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WEAK-OPEN COMPACT IMAGES OF METRIC SPACES

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Abstract. The main results of this paper are that

- (1) a space X is g -developable if and only if it is a weak-open π image of a metric space, one consequence of the result being the correction of an error in the paper of Z. Li and S. Lin;
- (2) characterizations of weak-open compact images of metric spaces, which is another answer to a question in in the paper of Y. Ikeda, C. liu and Y. Tanaka.

Keywords: g -developable, π -mapping, weak-open mapping, CWC-map, uniform weak base

MSC 2000: 54D55, 54E15, 54E40, 54E99

1. INTRODUCTION

A. Arhangel'skii [4] showed that a T_1 -space X has a uniform base if and only if it is an open compact image of a metric space. As is well known, a space X has a uniform base if and only if it is a metacompact developable space. R. Heath [9] showed that a T_1 -space X is developable if and only if it is an open π image of a metric space. In this paper characterizations of weak-open compact (π) images of metric spaces are given in terms of the concept of g -developable spaces, which are similar to Arhangel'skii's (Heath's) above result.

In this paper, all spaces are T_2 and all mappings are continuous and surjective. \mathbb{N} denotes the set of all natural numbers.

Definition 2.1. A cover $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ of a space X is called a weak base [5] for X if it has the following properties:

- (1) $x \in \bigcap \mathcal{P}_x$ for each $x \in X$,
- (2) if $P_1, P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ with $P \subset P_1 \cap P_2$,
- (3) for $U \subset X$, U is open in X if and only if for each $x \in U$ there exists $P \in \mathcal{P}_x$ with $P \subset U$.

\mathcal{P}_x is called a weak neighborhood base of x in X . If each \mathcal{P}_x is countable for $x \in X$, then X is called g -first countable.

For a space X , let g be a map defined on $\mathbb{N} \times X$ to the power-set of X such that $x \in g(n, x)$ and $g(n+1, x) \subset g(n, x)$ for each n and x ; a subset U of X is open if for each $x \in U$ there is an n such that $g(n, x) \subset U$. We call such a map a CWC-map (countable weakly-open covering map).

As mentioned in [11], X is g -first countable if X has a CWC-map g such that if $x_n \in g(n, x)$ for each n , then the sequence $\{x_n\}$ converges to x .

Definition 2.2 ([11]). A space is g -developable if it has a CWC-map g with the following property: If $x, x_n \in g(n, y_n)$ for each n , then the sequence $\{x_n\}$ converges to x .

As stated in [19], if X has a g -first countable CWC-map g such that $\{st(x, \mathcal{G}_n) : x \in X, n \in \mathbb{N}\}$ is a weak base for X , then X is g -developable (where $\mathcal{G}_n = \{g(n, x) : x \in X\}$). Conversely, let g be a g -developable CWC-map for a space X , then $\{st(x, \mathcal{G}_n) : x \in X, n \in \mathbb{N}\}$ is a weak base for X . Thus, a g -developable space is symmetrizable.

Definition 2.3. Let $f: X \rightarrow Y$ be a mapping.

- (1) f is called a π -mapping [1] if (X, d) is a metric space and $d(f^{-1}(y), X \setminus f^{-1}(U)) > 0$ for each $y \in Y$ and a neighborhood U of y in Y .
- (2) f is weak-open [21] if there is a weak base $\mathcal{B} = \bigcup \{\mathcal{B}_y : y \in Y\}$ for Y and there is an $x(y) \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of $x(y)$ in X , $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.
- (3) f is called a sequence-covering mapping [19] if each convergent sequence from Y is the image of some convergent sequence from X .
- (4) f is called a 1-sequence-covering mapping [14] if for each $y \in Y$, there is an $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y , then there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.
- (5) f is called a σ -locally finite mapping [16], if there exists a base \mathcal{B} for X such that $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is σ -locally finite in Y .

Obviously, every compact mapping on a metric space is a π -mapping, and each 1-sequence-covering mapping is a sequence-covering mapping. It is easy to check that a weak-open mapping is a quotient mapping.

Theorem 2.4. *A space X is g -developable if and only if it is a weak-open π image of a metric space.*

Proof. Necessity. Suppose that X has a CWC-map g satisfying Definition 2.2. Set $\mathcal{G}_n = \{g(n, x) : x \in X\} = \{G_\alpha : \alpha \in A_n\}$ for each $n \in \mathbb{N}$. Let A_n be the space with discrete topology. Put $M = \{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x(\alpha) \text{ in } X\}$, and equip M with the subspace topology induced from the usual product topology of the discrete spaces A_n . We define a distance function d on M as follows. For each $\alpha, \beta \in M$, if $\alpha = \beta$, then $d(\alpha, \beta) = 0$, if $\alpha \neq \beta$, then $d(\alpha, \beta) = 1/n$, where n is the smallest natural number such that $\alpha_n \neq \beta_n$. Hence d is a metric on M . We define $f: M \rightarrow X$ by $f(\alpha) = x(\alpha)$. It is easy to check that f is continuous and surjective.

(1) f is a weak-open mapping. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n, x) = G_{\gamma_n}$, then $\gamma = (\gamma_n) \in M$ and $\gamma \in f^{-1}(x)$. Set $B_n = \{\beta \in M : \text{the } i\text{-th coordinate of } \beta \text{ is } \gamma_i, \text{ for all } i \leq n\}$. It is easy to check that $\{B_n : n \in \mathbb{N}\}$ is a local decreasing base of $\gamma = (\gamma_i)$ in M , and $f(B_n) = \bigcap_{i \leq n} G_{\gamma_i} = G_{\gamma_n}$. Let U be any open neighborhood of γ in M ; then there exists $m \in \mathbb{N}$ such that $B_m \subset U$. Hence $f(B_m) \subset f(U)$. If we set $\mathcal{B}_x = \{g(n, x), n \in \mathbb{N}\}$ and $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$, then \mathcal{B} is a weak base for X which satisfies Definition 2.3 (2). Therefore f is a weak-open mapping.

(2) f is a π mapping. Suppose that $x \in X$ and U is an open neighborhood of x . It is not difficult to show that $\text{st}(x, \mathcal{G}_n) \subset U$ for some $n \in \mathbb{N}$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $\alpha_i = \beta_i$ for all $i \leq n$. Hence $f(\beta) = \bigcap_{i \in \mathbb{N}} G_{\beta_i} = G_{\beta_n} \subset U$. Thus $d(f^{-1}(x), X \setminus f^{-1}(U)) \geq 1/n$. It follows that f is a π mapping.

Sufficiency. Suppose X is a weak-open π image of a metric space M under f . Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be the weak base for X satisfying Definition 2.3 (2). For each $x \in X$, there is $\alpha(x) \in f^{-1}(x)$ satisfying Definition 2.3 (2). Set $g(n, x) = f(B(\alpha(x), 1/n))$ for each $x \in X$ and $n \in \mathbb{N}$, where $B(\alpha, 1/n) = \{\beta \in M : d(\alpha, \beta) < 1/n\}$.

(1) g is a g -first countable CWC-map for X . Suppose that U is a subset of X such that for each $x \in U$ there is n with $g(n, x) \subset U$. As f is weak-open and $B(\alpha(x), 1/n)$ is an open neighborhood of $\alpha(x)$, there is $P \in \mathcal{P}_x$ such that $P \subset f(B(\alpha(x), 1/n)) \subset U$. So U is open in X . On the other hand, suppose that U is an open subset of X . For each $x \in U$, we then have $B(\alpha(x), 1/m) \subset f^{-1}(U)$ for some $m \in \mathbb{N}$. Hence $g(m, x) = f(B(\alpha(x), 1/m)) \subset U$.

(2) g possesses the desired property: if $x, x_n \in g(n, y_n)$ for each n , then the sequence $\{x_n\}$ converges to x . Suppose that U is an open neighborhood of x in X . Then there is $n \in \mathbb{N}$ such that $d(f^{-1}(x), X \setminus f^{-1}(U)) > 1/n$. Taking a natural number $m \geq n$, if $x \in f(B(\alpha(z), 1/m)) = g(m, z)$, then $(f^{-1}(x) \cap B(\alpha(z), 1/m)) \neq \emptyset$. Suppose $B(\alpha(z), 1/m) \not\subseteq f^{-1}(U)$, then $d(f^{-1}(x), X \setminus f^{-1}(U)) < 2/m \leq 1/n$, a contradiction. Hence $B(\alpha(z), 1/m) \subset f^{-1}(U)$, $f(B(\alpha(z), 1/m)) \subset U$. So $\text{st}(x, \mathcal{G}_m) \subset U$, hence x_n converges to x . Therefore X is g -developable. \square

Lemma 2.5 ([21]). *Let $f: X \rightarrow Y$ be a weak-open mapping. If X is first countable, then f is a 1-sequence-covering mapping.*

We recall that X is a Cauchy space [2] if X has a symmetric such that each convergent sequence is Cauchy. K. B. Lee [11] showed that a space X is g -develoable if and only if it is a Cauchy space. Y. Tanaka [20] characterized sequence-covering quotient π images of metric spaces by Cauchy spaces. Hence we have the following result.

Corollary 2.6. *For a space X , the following assertions are equivalent:*

- (1) X is a weak-open π image of a metric space.
- (2) X is a 1-sequence-covering quotient π image of a metric space.
- (3) X is a sequence-covering quotient π image of a metric space.
- (4) X is a Cauchy space.
- (5) X is a g -developable space.

Example 2.7. There exists a g -metrizable space which is not a weak-open π image of a metric space.

L. Foged [6] showed that there is a g -metrizable space which is not g -developable. By Theorem 2.4, the g -metrizable space is not a weak-open π image of a metric space.

Remark 2.8. Example 2.7 shows that Theorem 2.2 [13] is wrong. And Lemma 2.1 [13] is not correct either, see Example 16 [16].

Examining the proof of Theorem 2.2 [13], we see that (1) and (3) of the theorem are equivalent. Thus we have the following result.

Theorem 2.9 ([12], [13], [16]). *For a regular space X , the following assertions are equivalent:*

- (1) X is a weak-open σ -image of a metric space.
- (2) X is a 1-sequence-covering σ -image of a metric space.
- (3) X is a compact-covering quotient π σ -image of a metric space.
- (4) X is a g -metrizable space.

3. WEAK-OPEN COMPACT IMAGES OF METRIC SPACES

Definition 3.1 ([8]). Let \mathcal{P} be a cover of a space X . \mathcal{P} is called a cs-network if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x , then $\{x\} \cup \{x_n: n \geq m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.

Definition 3.2 ([3]). Let \mathcal{P} be a cover of a space X . \mathcal{P} is called a point-regular cover for X if $\{P \in (\mathcal{P})_x: P \subsetneq U\}$ is finite for each open neighborhood U of x .

\mathcal{P} is called a point-regular cs-network (weak base) for X if \mathcal{P} is called a point-regular cover and a cs-network (weak base) for X .

Definition 3.3 ([3]). Let \mathcal{P} be a cover of a space X . \mathcal{P} is called a uniform regular cover for X , if whenever \mathcal{P}' is a countable infinite subset of $(\mathcal{P})_x$, then \mathcal{P}' is a network for x in X .

Definition 3.4 ([7]). Let X be a space.

- (1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to the point x , then $\{x_n: n \geq m\} \subset P$ for some $m \in \mathbb{N}$.
- (2) Let $P \subset X$. P is called a sequential open subset in X if P is a sequential neighborhood of x for each $x \in P$.
- (3) X is called a sequential space if each sequential open subset in X is open.

For a space X , set $W(X) = \{x \in X: \{x\} \text{ is a weak neighborhood of } x\}$ and $\mathcal{W}(X) = \{\{x\}: x \in W(X)\}$.

Proposition 3.5. For a space X , the following assertions are equivalent:

- (1) X has a point finite g -developable CWC-map g (i.e. $\mathcal{G}_n = \{g(n, x): x \in X\}$ is point finite for each $n \in \mathbb{N}$).
- (2) X has a uniform weak base.

Proof. (1) \Rightarrow (2). Set $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$. For each $x \in X$, let \mathcal{G}' be a countable subset of $(\mathcal{G})_x$, and set $\mathcal{G}' = \{g(n_k, x_k): k \in \mathbb{N}\}$. Then $\{n_k: k \in \mathbb{N}\}$ is an infinite set. Suppose that U is an open neighborhood of x , then there is $m \in \mathbb{N}$ such that $\text{st}(x, \mathcal{G}_m) \subset U$. Taking $k \in \mathbb{N}$ with $n_k \geq m$, then $x \in g(n_k, x_k) \subset \text{st}(x, \mathcal{G}_m) \subset U$. Hence \mathcal{G} is a uniform weak base for X .

(2) \Rightarrow (1) Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a uniform weak base for X .

(i) \mathcal{P} is point countable. For $x \in X$, suppose that $(\mathcal{P})_x$ is not countable. For $y \neq x$, as \mathcal{P} is a uniform weak base for X , hence $\{P \in (\mathcal{P})_x: y \in P\}$ is finite. So there are an infinite subset $\{P_n: n \in \mathbb{N}\}$ of $(\mathcal{P})_x$, $x_n \in P_n \setminus \{x\}$ and $k \in \mathbb{N}$ such that $\text{ord}(x_n, (\mathcal{P})_x) = k$ for each n . Note that \mathcal{P} is a uniform weak base for X ,

hence $x_n \rightarrow x$. Since a weak base for X is a cs-network for X , there are a subset $\{F_n: n \in \mathbb{N}\}$ of $(\mathcal{P})_x$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_i}: i \geq j\} \subset F_j \subset X \setminus \{x_{n_i}: i < j\}$, $j \in \mathbb{N}$. So $\text{ord}(x_{n_j}, (\mathcal{P})_x) \geq j$, a contradiction. Therefore \mathcal{P} is point countable.

(ii) \mathcal{P} has the property (*): $\{R \in \mathcal{P}: P \subset R\}$ is finite for each $P \in \mathcal{P}$. Otherwise, there is an infinite subset $\{P_n: n \in \mathbb{N}\}$ of $\mathcal{P} \setminus \{P\}$ such that $P \subset P_n$ for each n . By (i), X is g -first countable, hence X is a sequential space. If P is a single point, then since a weak neighborhood is a sequential neighborhood, P is a sequential open subset of X , thus P is open in X and $P_n \subset P$ for some P_n , a contradiction. If P is not a single point, then we may suppose that $\{x, y\} \subset P$, thus there is $n \in \mathbb{N}$ such that $x \in P_n \subset X - \{y\}$, a contradiction.

(iii) Put $\mathcal{P}^m = \{H \in \mathcal{P}: \text{if } H \subset P \in \mathcal{P}, \text{ then } P = H\}$. Then \mathcal{P}^m is a point finite cover for X . For $P \in \mathcal{P}$, by Property (*), there is $H \in \mathcal{P}^m$ with $P \subset H$, thus \mathcal{P}^m is a cover for X . For $x \in X$, if $(\mathcal{P}^m)_x$ is infinite, we denote it by $\{H_n: n \in \mathbb{N}\}$. Since \mathcal{P} is a uniform weak base for X , there is a weak neighborhood $P \in \mathcal{P}$ of x and we may assume $P \subset H_1$. Taking $x_n \in (H_{n+1} \setminus H_1)$, then $x_n \rightarrow x$ and $\{x_n\} \cap P = \emptyset$, a contradiction. Therefore \mathcal{P}^m is a point finite cover for X .

(iv) Put $\mathcal{P}' = (\mathcal{P} \setminus \mathcal{P}^m) \cup \mathcal{W}(X)$. Then \mathcal{P}' is a uniform weak base for X . Let U be an open neighborhood of x and we may assume $x \notin \mathcal{W}(X)$. Then there are distinct elements P_1, P_2 and P of \mathcal{P}_x with $x \in P \subset P_1 \cap P_2 \subset P_1 \cup P_2 \subset U$. Thus $P \in \mathcal{P}'$, \mathcal{P}' is a weak base for X . Hence \mathcal{P}' is a uniform weak base for X .

(v) X has a point finite g -developable CWC-map g . Put $\mathcal{P}_1 = \mathcal{P}^m$, $\mathcal{P}_{n+1} = [(\mathcal{P} \setminus \bigcup\{\mathcal{P}_j: j \leq n\}) \cup \mathcal{W}(X)]^m$, $n \in \mathbb{N}$. It is easy to see that \mathcal{P}_{n+1} refines \mathcal{P}_n for each $n \in \mathbb{N}$, and $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ by the property (*). For each $x \in X$, $n \in \mathbb{N}$, if $x \in W(X)$, then we define $g(n, x) = \{x\}$. If $x \notin W(X)$, since each \mathcal{P}_n is point finite, we may assume that $\mathcal{P}_x = \{P_i: i \in \mathbb{N}\} = \{P_1, P_2, \dots, P_{k_1}, P_{k_1+1}, \dots, P_{k_2}, P_{k_2+1}, \dots, P_{k_n} \dots\}$ with $P_i \in \mathcal{P}_{m_n}$ when $k_{n-1} < i \leq k_n$ and $m_n < m_{n+1}$ (where $k_0 = 0$). We define $g(n, x) = \bigcap_{j \leq k_n} P_j$. Since each \mathcal{P}_n is point finite and \mathcal{P}_{n+1} refines \mathcal{P}_n for $n \in \mathbb{N}$, $\mathcal{G}_n = \{g(n, x): x \in X\}$ is point finite for each $n \in \mathbb{N}$. Let U be a subset of X such that for each $x \in U$, $g(n, x) \subset U$ for some $g(n, x)$. Then there is $P \in \mathcal{P}_x$ with $P \subset g(n, x)$. Hence U is open in X , thus g is a g -first CWC-map for X . To complete the proof, we need only to show that $\{\text{st}(x, \mathcal{G}_n): x \in X, n \in \mathbb{N}\}$ is a weak base for X . Let U be an open neighborhood of x and suppose that $\text{st}(x, \mathcal{G}_n) \subsetneq U$ for each $n \in \mathbb{N}$. Then there are $x_n, x \in g(n, y_n)$ with $x_n \notin U$ for each $n \in \mathbb{N}$. Note that if $P_n \in \mathcal{P}_n$ then $\{P_n: n \in \mathbb{N}\}$ are different from each other. Thus there is an infinite subset \mathcal{P}'' of $(\mathcal{P})_x$ such that each element of \mathcal{P}'' contains some $g(n, y_n)$. Since \mathcal{P} is a uniform weak base for X , there is $P \in \mathcal{P}''$ such that $g(n, y_n) \subset P \subset U$ for some $g(n, y_n)$,

a contradiction. Hence $\text{st}(x, \mathcal{G}_n) \subset U$ for some $n \in \mathbb{N}$, and $\{\text{st}(x, \mathcal{G}_n): n \in \mathbb{N}\}$ is a weak base for X . \square

Theorem 3.6. *The following assertions are equivalent for a space X .*

- (1) X is a weak-open compact image of a metric space.
- (2) X has a point finite g -developable CWC-map g .

Proof. (1) \Rightarrow (2) Let X be a weak-open compact image of a metric space M under f . By Theorem 1.3.3 [15], there is a sequence $\{\mathcal{P}_n\}$ of open covers of M such that $\{\text{st}(K, \mathcal{P}_n): n \in \mathbb{N}\}$ is a neighborhood base of K in M for each $K \in \mathcal{K}(M)$ (where $\mathcal{K}(M)$ is the collection of all compact subsets of M). By the paracompactness of M , we may assume that each \mathcal{P}_n is local finite and \mathcal{P}_{n+1} refines \mathcal{P}_n for each $n \in \mathbb{N}$. Thus $f(\mathcal{P}_n)$ is a point finite cover of X for each $n \in \mathbb{N}$. Let $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ be a weak base for X and let $\alpha(x) \in f^{-1}(x)$ for $x \in X$ satisfying Definition 2.3 (2). For each $n \in \mathbb{N}$ and $x \in X$, there exists $P_n \in \mathcal{P}_n$ with $\alpha(x) \in P_n$. For this P_n there exists $B_n \in \mathcal{B}_x$ with $B_n \subset f(P_n)$. Put $g(n, x) = \bigcap_{j \leq n} B_j$, then $\mathcal{G}_n = \{g(n, x): x \in X\}$ is point finite for each $n \in \mathbb{N}$. Suppose that U is an open subset of X for $x \in U$, then $f^{-1}(x) \subset f^{-1}(U)$, so there exists $n \in \mathbb{N}$ such that $\text{st}(f^{-1}(x), \mathcal{P}_n) \subset f^{-1}(U)$. Hence $g(n, x) \subset \text{st}(x, \mathcal{G}_n) \subset \text{st}(x, f(\mathcal{P}_n)) \subset U$. Conversely, let U be a subset of X such that for each $x \in U$ there exists $n \in \mathbb{N}$ with $g(n, x) \subset U$. By the definition of $g(n, x)$, we have $B \subset g(n, x) \subset U$ for some $B \in \mathcal{B}_x$. As \mathcal{B} is a weak base for X , U is open in X . Therefore g is a g -first countable CWC-map for X . Similarly, as shown above, $\{\text{st}(x, \mathcal{G}_n): x \in X, n \in \mathbb{N}\}$ is a weak base for X , so g is a point finite g -developable CWC-map for X .

(2) \Rightarrow (1) Put $\mathcal{G}_n = \{g(n, x), x \in X\}$, $n \in \mathbb{N}$, then each \mathcal{G}_n is point finite. It is easy to check that $\{G_i: i \in \mathbb{N}\}$ is a network of x if $x \in G_i \in \mathcal{G}_i$ for each $i \in \mathbb{N}$. Set $\mathcal{G}_n = \{G_\alpha: \alpha \in A_n\}$, $n \in \mathbb{N}$, and $M = \{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n: \{G_{\alpha_n}: n \in \mathbb{N}\}$ forms a network at some point $x(\alpha)$ in $X\}$, and equip M with the subspace topology induced from the usual product topology of the discrete spaces A_n . Then M is a metric space. We define $f: M \rightarrow X$ by $f(\alpha) = x(\alpha)$. It is easy to see that f is a continuous and surjective mapping.

(i) f is a weak-open mapping. As in the proof of Theorem 2.4, f is a weak-open mapping

(ii) f is a compact mapping. For each $x \in X$, $i \in \mathbb{N}$, set

$$C_i = \{\alpha \in A_i: x \in G_\alpha \in \mathcal{G}_i\};$$

then $\prod_{i \in \mathbb{N}} C_i$ is a compact subset of $\prod_{i \in \mathbb{N}} A_i$. If $\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} C_i$, then $x \in \bigcap_{i \in \mathbb{N}} G_{\alpha_i}$, $\alpha \in M$ and $f(\alpha) = x$. So $\prod_{i \in \mathbb{N}} C_i \subset f^{-1}(x)$. On the other hand, if $\alpha = (\alpha_i) \in f^{-1}(x)$,

then $x \in \bigcap_{i \in \mathbb{N}} G_{\alpha_i}$. Hence $\alpha \in \prod_{i \in \mathbb{N}} C_i$, $f^{-1}(x) \subset \prod_{i \in \mathbb{N}} C_i$. Thus $f^{-1}(x) = \prod_{i \in \mathbb{N}} C_i$, and f is a compact mapping. \square

Corollary 3.7. *The following assertions are equivalent for a space X .*

- (1) X is a weak-open compact image of a metric space.
- (2) X is a 1-sequence-covering quotient compact image of a metric space.
- (3) X is a sequence-covering quotient compact image of a metric space.
- (4) X has a point-regular weak base.
- (5) X is a sequential space with a point-regular cs-network.
- (6) X has a uniform weak base.
- (7) X has a point finite g -developable CWC-map.

(3) \Leftrightarrow (4) was shown in [10], (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) were shown by S. Lin and P. Yan [17], which answered a question from [10]: For a sequential space X with a point-regular cs-network, characterize X by means of a nice image of a metric space. Thus Corollary 3.7 is another answer to the question.

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