o o f. The colouring φ in Figure 2.3 gives us the lower bound 15.

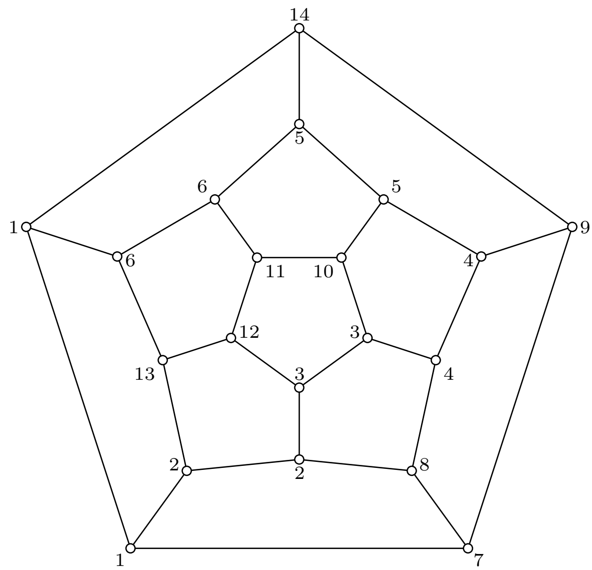

Figure 2.3

To prove the opposite inequality suppose that there is a 15-colouring φ of D that has no rainbow face. Let us fix one vertex for each colour c, $c \in [1, 15]$. The remaining five vertices create the set $X = \{x_1, x_2, x_3, x_4, x_5\}$. Consider the colour set $\varphi(X)$. There are four possibilities for the cardinality of the set $\varphi(X)$.

1. Let $|\varphi(X)| = 5$. Then in D there are five pairs of vertices, each pair of the same colour. One pair can enforce at most two non-rainbow faces. So at most 10 non-rainbow faces are in D in this case, a contradiction because D has twelve faces.

2. Let $|\varphi(X)| = 4$. Now there is one triple of vertices of the same colour and three pairs. Because D is trivalent one triple can enforce at most three non-rainbow faces. So in this case there are at most 9 non-rainbow faces; again we have a contradiction.

3. Let $|\varphi(X)| = 3$. In this case we have in D either two triples and one pair of vertices of the same colour or one quadruple and two pairs of vertices of the same colours. In both cases there are in D at most 9 non-rainbow faces, a contradiction.

4. Let $|\varphi(X)| \leqslant 2$. Analogously as above one can show that there are at most 11 non-rainbow faces in D, a contradiction. □

3. Prisms

The n-sided prism D_n, i.e. the $(4,4,n)$-polyhedron, $n \geqslant 3$, is a generalization of the cube. It consists of the vertex set $V = \{a_1, a_2, \dots, a_n, b_1, \dots, b_n\}$ and the edge set $E = \{\{a_i, a_{i+1}\} \cup \{b_i, b_{i+1}\} \cup \{a_i, b_i\},\ i = 1, \dots, n$, indices modulo $n\}$. The set of faces of D_n consists of two n-gonal faces $\alpha = [a_1 a_2 \dots a_n]$ and $\beta = [b_1 b_2 \dots b_n]$ and n quadrangles $[a_i a_{i+1} b_{i+1} b_i]$ for any $i = 1, 2, \dots, n$, indices modulo n.

Theorem 3.1. *Let D_n be an n-sided prism, $n \geqslant 3$. Then*

$$\mathrm{rb}(D_n) = \begin{cases} \dfrac{3n}{2} & \text{if } n \text{ is even,} \\ \dfrac{3n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

P r o o f. Lower bounds.

1. Let $n = 2k$, $k \geqslant 2$. In this case D_{2k} has $4k$ vertices and $2k+2$ faces. We partition the face set of D_{2k} into $k+1$ pairs of faces so that each pair π_i of faces has an edge e_i in common. The edges $e_1, \dots, e_{k+1}$ create a matching. Now we colour the vertices of D_{2k} as follows. The vertices incident with the edge e_i obtain the colour i. So we have used $k+1$ colours $\{1, 2, \dots, k+1\}$. The remaining $4k - 2k - 2 = 2k - 2$ vertices are coloured with colours $k+2, \dots, 3k-1$. It is easy to see that this colouring with $3k-1$ colours do not force any rainbow face. Hence, in this case, $\mathrm{rb}(D_n) \geqslant 3k$.

2. Let $n = 2k+1$, $k \geqslant 1$. Then the following colouring φ gives the required lower bound: $\varphi(a_1) = \varphi(b_1) = 1$, $\varphi(a_{2i}) = \varphi(b_{2i}) = i$ for $i = 1, \dots, k$, $\varphi(a_{2i+1}) = k + i$ and $\varphi(b_{2i+1}) = 2k + i$ for all $i = 1, 2, \dots, k$. Again $\mathrm{rb}(D_n) \geqslant 3k+1$ because there is no rainbow face under this colouring.

U p p e r b o u n d s.

1. Let $n = 2k$. D_{2k} contains a set S of k mutually disjoint quadrangles that cover all vertices of D_{2k}. Suppose there is a $3k$-colouring φ of D_{2k} such that there is no rainbow quadrangle and no $2n$-gonal rainbow face. This means that each quadrangle from S has at most three different colours. Moreover, the colour sets of different quadrangles from S are disjoint, otherwise we have a contradiction with the number of colours used. Therefore on each quadrangle of S there are exactly three colours. If any quadrangle has two non neighbouring vertices coloured with the same colour, there is exactly one face that these vertices force not to be rainbow. If on a quadrangle there is a monochromatic edge then this edge enforces another neighbouring face not to be rainbow. Let us check what is the maximum number of non-rainbow faces in D_{2k} under the colouring φ. The quadrangular faces from S

can cause that at most k other faces not in S are non-rainbow. Therefore there are in D_{2k} at most $2k$ non-rainbow faces, a contradiction because D_{2k} has $2k+2$ faces together.

2. Let $n = 2k+1$. D_{2k+1} contains $2k+3$ faces and $4k+2$ vertices. Suppose there is a colouring φ of vertices of D_{2k+1} with $\frac{1}{2}(3n-1) = 3k+1$ colours such that there is no rainbow face there. It is easy to see that there is a set S of k quadrangular faces that are mutually vertex disjoint and none of them contains any of the vertices a_{2k+1} and b_{2k+1}, which together with the edge $a_{2k+1}b_{2k+1}$ cover all vertices of the prism D_{2k+1}.

Let $\varphi(a_{2k+1}) = \varphi(b_{2k+1})$. It is easy to see that every quadrangle from S is coloured exactly with three colours and the sets of colours used on two distinct quadrangles of S are disjoint. These quadrangles together with the edge $a_{2k+1}b_{2k+1}$ provide at most $2k+2$ non-rainbow faces and hence at least one face must be rainbow, a contradiction.

If $\varphi(a_{2k+1}) \neq \varphi(b_{2k+1})$ then, analogously as above, we can show that at least $k-1$ quadrangles from S have vertices coloured with exactly three colours and at most one is such that its vertices are coloured with exactly two colours which are not on other quadrangles. The last one together with its neighbours can enforce at most three non-rainbow faces. Hence in the $(2k+1)$-prism D_{2k+1} there are at most $2(k-1)+3 = 2k+1$ non-rainbow faces. Because D_{2k+1} has exactly $2k+3$ faces, at least two of them must be rainbow, a contradiction. □

4. Antiprisms

An n-sided antiprism A_n is the $(3,3,3,n)$-polyhedron defined as follows:

The vertex set is $V(A_n) = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$, the edge set $E(A_n) = \{\{a_i a_{i+1}\} \cup \{b_i b_{i+1}\} \cup \{a_i b_i\} \cup \{a_i b_{i-1}\},\ i = 1, \ldots, n$, indices modulo $n\}$, the face set of A_n consists of two n-gonal faces α and β and $2n$ triangular faces (triangles).

Theorem 4.1. *Let $n \geqslant 4$ be an integer. Then*

$$\mathrm{rb}(A_n) = n.$$

Proof. The following colouring φ does not contain any rainbow face: $\varphi(a_1) = \varphi(b_1) = \varphi(a_n) = \varphi(b_n) = 1$, $\varphi(a_i) = \varphi(b_i) = i$ for every $i = 2, 3, \ldots, n-1$. This proves that $\mathrm{rb}(A_n) \geqslant n$.

To prove the opposite inequality let us show first the following

Claim:

$$\mathrm{rb}(A_4) = 4.$$

P r o o f. From the above we have $\mathrm{rb}(A_4) \geqslant 4$. Suppose there is a 4-colouring φ of the vertices of A_4 such that there is no rainbow face in A_4. Consider two cases:

1. Each colour is used at least twice. As A_4 has 8 vertices so each colour is used exactly twice. Consider a 4-face $\alpha = [a_1a_2a_3a_4]$. Without loss of generality let $\varphi(a_1) = \varphi(a_2) = 1$ then $\varphi(b_4) = \varphi(b_1) = 2$ and $\varphi(b_2) \in \{1, 2\}$, a contradiction. If $\varphi(a_1) = \varphi(a_3) = 1$, then $\varphi(a_2) = 2 = \varphi(b_1)$ and $\varphi(b_2) \notin \{1, 2\}$ and we have a rainbow face $[a_2a_3b_2]$, a contradiction.

2. Let, without loss of generality, the colour 1 be used exactly once, i.e. let $\varphi(b_1) = 1$. Then $\varphi(a_1) = \varphi(a_2) = \varphi(b_2) = \varphi(b_4) = 2$. One of the colours 3 and 4 has to be used exactly once. Suppose (without loss of generality) $\varphi(b_3) = 3$; then all neighbours of b_3 must be of the colour 2, a contradiction because the colour 4 is not used. If $\varphi(a_3)$ or $\varphi(a_4) = 3$ we again obtain a contradiction in an analogous way. □

Next, let $n \geqslant 5$. To prove the opposite inequality suppose that there is such a surjective mapping $\varphi\colon V \to [1, n]$ that does not involve any rainbow face.

1. Let each colour from $[1, n]$ be used at least twice. Because A_n has exactly $2n$ vertices each colour has to be used exactly two times. As α is not a rainbow face there are on α two vertices x and y coloured with the same colour, say, n i.e. $\varphi(x) = \varphi(y) = n$.

Observe that all neighbours of x have to be coloured with colour a or n and all neighbours of y with colour b or n. As $|N[x] \cup N[y]| \geqslant 7$ we immediately have a contradiction. Here $N[z]$ denotes a closed neighbourhood of the vertex z.

2. There is a colour, say n, which is used exactly once. Let this colour be used on a vertex a_n of the face α, i.e. $\varphi(a_n) = n$. Clearly all vertices adjacent to a_n at A_n are coloured with the same colour, say 1, i.e. $\varphi(a_{n-1}) = \varphi(b_{n-1}) = \varphi(b_n) = \varphi(a_1) = 1$.

2.1. Let, without loss of generality, there be at least two colours on β that are used at least twice or one colour that is used three times. Then if we delete vertices a_n and b_n and add edges $b_{n-1}b_1$, $a_{n-1}a_1$ and a_1b_{n-1} we obtain an antiprism A_{n-1} whose vertices are coloured with $n-1$ colours without any rainbow face. Next we can repeat our consideration. In the end we get a contradiction because $\mathrm{rb}(A_4) = 4$.

2.2. Let, without loss of generality, there be exactly one colour on β that is used precisely twice, all the other colours being used at most once at β. Because of symmetry we can suppose that the same is true for the face α. This means that colour 1 is used twice on α and two times on β. If there is another colour, say $n-1$, used exactly once at β and not used at α, then there must be another colour used four times in the colouring φ on A_n and we have a contradiction. So any other colour different from 1 and n is used exactly twice on A_n. So we have coloured at least $2(n-2) + 4 + 1 = 2n + 1$ vertices, a contradiction. □

5. Archimedean solids

A description of Archimedean solids is very nicely elaborated in the book of Cromwell [3]. The set of Archimedean solids consists of thirteen polyhedra:

1. the $(3,6,6)$-polyhedron known as the truncated tetrahedron,
2. the $(3,8,8)$-polyhedron, i.e. the truncated cube,
3. the $(3,10,10)$-polyhedron, i.e. the truncated dodecahedron,
4. the $(4,6,6)$-polyhedron, i.e. the truncated octahedron,
5. the $(4,6,8)$-polyhedron, i.e. the great rhomb-cub-octahedron,
6. the $(4,6,10)$-polyhedron, i.e. the great rhom-icosi-dodecahedron,
7. the $(5,6,6)$-polyhedron, i.e. the truncated icosahedron known also as the buckminster fulleren,
8. the $(3,4,3,4)$-polyhedron, i.e. the cub-octahedron,
9. the $(3,4,4,4)$-polyhedron, i.e. the rhomb-cub-octahedron,
10. the $(3,4,5,4)$-polyhedron, i.e. the rhom-icosi-dodecahedron,
11. the $(3,5,3,5)$-polyhedron, i.e. the icosi-dodecahedron,
12. the $(3,3,3,3,4)$-polyhedron, i.e. the snub cube,
13. the $(3,3,3,3,5)$-polyhedron, i.e. the snub dodecahedron.

In the theorems below we will write in brackets instead of the name of a polyhedron the cyclic sequence that characterizes it.

Theorem 5.1.
(i) $\mathrm{rb}(3,6,6)=9$,
(ii) $\mathrm{rb}(3,8,8)=17$,
(iii) $\mathrm{rb}(3,10,10)=41$.

P r o o f. We prove only the case (iii). The cases (i) and (ii) can be done analogously and we let them to the reader. Recall that the $(3,6,6)$-polyhedron contains 12 vertices and 8 faces. The $(3,8,8)$-polyhedron has 24 vertices and 14 faces.

The $(3,10,10)$-polyhedron P has 60 vertices, 20 triangles and 12 decagons. To establish the lower bound consider the set of face-disjoint pairs of triangle-dodecahedrons having an edge in common. So we obtain twelve edges $e_1,\ldots,e_{12}$. On any of the remaining triangles we choose one edge. We obtain another set $\{e_{13},\ldots,e_{20}\}$ of edges. Notice that these two sets form a matching. Let us colour the endvertices of the edge e_i with colour i for every $i\in[1,20]$. We colour the remaining 20 vertices of P with colours $21,\ldots,40$. Observe that no face under this colouring is rainbow.

To show that any 41-colouring of P enforces a rainbow face consider the set of all triangles. These triangles cover all vertices of P. Because there are 20 triangles in this set and any colouring uses 41 colours so there must be one with three colours, and we are done. □

Theorem 5.2.
(i) $\mathrm{rb}(4,6,6) = 18$,
(ii) $\mathrm{rb}(4,6,8) = 36$,
(iii) $\mathrm{rb}(4,6,10) = 90$.

Proof. The idea of the proof is the same in all three cases. We will prove the case (iii). The remaining two cases are left to the reader. The $(4,6,10)$-polyhedron R consists of 120 vertices, 30 quadrangles, 20 hexagons and 12 decagons. Recall that the $(4,6,6)$-polyhedron has 24 vertices and 14 faces. The $(4,6,8)$-polyhedron has 48 vertices and 26 faces.

A colouring giving the lower bound for the rainbowness is obtained as follows. Partition the faces of R into face-disjoint pairs of faces having an edge in common. As R is cubic and having 62 faces we obtain 31 pairs with common edges $e_1, \dots, e_{31}$. These edges form a matching. Let us colour the end vertices of the edge e_i with the colour i for every $i \in [1, 31]$. The remaining $120 - 62 = 58$ vertices are coloured successively with different colours $32, \dots, 89$. Clearly no face under this colouring is rainbow.

To obtain the upper bound 90 we have to show that every vertex 90-colouring of R enforces a rainbow face. Suppose there is a 90-colouring φ under which no face of R is rainbow. Observe that the set of all quadrangular faces covers all vertices of R. Because no quadrangle is rainbow the set of colours used at any quadrangle consists of at most three colours. As φ uses 90 colours, the colour set of any quadrangle consists of exactly three colours and, moreover, colour sets of distinct quadrangular faces are disjoint. This means that any non-rainbow face which is not a quadrangle must be incident with a monochromatic edge. The number of monochromatic edges is at most 30 but the number of non quadrangular faces is 32 and we have a contradiction. □

Theorem 5.3.

$$\mathrm{rb}(5,6,6) = 45.$$

Proof. **1.** The $(5,6,6)$-polyhedron M has 32 faces, twelve pentagons and twenty hexagons. It has 60 vertices. The faces are paired into 16 mutually face-disjoint pairs in such a way that each pair of faces shares an edge in common. M is a cubic graph, therefore we obtain a set $S = \{e_1, \dots, e_{16}\}$ of sixteen edges that form a matching. The endvertices of the edge e_i are coloured with the colour i for every $i \in [1, 16]$. The remaining $60-32 = 28$ vertices are coloured successively with colours $17, \dots, 44$. The result is a 44-colouring of M that does not contain any rainbow face.

2. Next we show that any 45-colouring of M enforces a rainbow face. Suppose that there is a 45-colouring of M having no rainbow face. The set of all twelve pentagons covers all 60 vertices of M. Each pentagon contains at least two vertices of the same colour. This means that there are at most 12 monochromatic edges whose 24 endvertices are coloured with at most 12 colours. The remaining 36 vertices are coloured with at least 33 colours. Hence there are at most 3 additional monochromatic edges. Each monochromatic edge on a pentagon can enforce at most one non-rainbow hexagon. But there are at most 3 monochromatic edges that can enforce at most six additional non-rainbow hexagons. Altogether we have at most 18 non rainbow hexagons. But M has 20 hexagons, which means that there must be at least one rainbow hexagon, a contradiction. □

Theorem 5.4.

$$\mathrm{rb}(3,4,3,4) = 6.$$

P r o o f. For a lower bound see colouring φ at Figure 5.1 where $\varphi(a_2) = 2$, $\varphi(a_4) = 3$, $\varphi(b_1) = 4$, $\varphi(b_3) = 5$ and $\varphi(x) = 1$ for all other vertices x.

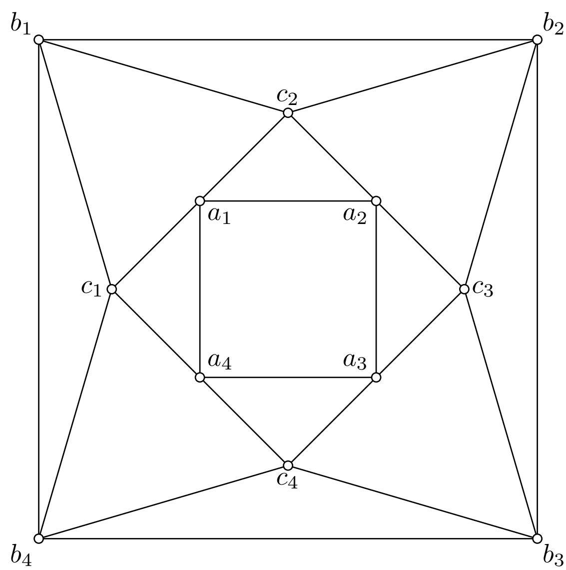

Figure 5.1

To prove the upper bound 6 suppose there is a vertex 6-colouring φ with no rainbow face at $(3,4,3,4)$-polyhedron

1. Let each colour be used at least (and hence exactly) twice. Without loss of generality we can suppose that $\varphi(a_1) = \varphi(a_2) = 1$, $\varphi(b_1) = \varphi(c_2) = 2$. Then $\varphi(c_1) = 3$ and consequently $\varphi(b_4) = \varphi(a_4) = 3$, a contradiction.

2. Let, without loss of generality, the colour 6 be used exactly once. Let $\varphi(c_2) = 6$. Then $\varphi(a_1) = \varphi(a_2) = 1$ and $\varphi(b_1) = \varphi(b_2) = 2$. This implies $\varphi(c_1) \in \{1, 2\}$ and $\varphi(c_3) \in \{1, 2\}$. So we have $\{3, 4, 5\} \subseteq \{\varphi(a_3), \varphi(a_4), \varphi(c_4), \varphi(b_3), \varphi(b_4)\}$ and one of the faces $[c_1b_4c_4a_4]$, $[a_3a_4c_4]$, $[b_3b_4c_4]$ or $[a_3c_4b_3c_3]$, is rainbow, a contradiction. □

Theorem 5.5.

$$\mathrm{rb}(3, 4, 4, 4) = 12.$$

P r o o f. In Figure 5.2 there is a graph of the $(3, 4, 4, 4)$-polyhedron R. Let for the purposes of the proof the vertices of this $(3, 4, 4, 4)$-polyhedron be denoted in accordance with Figure 5.2. The following 11-colouring φ provides no rainbow face of R: $\varphi(u_1) = \varphi(u_2) = \varphi(x_1) = 5$, $\varphi(u_6) = 6$, $\varphi(u_3) = \varphi(u_5) = \varphi(v_3) = \varphi(v_5) = \varphi(w_3) = \varphi(w_5) = \varphi(x_3) = \varphi(x_5) = 1$, $\varphi(u_4) = \varphi(x_4) = 4$, $\varphi(v_1) = \varphi(v_2) = 7$, $\varphi(v_4) = 2$, $\varphi(v_6) = 8$, $\varphi(w_1) = \varphi(w_2) = 9$, $\varphi(w_4) = 3$, $\varphi(w_6) = 10$, $\varphi(x_2) = \varphi(x_6) = 11$. This gives the lower bound 11.

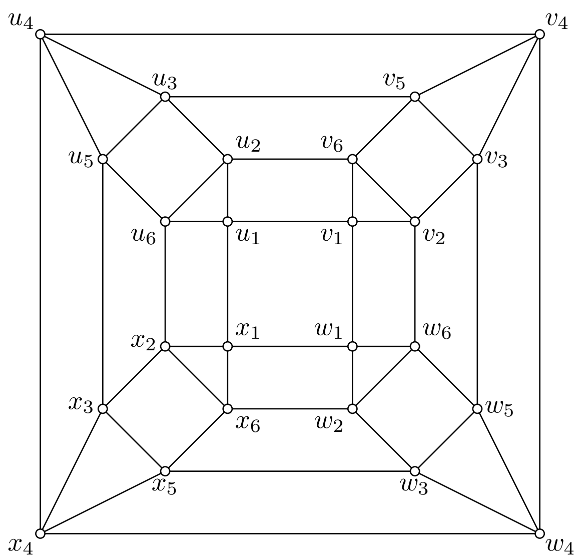

Figure 5.2

Suppose there is a surjective 12-colouring $\varphi\colon V \to [1, 12]$ such that no face of the $(3, 4, 4, 4)$-polyhedron is rainbow.

Let $\mathscr{U} = \{u_1, v_1, w_1, x_1\}$ and $X = \{u_2, u_6, v_2, v_6, w_2, w_6, x_2, x_6\}$ be sets of vertices of the polyhedron R. Denote by K the subgraph of R induced on the vertex set $\mathscr{U} \cup X$ and by K' the subgraph induced on the remaining vertices of R. Evidently K and K' are isomorphic. Let $\varphi(K)$ denote the set of colours used at the vertices of K. We say that an edge is monochromatic in X if the subgraph of R induced by

X contains a monochromatic edge. It is easy to see that $5 \leqslant |\varphi(K)| \leqslant 7$. First, we prove four lemmas. □

Lemma 1. *If* $|\varphi(K)| = 7$ *then* X *contains at most one monochromatic edge.*

P r o o f. Let us first show that $|\varphi(X)| \geqslant 6$. Suppose that $|\varphi(X)| \leqslant 5$. Then there are two colours, say 6 and 7, that are not in $\varphi(X)$. Then, without loss of generality, let $\varphi(u_1) = 6$ and $7 \in \{\varphi(v_1), \varphi(w_1)\}$.

1. Let $\varphi(v_1) = 7$. Then $\varphi(u_2) = \varphi(u_6) = \varphi(v_2) = \varphi(v_6) = 1$. The remaining four colours 2, 3, 4 and 5 must be used at two triangles $[w_1w_2w_6]$ and $[x_1x_2x_6]$, exactly two on each. Since $\varphi(\{w_1, w_2, w_3\}) \cap \varphi(\{x_1, x_2, x_3\}) = \emptyset$, the quadrangle $[u_1v_1w_1x_1]$ is rainbow, a contradiction.

2. Let $\varphi(w_1) = 7$. Then $\varphi(u_2) = \varphi(u_6) = 1$ and $\varphi(w_2) = \varphi(w_6)$.

2.1. If $\varphi(w_2) = 1$ then, without loss of generality, $\varphi(\{v_1, v_2, v_6\}) = \{2, 3\}$ and $\varphi(\{x_1, x_2, x_6\}) = \{4, 5\}$ and again $[u_1v_1w_1x_1]$ is rainbow, a contradiction.

2.2. If $\varphi(w_2) \neq 1$ then, without loss of generality, $\varphi(w_2) = 2$ and $\varphi(\{v_1, v_2, v_6\}) = \{3, 4\}$. But then either $[u_1v_1v_6u_2]$ or $[v_1v_2w_6w_1]$ is rainbow, a contradiction.

Suppose that there are at least two monochromatic edges in X. Because now $|\varphi(X)| = 6$ there is exactly one colour, say 7, which is not in X. Let $\varphi(u_1) = 7$. Then $\varphi(u_2) = \varphi(u_6) = 1$. As the remaining five colours must be used as well we have, without loss of generality, $\varphi(v_1) = \varphi(v_6) = 2$, $\varphi(v_2) = 3$ and $\varphi(\{x_1, x_2, x_6\}) = \{4, 5\}$ or $\varphi(\{w_1, w_2, w_6\}) = \{4, 5\}$. In the first case $\varphi(x_1) = \varphi(x_2) = 4$, $\varphi(x_6) = 5$ and $\varphi(w_1) \in \{2, 4, 7\}$. Then, because of the second monochromatic edge in X, $\varphi(w_2) = \varphi(w_6) = 6$, a contradiction because one of the faces $[v_1v_2w_6w_1]$, $[w_1w_2x_6x_1]$ is rainbow.

In the second case $\varphi(w_1) = \varphi(w_6) = 4$, $\varphi(w_2) = 5$ and $\varphi(x_1) \in \{2, 4, 7\}$, $6 \in \{\varphi(x_2), \varphi(x_6)\}$. If $\varphi(x_2) = 6$ then $\varphi(x_1) = 7$ and, consequently, $\varphi(x_6) \in \{5, 6\}$, a contradiction because either $[x_1x_2x_6]$ or $[x_1w_1w_2x_6]$ is rainbow. □

Lemma 2. *If* $|\varphi(K)| = 6$ *then* X *contains at most three monochromatic edges.*

P r o o f. First we show that $|\varphi(X)| \geqslant 4$. Clearly $|\varphi(X)| \geqslant 3$, otherwise the face $[u_1v_1w_1x_1]$ is rainbow. If $|\varphi(X)| = 3$ then $|\varphi(\mathscr{U})| = 3$ and $\varphi(X) \cap \varphi(\mathscr{U}) = \emptyset$.

Let $\varphi(X) = \{1, 2, 3\}$ and $\varphi(\mathscr{U}) = \{4, 5, 6\}$. Let, without loss of generality, $\varphi(u_1) = 4$, $\varphi(v_1) = 5$, $\varphi(w_1) = 6$. Then $\varphi(u_2) = \varphi(u_6) = \varphi(v_2) = \varphi(v_6) = \varphi(w_2) = \varphi(w_6) = 1$, $\varphi(x_2) = 2$, $\varphi(x_6) = 3$ and one of the faces $[x_1x_2x_6]$, $[u_1v_1w_1x_1]$ is rainbow.

If $|\varphi(X)| > 4$ then there is nothing to prove. So let $\varphi(X) = [1, 4]$ and there are at least four monochromatic edges in X. Then there are two colours, say 5 and 6 that are not present in $\varphi(X)$. Suppose $\varphi(u_1) = 5$. Then, without loss of generality, $6 \in \{\varphi(v_1), \varphi(w_1)\}$.

1. If $\varphi(v_1) = 6$ then, without loss of generality, $\varphi(u_2) = \varphi(u_6) = \varphi(v_2) = \varphi(v_6) = 1$, $\varphi(w_1) = \varphi(w_6) = 2$, $\varphi(w_2) = 3$ and consequently $\varphi(x_1) \in \{2, 5, 6\}$, $4 \in \{\varphi(x_2), \varphi(x_6)\}$.

1.1. If $\varphi(x_2) = 4$ then $\varphi(x_1) = 5$, $\varphi(x_6) = 4$ and the face $[w_1w_2x_6x_1]$ is rainbow.

1.2. If $\varphi(x_6) = 4$ then $\varphi(x_1) = 2$, $\varphi(x_2) = 4$ because of the fourth monochromatic edge on X. Again a contradiction because of the face $[u_1u_6x_2x_1]$ is rainbow.

2. If $\varphi(w_1) = 6$ then, without loss of generality, $\varphi(u_2) = \varphi(u_6) = 1$ and $\varphi(w_2) = \varphi(w_6)$.

2.1. If $\varphi(w_2) = 1 = \varphi(w_6)$ then, without loss of generality, $\{2, 3\} \subseteq \{\varphi(v_1), \varphi(v_2), \varphi(v_6)\}$. If $\varphi(v_1) \neq \varphi(v_2)$ then $[v_1, v_2, w_1, w_6]$ is rainbow. If $\varphi(v_1) \neq \varphi(v_6)$ then $[u_1u_2v_1v_6]$ is rainbow.

2.2. If $\varphi(w_2) \neq 1$, say $\varphi(w_2) = 2 = \varphi(w_6)$ then, without loss of generality, $\varphi(v_6) = 3$ and $\varphi(v_1) \in \{1, 3, 5\}$. If $\varphi(v_2) = 4$ then $[v_1v_2w_6w_1]$ is rainbow, a contradiction. If $\varphi(v_2) \neq 4$ then $4 \in \{\varphi(x_2), \varphi(x_6)\}$ and consequently $\varphi(v_2) = 2$ and $\varphi(x_2) = 1$ or $\varphi(x_6) = 2$ because of at least four monochromatic edges in X.

2.2.1. If $\varphi(x_2) = 1$ then $\varphi(x_6) = 4$ and either $[w_1w_2x_6x_1]$ or $[u_1v_1w_1x_1]$ is rainbow.

2.2.2. If $\varphi(x_6) = 2$ then $\varphi(x_2) = 4$ and either $[u_1x_1x_2u_6]$ or $[u_1v_1w_1x_1]$ is rainbow. □

Lemma 3. *If $|\varphi(K)| = 5$ then X contains at most five monochromatic edges.*

P r o o f. First we show that $|\varphi(X)| \geqslant 3$. If $|\varphi(X)| = 2$ then $|\varphi(\mathscr{U})| = 3$ and, without loss of generality, $\varphi(u_1) = 1$, $\varphi(v_1) = 2$, $\varphi(w_1) = 3$, $\varphi(x_1) \in \{1, 2, 3\}$, $\varphi(u_2) = \varphi(u_6) = \varphi(v_2) = \varphi(v_6) = \varphi(w_2) = \varphi(w_6) = 4$, $\varphi(x_2) = \varphi(x_6) = 5$. Consequently, at least one of the faces $[u_1x_1x_2u_6]$ or $[x_1w_1w_2x_6]$ is rainbow, a contradiction.

As the subgraph of R induced by the set X is a cycle on 8 vertices and the colour set $\varphi(X)$ has at least three colours, it is easy to see that there are at most five monochromatic edges in X. □

Lemma 4. *Let $|\varphi(K)| = 7$ and $|\varphi(X)| = 6$. Then there is a unique 7-colouring of K (up to permutations of colours and symmetries of K), namely $\varphi(u_1) = 1$, $\varphi(u_2) = \varphi(u_6) = 2$, $\varphi(v_1) = \varphi(v_6) = 3$, $\varphi(v_2) = 4$, $\varphi(w_1) = \varphi(w_6) = \varphi(x_1) = \varphi(x_2) = 5$, $\varphi(w_2) = 6$, and $\varphi(x_6) = 7$.*

P r o o f. Let T_u be a triangle $[u_1u_2u_6]$, let the triangles T_v, T_w and T_x are defined analogously. Because $|\varphi(X)| = 6$ there must be at least one colour, say 1, that is not in $\varphi(X)$. Let, without loss of generality, $\varphi(u_1) = 1$. Then seven colours of K must be distributed so that, without loss of generality, $\varphi(T_u) = \{1, 2\}$, $\varphi(T_v) = \{3, 4\}$, $\varphi(T_w \cup T_x) \supseteq \{5, 6, 7\}$. The straightforward case by case analysis shows the only the case when no rainbow face appears in K is that one of the Lemma. □

Let C be the cycle induced by the vertices of the set X in R and let C' be the cycle induced by the set $X' = \{u_5, u_3, v_5, v_3, w_5, w_3, x_5, x_3\}$. Between cycles C and C' in R there is a set S of eight quadrangles. Next we show that at least one face of S is rainbow. Because of Lemmas 1, 2 and 3 there are four cases.

1. Let $|\varphi(K)| = |\varphi(K')| = 7$. Then $|\varphi(K) \cap \varphi(K')| = 2$ and there are three monochromatic edges between C on S' and no ones on C and C' or at most two monochromatic edges between C and C', at most one on C and at most one on C'. These monochromatic edges enforce at most six faces of S not to be rainbow. So at least one face of S is rainbow.

2. Let $|\varphi(K)| = 7$ and $|\varphi(K')| = 6$ or $|\varphi(K)| = 6$ and $|\varphi(K')| = 7$. Then $|\varphi(K) \cap \varphi(K')| = 1$ and there are at most two monochromatic edges between C and C'. These edges with at most four monochromatic edges that are together on C and C' can enforce at most seven faces of S not to be rainbow except the case when $|\varphi(K)| = 7$ and $|\varphi(X)| = 6$ or $|\varphi(K')| = 7$ and $\varphi(X') = 6$. In these cases all faces of S can appear not be rainbow. In these cases, due to Lemma 4, there is a unique 7-colouring of K or K', respectively. It is easy to see that every extension of this 7-colouring to a 12-colouring of the whole graph R leads to a rainbow face on R.

3. Let $\varphi(K)| = 7$ and $|\varphi(K')| = 5$ or $|\varphi(K)| = 5$ and $|\varphi(K')| = 7$. In this case $\varphi(K) \cap \varphi(K') = \emptyset$ and therefore there is no monochromatic edge between C and C', and on $C \cup C'$ there are at most six monochromatic edges. These monochromatic edges can enforce at most six faces not to be rainbow of the eight of S, a contradiction.

4. Let $|\varphi(K)| = |\varphi(K')| = 6$. In this case $\varphi(K) \cap \varphi(K') = \emptyset$ and on $C \cup C'$ there are at most six monochromatic edges and therefore there are at most six non-rainbow faces in S. Again a contradiction.

Theorem 5.6.

$$31 \leqslant \mathrm{rb}(3, 4, 5, 4) \leqslant 35.$$

P r o o f. For the lower bound see Figure 5.3.

To prove the upper bound let us suppose that there exists a non-rainbow 35-colouring φ of the $(3, 4, 5, 4)$-graph S. There are three possibilities of the distribution of colours to the vertices of S when considering triangular faces of S.

1. 17 triangles $T_1, \ldots, T_{17}$ with vertices of two colours having disjoint colour sets $\varphi(T_i)$ and $\varphi(T_j)$ for distinct triangles T_i, T_j, i.e. $\varphi(T_i) \cap \varphi(T_j) = \emptyset$; moreover, one triangle T_{18} with one colour not contained in the previous triangles, and two triangles T_{19} and T_{20} with colours already involved in the above mentioned triangles.

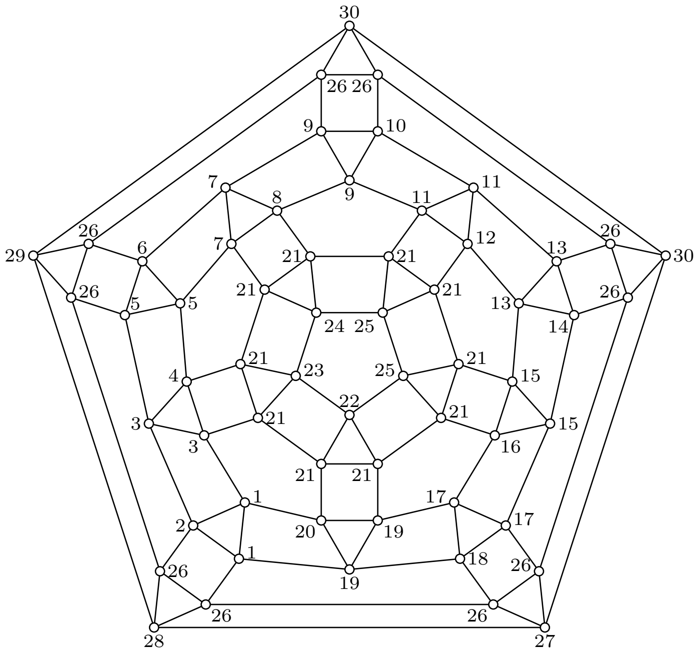

Figure 5.3

2. 16 triangles $T_1, \ldots, T_{16}$ with $|\varphi(T_i)| = 2$ for any $i \leqslant 16$ and $\varphi(T_i) \cap \varphi(T_j) = \emptyset$, triangles T_{17},T_{18}, T_{19} each having exactly one colour not appearing in other triangles and a triangle T_{20} with colour(s) already contained in previous triangles.

3. 15 triangles $T_1, \ldots, T_{15}$ with $|\varphi(T_i)| = 2$ for any $i \leqslant 15$ and $\varphi(T_i) \cap \varphi(T_j) = \emptyset$ and five triangles $T_{16}, \ldots, T_{20}$ each having exactly one colour not appearing in any of the triangles $T_1, \ldots, T_{15}$.

For the purposes of this proof let us associate with S a new graph $A(S)$ defined as follows. The vertex set of $A(S)$ is the set $\{T_1, \ldots, T_{20}\}$. Two vertices T_i, T_j create an edge T_iT_j in $A(S)$ if in S the triangles T_i and T_j are separated by a quadrangular face. It is easy to see that $A(S)$ is isomorphic with the graph of the dodecahedron. Furthermore there is a bijection between the sets of pentagons of $A(S)$ and the associated dodecahedron.

Consider now the third possibility. Let us call white the vertices of $A(S)$ associated with the triangles $T_1, \ldots, T_{15}$, the other five let us call black. If there is a pentagon in $A(S)$ incident only with white vertices then the corresponding pentagon of S is rainbow. The reason is that the triangles of S corresponding to two distinct vertices have disjoint colour sets.

If there is no pentagon incident only with white vertices in $A(S)$ then there is a configuration consisting of a black vertex x incident with three pentagons all other vertices of which are white. Because the colour set of the triangle corresponding to x consists of at most two colours and the colour sets of the triangles corresponding to the remaining 9 vertices of this configuration are mutually disjoint, at least one pentagon of S corresponding to a pentagon of the configuration is rainbow.

Analogously we proceed in the possibilities 1 and 2. In the first possibility the graph $A(S)$ consists of 17 white and three black vertices. In the second possibility in $A(S)$ there are 16 white and 4 black vertices. It is easy to find in each of these cases at least one configuration considered above. □

Theorem 5.7.

$$14 \leqslant \mathrm{rb}(3,5,3,5) \leqslant 15.$$

Proof. Denote the $(3,5,3,5)$-polyhedron by Q. For the lower bound see Figure 5.4. For the purposes of this proof we introduce a new notion. So let us call a graph consisting of two triangles with a vertex in common a *clock*.

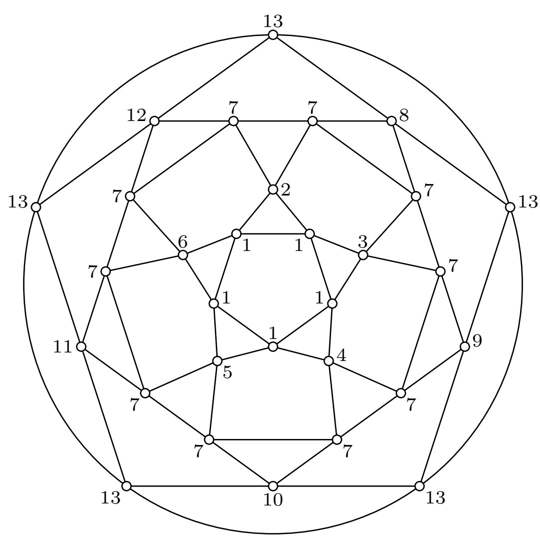

Figure 5.4

Observe that there is a set of six clocks that cover all vertices of the graph Q. Let us call them $H_1, \ldots, H_6$ and denote the family of these graphs by $\mathscr{H}$. The triangular faces of Q that are not in the clocks from $\mathscr{H}$ form a family $\mathscr{T} = \{T_1, \ldots, T_8\}$ containing exactly eight triangles. Notice that any triangle $T \in \mathscr{T}$ shares a vertex with exactly three members of $\mathscr{H}$. We can now associate with Q a bipartite graph

B with the vertex set $\mathscr{H} \cup \mathscr{T}$. The edge H_iT_j exists in B if and only if H_i shares a vertex with T_j.

To prove the upper bound let us consider a 15-colouring $\varphi\colon V(Q) \to [1, 15]$ that enforces no rainbow face in Q. One can suppose (without loss of generality) that the colours are distributed in $\mathscr{H}$ as follows. There are two possibilities:

1. $\varphi(H_1) = \{1, 2, 3\}$, $\varphi(H_2) = \{4, 5, 6\}$, $\varphi(H_3) = \{7, 8, 9\}$, $\varphi(H_4) = \{10, 11, 12\}$, $\varphi(H_5) = \{13, 14\}$ or $\{i, 13, 14\}$ for some $i \leqslant 12$, and $\varphi(H_6) = \{15\}$ or $\{k, 15\}$ or $\{j, k, 15\}$ for some $j \leqslant 12$ and $k \leqslant 14$.

2. $\varphi(H_1)$, $\varphi(H_2)$ and $\varphi(H_3)$ are as above, $\varphi(H_4) = \{10, 11\}$ or $\{i, 10, 11\}$ for some $i \leqslant 9$, $\varphi(H_5) = \{12, 13\}$ or $\{j, 12, 13\}$ for some $j \leqslant 9$, and finally $\varphi(H_6) = \{14, 15\}$ or $\{k, 14, 15\}$ for some $k \leqslant 9$.

Consider first the possibility **1**. If there is in B a vertex T_m adjacent to vertices H_i, H_j and H_k for $1 \leqslant i < j < k \leqslant 4$ then the triangle T_m is rainbow. If this is not the case then the distance of H_5 and H_6 in B is $\operatorname{dist}(H_5, H_6) = 4$ and H_5 shares a triangle with the pair H_i and H_{i+1} for any $i = 1, 2, 3, 4$ (modulo 4). At least two of them are rainbow because on the 2-valent vertices of H_5 there are at least two distinct colours.

2. If there is a vertex T_k in B adjacent to H_1, H_2 and H_3 then T_k in Q is rainbow. If it is not the case then without loss of generality the vertex H_5 shares triangles with the pair H_i and H_{i+1} for every $i = 1, 2, 3, 4$ (modulo 4). Let these triangles be T_1, T_2, T_3 and T_4. If none of T_1 and T_2 is rainbow (which means that $i \in \varphi(H_2)$) then the triangle T_3 is rainbow because colour on it is from $\{7, 8, 9\}$, the second is from $\{10, 11\}$ and the third one is either $i \in \{4, 5, 6\}$ or from $\{12, 13\}$. □

Theorem 5.8.

$$\operatorname{rb}(3, 3, 3, 3, 4) = 10.$$

P r o o f. For the purposes of the proof let the vertices of the $(3, 3, 3, 3, 4)$-polyhedron be denoted as in Figure 5.5. The colouring φ which provides a 9-colouring without a rainbow face is as follows: $\varphi(u_4) = 1$, $\varphi(v_5) = 2$, $\varphi(w_2) = 3$, $\varphi(x_1) = 4$, $\varphi(v_1) = 5$, $\varphi(w_4) = 6$, $\varphi(x_5) = 7$, $\varphi(u_2) = 8$. The remaining vertices receive the colour 9. So we have $\operatorname{rb}(3, 3, 3, 3, 4) \geqslant 10$.

To obtain the opposite inequality suppose there is a 10-colouring φ of vertices that also has no rainbow face.

For the purposes of this proof let K be the configuration induced in Figure 5.5 by the vertex set $\{u_1, u_2, u_6, v_1, v_2, v_6, w_1, w_2, w_6, x_1, x_2, x_6\}$. It consists of the quadrangle $[u_1v_2w_1x_1]$ and 12 triangles. Let K' be the configuration induced by the remaining vertices of the $(3, 3, 3, 3, 4)$-polyhedron. Denote by T_u the triangle $[u_1u_2u_6]$,

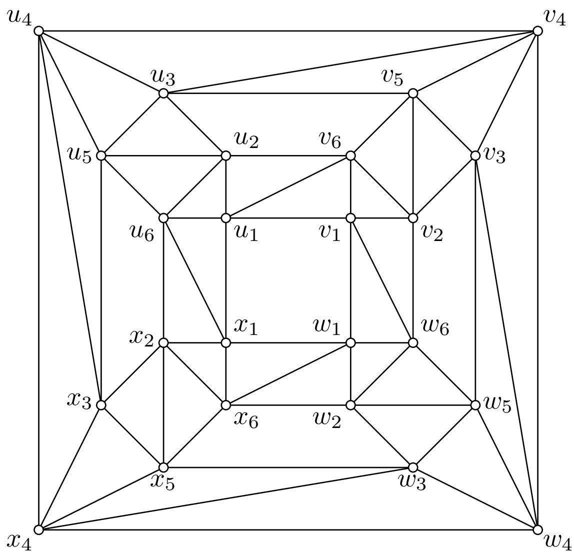

Figure 5.5

let the triangles T_v, T_w, T_x be defined analogously. Clearly $1 \leqslant |\varphi(T_y)| \leqslant 2$ for any $y \in \{u, v, w, x\}$.

Lemma 5.

$$|\varphi(K)| \leqslant 5.$$

P r o o f. Suppose $[1, 6] \subseteq \varphi(K)$. Because of the symmetries of K there are two possibilities to consider:

1. Let $|\varphi(T_u) \cup \varphi(T_v)| = 4$. Then without loss of generality let $\varphi(T_u) = \{1, 2\}$, $\varphi(T_v) = \{3, 4\}$. This implies that $\varphi(u_1) = \varphi(u_2) = 1$, $\varphi(u_6) = 2$, $\varphi(v_1) = \varphi(v_6) = 3$, $\varphi(v_2) = 4$, $\varphi(w_6) \in \{3, 4\}$, $\varphi(x_1) \in \{1, 2\}$.

1.1. Let $\varphi(x_1) = 1$, then $\varphi(x_2) \in \{1, 2\}$ and $\varphi(x_6) = 5$ since otherwise $\{\varphi(w_1), \varphi(w_2)\} = \{5, 6\}$ and the face T_w is rainbow. This implies $\varphi(w_1) \in \{1, 5\}$, $\varphi(w_2) = 6$ and T_w is again rainbow, a contradiction.

1.2. Let $\varphi(x_1) = 2$, then $\varphi(w_1) = 3$, $\varphi(x_6) \in \{2, 3\}$ and at least one of T_x, T_w, or $[w_1 w_2 x_6]$ is rainbow.

2. Let $|\varphi(T_u) \cup \varphi(T_v)| = 3$ and $|\varphi(T_u) \cup \varphi(T_x)| = 3$. Then without loss of generality $\varphi(T_u) = \{1, 2\}$, $\varphi(T_w) = \{4, 5\}$, $3 \in \varphi(T_v)$ and $6 \in \varphi(T_x)$.

2.1. Let $\varphi(u_1) = \varphi(u_2) = 1$, $\varphi(u_6) = 2$. Then $\varphi(x_1) \in \{1, 2\}$ and without loss of generality $\varphi(w_1) = 4$.

2.1.1. If $\varphi(w_2) = 4$, $\varphi(w_6) = 5$ then $\varphi(v_1) \in \{4, 5\}$, $\varphi(v_2) = 3$ and consequently $\varphi(v_1) = \varphi(v_6) = 5$ and $\varphi(x_1) = 1$. Then $\varphi(x_2) = 6$ or $\varphi(x_6) = 6$. In the former case the triangle $[u_6x_1x_2]$ is rainbow, in the latter the triangle $[w_1x_1x_6]$ is rainbow.

2.1.2. If $\varphi(w_2) = 5$, $\varphi(w_6) \in \{4, 5\}$ then $\varphi(x_6) \in \{4, 5\}$. Because now $\varphi(x_2) = 6$ the triangle T_x is rainbow.

2.2. Let $\varphi(u_1) = 1$ and $\varphi(u_2) = \varphi(u_6) = 2$. Then $\varphi(v_6) \in \{1, 2\}$, $\varphi(x_1) \in \{1, 2\}$. Let $\varphi(w_1) = 4$, then also $5 \in \{\varphi(w_2), \varphi(w_6)\}$.

2.2.1. If $\varphi(w_2) = 5$ then $\varphi(x_6) \in \{4, 5\}$ and $\varphi(x_2) = 6$, and T_x is rainbow.

2.2.2. If $\varphi(w_6) = 5$ then $\varphi(v_1) \in \{4, 5\}$ and $\varphi(v_2) = 3$, and T_v is rainbow.

In all cases we have obtained a contradiction, hence $|\varphi(K)| \leqslant 5$. □

As φ is a 10-colouring and the configuration K' is isomorphic to K, we have, due to Lemma 5, $\varphi(K) \cap \varphi(K') = \emptyset$ and $|\varphi(K)| = |\varphi(K')| = 5$. Let C be the cycle induced by the vertex set $X = \{u_2, u_6, x_2, x_6, w_2, w_6, v_2, v_6\}$ and C' the cycle induced by the vertex set $Y = \{u_3, u_5, x_3, x_5, w_3, w_5, v_3, v_5\}$. Between these two cycles there is a set S of twelve faces altogether. As $\varphi(X) \cap \varphi(Y) = \emptyset$ the fact that the faces of S are not rainbow must be caused by monochromatic edges on C and C', one monochromatic edge enforces one face of S not to be rainbow. As on the vertices of C (and above on C') there must be at least three colours there are at most five monochromatic edges on $C(C')$. Because there are, in S, twelve faces we have a contradiction. □

Theorem 5.9.

$$19 \leqslant \mathrm{rb}(3, 3, 3, 3, 5) \leqslant 30.$$

Proof. The snub dodecahedron, i.e. the $(3, 3, 3, 3, 5)$-polyhedron P has 60 vertices, 92 faces and 150 edges. Its graph is in Figure 5.6 where one can find a 18-colouring that involves no rainbow face. This shows $\mathrm{rb}(P) \geqslant 19$.

To prove the upper bound let us suppose that there is a surjective 30-colouring φ of vertices of P that does not contain any rainbow face. For the purpose of this proof let the configuration K be the set of all faces of the $(3, 3, 3, 3, 5)$-polyhedron P that have a vertex in common with a given pentagonal face of P. Analogously to the proof of Lemma 5 one can prove that for the set of colours that appear at the vertices of K we have $|\varphi(K)| \leqslant 7$.

Let the configuration R be a ring of 30 triangles of P bounded with two vertex disjoint cycles C_x and C_y of length 15 on vertex sets $\{x_1, x_2, \ldots, x_{15}\}$ and $\{y_1, y_2, \ldots, y_{15}\}$. The edge set of R consists of the edges x_jy_j, x_jx_{j+1}, y_jy_{j+1} for any $j \in [1, 15]$, indices modulo 15, $x_{3i+2}y_{3i+1}$, $x_{3i+3}y_{3i+2}$, $x_{3i+3}y_{3i+4}$ for $i \in [0, 4]$, indices modulo 15. Notice that R shares edges with 10 pentagonal faces of P, exactly two with each of these pentagons. Let D_i be a subgraph of R induced on the vertex set $\{x_{3i+1}, x_{3i+2}, x_{3i+3}, y_{3i+1}, y_{3i+2}, y_{3i+3}\}$, $i \in [0, 4]$. It is easy to see that $|\varphi(D_i)| \leqslant 4$,

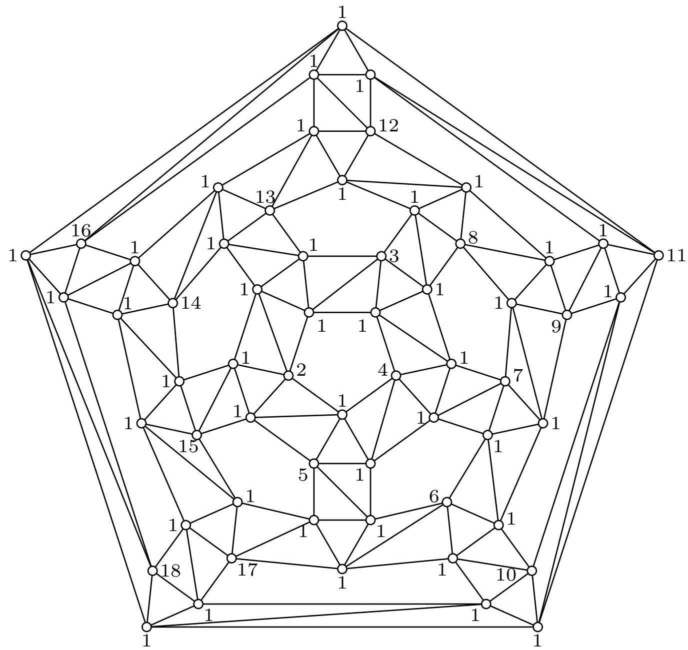

Figure 5.6

$|\varphi(D_i \cup D_{i+1})| \leqslant 7$, $|\varphi(D_i \cup D_{i+1} \cup D_{i+2})| \leqslant 10$ and $|\varphi(D_i \cup D_{i+1} \cup D_{i+2} \cup D_{i+3})| \leqslant 13$, indices modulo 5. The colouring of D_{i+4} depends on the colourings of D_i and D_{i+3}. In every possible case there are at most two additional colours in $\varphi(D_{i+4})$. So we have $|\varphi(R)| \leqslant 15$.

To complete the proof of the theorem observe that there are three mutually vertex-disjoint configurations, two configurations K and one configuration R that cover all vertices of P. Because φ is a 30-colouring at least one of the above mentioned configurations K or R must contain a rainbow face, a contradiction. □

6. One more semiregular polyhedron

There is one more $(3,4,4,4)$-polyhedron that is not Archimedean. Its graph is drawn in Figure 6.1. It was discovered by Ashkinuze [1], see also [5] or [6]. For this polyhedron we are able to prove.

Theorem 6.1. *Let A be the $(3,4,4,4)$-polyhedron of Ashkinuze. Then*

$$\mathrm{rb}(A) = 12.$$

P r o o f. In Figure 6.1 there is a 11-colouring of A without any rainbow face. This yields $\mathrm{rb}(A) \geqslant 12$.

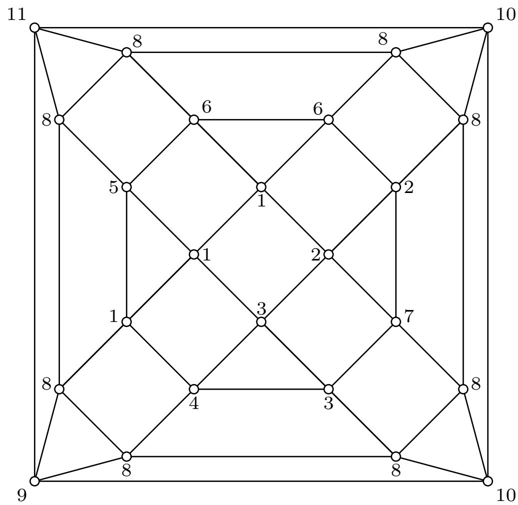

Figure 6.1

To show that $\mathrm{rb}(A) \leqslant 12$ we proceed in the same way as in the proof of Theorem 5.5. Details are left to the reader. □

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References

[1] *V. G. Ashkinuze*: On the number of semiregular polyhedra. Mat. Prosvesc. *1* (1957), 107–118. (In Russian.)
[2] *H. S. M. Coxeter*: Regular Potytopes. Dover Pub. New York, 1973.
[3] *P. R. Cromwell*: Polyhedra. Cambridge University Press, 1997.
[4] *L. Fejes Tóth*: Regular Figures. Pergamon Press, Oxford, 1964.
[5] *B. Grünbaum*: Convex Polytopes (2nd edition). Springer Verlag, 2004.
[6] *E. Jucovič*: Convex Polyhedra. Veda, Bratislava, 1981. (In Slovak.)
[7] *S. Negami*: Looseness ranges of triangulations on closed surfaces. Discrete Math. *303* (2005), 167–174. zbl
[8] *J. Zaks*: Semi-regular polyhedra and maps. Geom. Dedicata *7* (1978), 465–478.

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