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SHAPE-PRESERVING PROPERTIES AND ASYMPTOTIC BEHAVIOUR OF THE SEMIGROUP GENERATED BY THE BLACK-SCHOLES OPERATOR

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Abstract. The paper is devoted to a careful analysis of the shape-preserving properties of the strongly continuous semigroup generated by a particular second-order differential operator, with particular emphasis on the preservation of higher order convexity and Lipschitz classes. In addition, the asymptotic behaviour of the semigroup is investigated as well. The operator considered is of interest, since it is a unidimensional Black-Scholes operator so that our results provide qualitative information on the solutions of classical problems in option pricing theory in Mathematical Finance.

Keywords: strongly continuous semigroups, differential operators, positive linear operators, Black-Scholes operator

MSC 2000: 47D06, 47E05, 41A35, 41A36

1. Introduction and notation

The present paper is focused upon studying qualitative and asymptotic properties of the strongly continuous semigroup $(S_m(t))_{t\geqslant 0}$ generated by a second-order differential operator of the form

$$Lu(x) := \frac{\sigma^2}{2}x^2u''(x) + rxu'(x) - ru(x) \quad (x \geqslant 0,\ \sigma > 0,\ r \geqslant 0), \tag{1.1}$$

acting on a suitable domain, in the setting of weighted continuous function spaces.

Similar operators frequently occur in Mathematical Finance: really, they typically arise when setting up theoretical models in no-arbitrage pricing theory. In this respect (1.1) may be regarded as a unidimensional *Black-Scholes operator* in its

The paper is dedicated to Professor Luigi Albano on the occasion of his 70th birthday.

simplest form, with volatility and riskless interest rate constant over time and equal to σ and r, respectively.

The existence of the semigroup $(S_m(t))_{t\geqslant 0}$ generated by (1.1), together with its deep connection with suitable Markov processes, has been established in [3]. In the same framework, in [1] and [2] the authors, according with a general scheme of investigation of the interplay between constructive approximation processes and degenerate evolution problems (see, for instance, [1]–[5] and many of the references quoted therein), have shown that $(S_m(t))_{t\geqslant 0}$ may be written down in terms of suitable Post-Widder-type operators Q_n, as described in (2.2).

Our purpose hereby is to emphasize how representation (2.2), far from being merely a theoretical result, turns out to be a powerful tool when investigating qualitative properties of the semigroup and, consequently, of the (mild or strong) solution of the PDE problem (2.6), which, as it is well-known, is given by $S_m(t)f$ at any time $t > 0$.

From this viewpoint the paper is organized as follows. We first provide an integral representation of $(S_m(t))_{t\geqslant 0}$ by using some methods developed, in a less general case, in [4] and based essentially upon the study of the stochastic differential equation associated with L. In this way we are able to determine the existence of an (oblique) asymptote of $S_m(t)f$ at $+\infty$.

In the remaining part of the paper, referring solely to the representation (2.2) of the semigroup through the Q_n's, we show how each $S_m(t)$ preserves higher order convexity and Lipschitz classes: here we apply some general results (developed in [5] and using the notion of total positivity as a starting point) about the preservation properties of positive linear operators having a suitable integral representation and of their limiting strongly continuous semigroups, if any (see [5, Theorem 2.7 and, in particular, representation (2.10)]).

A finer analysis is devoted to the properties of $S_m(t)f$, f being a convex function with linear growth; the asymptotic behaviour of the semigroup as $t \to +\infty$ is investigated, as well.

Finally, an application to the classical Black-Scholes problem for European call and put options is presented: in this respect, we point out how some of our results perfectly match the analogous ones supplied in [7] and [9] but proved through quite different techniques.

The notation we use throughout the paper is standard enough. If $k \geqslant 1$ is an integer and I is a real interval, by $C^k(I)$ we denote the vector space of all real-valued k-times continuously differentiable functions on I. As usual, $C(I)$ and $L^1(I)$ stand for the vector space of all real-valued continuous and Lebesgue integrable, respectively, functions on I. For any real $\lambda \geqslant 0$, we denote by e_λ the power function $e_\lambda(x) := x^\lambda$ $(x \geqslant 0)$.

2. The semigroup

For every integer $m \geqslant 1$ let us set

$$E_m^0 := \Big\{ f \in C([0,+\infty[) \colon \lim_{x\to+\infty} \frac{f(x)}{1+x^m} = 0 \Big\},$$

which becomes a Banach space with respect to the weighted norm $\|f\|_m := \sup_{x\geqslant 0}(|f(x)|/(1+x^m))$. Given $\sigma > 0$ and $r \geqslant 0$, consider the differential operator

$$Lu(x) := \frac{\sigma^2}{2} x^2 u''(x) + rxu'(x) - ru(x) \quad (x \geqslant 0)$$

acting on the domain

$$D_m(L) := \Big\{ u \in E_m^0 \cap C^2(]0,+\infty[) \colon \lim_{x\to 0^+} \Big(\frac{\sigma^2}{2}x^2u''(x) + rxu'(x)\Big) = \lim_{x\to+\infty} \frac{1}{1+x^m}\Big(\frac{\sigma^2}{2}x^2u''(x) + rxu'(x)\Big) = 0 \Big\}.$$

For each integer $n \geqslant r/\sigma^2$, let us denote by Q_n the positive linear operator defined as

$$Q_n f(x) := \Big(1 - \frac{r}{n\sigma^2}\Big)\Big(\frac{n^2\sigma^2}{(n\sigma^2+r)x}\Big)^n \frac{1}{\Gamma(n)} \times \int_0^{+\infty} \exp\Big(-\frac{n^2\sigma^2 u}{(n\sigma^2+r)x}\Big) u^{n-1} f(u)\,\mathrm{d}u, \tag{2.1}$$

where $f \in E_m^0$ and $x \geqslant 0$.

The main results concerning $(L, D_m(L))$ and the sequence $(Q_n)_{n\geqslant r/\sigma^2}$ are listed below.

Theorem 2.1 ([3]). *The operator $(L, D_m(L))$ is the generator of a strongly continuous positive semigroup $(S_m(t))_{t\geqslant 0}$ on E_m^0.*

Theorem 2.2 ([1]). *Let $m \geqslant 2$. Then for each $f \in E_m^0$ and $t \geqslant 0$,*

$$S_m(t)f = \lim_{n\to+\infty} Q_n^{k(n)} f \quad \text{in } E_m^0, \tag{2.2}$$

where $(k(n))_{n\geqslant 1}$ is an arbitrary sequence of positive integers such that

$$\lim_{n\to+\infty} k(n)/n = \sigma^2 t.$$

It is extremely useful for the sequel to obtain an explicit representation of the semigroup $(S_m(t))_{t\geqslant 0}$ which is easier to handle than (2.2). This is actually provided in the next theorem, which, however, covers an interest on its own.

Theorem 2.3. *Let $m \geqslant 1$ and $t > 0$. The operator $S_m(t)\colon E_m^0 \longrightarrow E_m^0$ has the integral representation*

$$(2.3) \qquad S_m(t)f(x) = \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\Bigl(x \cdot \exp\Bigl(\sigma u + \Bigl(r - \frac{\sigma^2}{2}\Bigr)t\Bigr)\Bigr) \exp(-u^2/2t)\,\mathrm{d}u,$$

where $f \in E_m^0$ and $x \geqslant 0$.

P r o o f. For $r = 0$ and $\sigma = 1$, the proof may be found in [4, Theorem 4.4]. In the general case $r \geqslant 0$, $\sigma > 0$, it runs essentially in the same way except for some slight changes somewhere and is therefore omitted for the sake of brevity; note that for each $t \geqslant 0$ (compare with [4, Formula (5), p. 270])

$$(2.4) \qquad \|S_m(t)\| = \exp\Bigl((m-1)\Bigl(\frac{m\sigma^2}{2} + r\Bigr)t\Bigr).$$

Moreover, since $Le_\lambda = (\lambda - 1)\bigl(\frac{1}{2}\lambda\sigma^2 + r\bigr)e_\lambda$ $(0 \leqslant \lambda < m)$, a standard argument from semigroup theory yields

$$(2.5) \qquad S_m(t)e_\lambda = e_\lambda \cdot \exp\Bigl((\lambda - 1)\Bigl(\frac{\lambda\sigma^2}{2} + r\Bigr)t\Bigr) \quad \text{for all } t \geqslant 0.$$

□

Remark. For a given $f \in E_m^0$, the function $u(x,t) := S_m(t)f(x)$ is a *mild solution* of the Black-Scholes equation

$$\begin{cases} u_t(x,t) = \dfrac{1}{2}\sigma^2 x^2 u_{xx}(x,t) + rxu_x(x,t) - ru(x,t) & (x \geqslant 0,\ t > 0), \\ u(x,0) = f(x) \quad (x \geqslant 0), \end{cases}$$

with $f \in E_m^0$ as the initial datum.

Observe that, according to the probabilistic scheme which underlies the Black-Scholes model, the above solution may also be obtained directly, in a different way, by using the Feynman-Kac formula, i.e., through merely probabilistic techniques (see, e.g., [11, Theorem 8.6, p. 128]).

Consider the above semigroup $(S_m(t))_{t\geqslant 0}$ for $m\geqslant 1$.

Theorem 3.1. *If $f\in E_m^0$ has the asymptote $y=ax+b$ as $x\to+\infty$, then each $S_m(t)f$ has the asymptote $y=ax+b\mathrm{e}^{-rt}$ as $x\to+\infty$.*

Proof. The assertion being clearly obvious for $t=0$, let us choose $t\geqslant 0$; using Theorem 2.3 we compute

$$\tag{3.1}\begin{aligned}\lim_{x\to+\infty}\frac{S_m(t)f(x)}{1+x} = \lim_{x\to+\infty}\frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{f(x\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t})}{1+x\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}}\\ \times\frac{1+x\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}}{1+x}\mathrm{e}^{-\frac{1}{2}u^2/t}\,\mathrm{d}u.\end{aligned}$$

The function $y\mapsto|f(y)|/(1+y)$ is continuous on $[0,+\infty[$ and converges to $|a|$ when $y\to+\infty$ by assumption; therefore it is bounded (let us say, by M) on $[0,+\infty[$. On the other hand, for any given $x\geqslant 0$

$$\frac{1+x\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}}{1+x}\leqslant\max\{1,\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}\}\quad\text{for all }u\in\mathbb{R}$$

so that the absolute value of the integrand in (3.1) is bounded from above by $M\cdot\psi$, where $\psi(u):=\sup\{\mathrm{e}^{-\frac{1}{2}u^2/t},\mathrm{e}^{-(u-\sigma t)^2/(2t)+rt}\}$ $(u\in\mathbb{R})$.

Since $\psi\in L^1(\mathbb{R})$, the application of the dominated convergence theorem in (3.1) immediately yields

$$\begin{aligned}\lim_{x\to+\infty}\frac{S_m(t)f(x)}{1+x}&=\frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}}\int_{\mathbb{R}}a\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{-\frac{1}{2}u^2/t}\,\mathrm{d}u\\ &=a\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\mathrm{e}^{-(u-\sigma t)^2/(2t)}\,\mathrm{d}u=a\end{aligned}$$

or, equivalently,

$$\tag{3.2}\lim_{x\to+\infty}\frac{S_m(t)f(x)}{x}=a.$$

Now it is easily seen that for any $x\geqslant 0$

$$\tag{3.3} S_m(t)f(x)-ax=\frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\big(f(x\cdot\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t})-ax\cdot\mathrm{e}^{\sigma u+(r-\frac{1}{2}\sigma^2)t}\big)\mathrm{e}^{-\frac{1}{2}u^2/t}\,\mathrm{d}u.$$

The function $y \mapsto |f(y) - ay|$ is continuous on $[0, +\infty[$, tends to $|b|$ as $y \to +\infty$, whence it is bounded on $[0, +\infty[$. The dominated convergence theorem does the job as before and we get

$$\lim_{x \to +\infty} (S_m(t)f(x) - ax) = \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}} \int_{\mathbb{R}} b\mathrm{e}^{-\frac{1}{2}u^2/t}\,\mathrm{d}u = b\mathrm{e}^{-rt},$$

which, together with (3.2), accomplishes the proof. □

4. Total positivity and preservation of higher order convexity and Lipschitz classes

In this section we shall use the same notation, definitions and results as those appearing in [5]. It is not difficult to see that the Q_n's defined in (2.1) are positive linear operators with totally positive kernels (see [5, Section 2, Example 3.4]).

A direct computation gives

$$Q_n e_k = a_{nk} e_k \quad \text{for every} \quad k = 0, 1, \ldots, m-1 \tag{4.1}$$

where explicitly

$$a_{nk} := \Big(1 - \frac{r}{n\sigma^2}\Big)\Big(1 + \frac{r}{n\sigma^2}\Big)^k \prod_{j=0}^{k-1}\Big(1 + \frac{j}{n}\Big). \tag{4.2}$$

Consequently (see [5, (2.8)]),

$$l_k := \lim_{n \to +\infty} (a_{nk})^n = \exp\bigg((k-1)\Big(\frac{r}{\sigma^2} + \frac{k}{2}\Big)\bigg). \tag{4.3}$$

Now let $q \geqslant 0$ be an integer. A function $f \in E_m^0$ is called q-convex if all its divided differences on $q + 1$ points in $[0, +\infty[$ are nonnegative; if f is q-times continuously differentiable, then f is q-convex iff $f^{(q)} \geqslant 0$ on $[0, +\infty[$.

If $M > 0$, we say that $f \in E_m^0$ belongs to the Lipschitz class $\mathrm{Lip}_q(M)$ if

$$|\Delta_h^q f(x)| \leqslant M h^q$$

for all $x \geqslant 0$ and $h > 0$; here, as usual, $\Delta_h^q f(x)$ denotes the q-th order difference of f with step h at x, i.e.,

$$\Delta_h^q f(x) := \sum_{i=0}^{q} (-1)^i \binom{q}{i} f(x + (q-i)h).$$

Taking into account (4.1), (4.2) and (4.3), from [5, Theorem 2.7] and the subsequent remarks, we deduce the next result concerning some shape-preserving properties of the semigroup $(S_m(t))_{t \geqslant 0}$.

Theorem 4.1. *Let $t \geqslant 0$. Then the following statements hold true:*

(a) *If $f \in E_m^0$ is q-convex, then $S_m(t)f$ is q-convex for all $q = 0, 1, \ldots, m$.*

(b) *$S_m(t)(E_m^0 \cap \operatorname{Lip}_q(M)) \subset \operatorname{Lip}_q(M \cdot \exp((q-1)(\frac{1}{2}q\sigma^2 + r)t))$ for all $M > 0$ and $q = 0, 1, \ldots, m-1$. In particular, $S_m(t)(E_m^0 \cap \operatorname{Lip}_1(M)) \subset \operatorname{Lip}_1(M)$.*

This result can be viewed as expressing the qualitative properties of the solution of problem (2.6) (see [5, Remark 2.10]).

We also observe that in [1] and [2] the authors proved (a) for $q = 1, 2$ and the inclusion

$$S_m(t)(E_m^0 \cap \operatorname{Lip}_1(M)) \subset \operatorname{Lip}_1\Big(M \cdot \exp\Big(\frac{2rt}{\sigma^2}\Big)\Big),$$

which should be compared with the corresponding (b) just stated above.

The preservation of 2-convexity is a financially significant qualitative property of the solution of (2.6); the corresponding financial interpretation is presented in [7] (see also [9, Remark 3.17 (2), p. 108]).

Remark. Two different approaches are discussed in [7] in order to prove the preservation of 2-convexity. By virtue of [5, Theorem 2.3], the above Theorem 4.1 may also be proved without passing through the representation (2.2), but simply checking out the total positivity of the kernel in the integral representation (2.3) and taking into account (2.5). The existence of the explicit integral representation (2.3) of the semigroup is, in this sense, a very lucky circumstance: in general, we are not aware of the existence of such integral representations for the semigroups considered in [5], where, consequently, the preservation properties are studied using only representations of the type (2.2) (see [5, Theorem 2.7 and formula (2.10)]).

For other preservation results in similar frameworks we refer the reader, for instance, to [6], [8] and [10].

5. Convex functions with linear growth

In this section we consider 2-convex functions, i.e., classical convex functions.

Proposition 5.1. *Let $f \in E_m^0$ be a convex function with $f(0) \leqslant 0$ and $t \geqslant s \geqslant 0$. Then $S_m(t)f \geqslant S_m(s)f \geqslant f$.*

Proof. For a given $x_0 > 0$ there exist $p, q \in \mathbb{R}$ such that $pe_1 + qe_0 \leqslant f$ and $px_0 + q = f(x_0)$. According to (2.5), we already know that for all $t \geqslant 0$

$$S_m(t)e_0 = \mathrm{e}^{-rt}e_0, \quad S_m(t)e_1 = e_1, \tag{5.1}$$

and therefore, since $q \leqslant f(0) \leqslant 0$, one gets

$$pe_1 + qe_0 \leqslant pe_1 + q\mathrm{e}^{-rt}e_0 = S_m(t)(pe_1 + qe_0) \leqslant S_m(t)f.$$

In particular,

$$f(x_0) = px_0 + q = (pe_1 + qe_0)(x_0) \leqslant S_m(t)f(x_0),$$

which, together with $S_m(t)f(0) = \mathrm{e}^{-rt}f(0) \geqslant f(0)$ (see (2.3)), yields $S_m(t)f \geqslant f$ for all $t \geqslant 0$.

Now choose $t \geqslant s \geqslant 0$ and recall that, due to (a) in Theorem 4.1, the function $S_m(s)f$ is convex as well, satisfying, in addition, $S_m(s)f(0) = \mathrm{e}^{-rs}f(0) \leqslant 0$. Applying what was just proved to $S_m(s)f$ gives

$$S_m(t-s)(S_m(s)f) \geqslant S_m(s)f,$$

i.e., $S_m(t)f \geqslant S_m(s)f \geqslant f$, which is the desired conclusion. □

Proposition 5.2 (Asymptotic behaviour). *Let $a, b, c \in \mathbb{R}$ and $f \in E_m^0$ be such that*

$$ae_1 + be_0 \leqslant f \leqslant ae_1 + ce_0. \tag{5.2}$$

Then for all $t \geqslant 0$

$$b\mathrm{e}^{-rt}e_0 \leqslant S_m(t)f - ae_1 \leqslant c\mathrm{e}^{-rt}e_0 \tag{5.3}$$

and, consequently,

$$\lim_{t\to+\infty} S_m(t)f = ae_1 \quad \textit{uniformly on } [0, +\infty[. \tag{5.4}$$

P r o o f. It is a direct consequence of (5.1) and of the positivity of the semigroup. □

Let us remark that all the functions f described in [9, Figs. 3.5–3.8, pp. 148–149] satisfy (5.2) with suitable constants $a, b, c \in \mathbb{R}$.

In the next proposition, which basically follows from Theorem 3.1 and Propositions 5.1 and 5.2, we collect the main properties of $S_m(t)f$, f being a convex function with linear growth.

Corollary 5.1. *Let $f \in E_m^0$ be a convex function satisfying* (5.2)*. Then the limit*

$$l := \lim_{x\to+\infty} (f(x) - ax) \tag{5.5}$$

exists and is finite; in addition, the following statements hold true:

(a) *For each $t \geqslant 0$, $S_m(t)f$ has the asymptote $y = ax + l\mathrm{e}^{-rt}$ as $x \to +\infty$.*

(b) *The sharp inequalities*

$$ae_1 + l\mathrm{e}^{-rt}e_0 \leqslant S_m(t)f \leqslant ae_1 + f(0)\mathrm{e}^{-rt}e_0 \quad (t \geqslant 0) \tag{5.6}$$

are satisfied and, consequently, (5.4) *is still valid.*

(c) *If, in addition, $f(0) \leqslant 0$, then the family of convex functions $(S_m(t)f)_{t\geqslant 0}$ is increasing with respect to t and satisfies $S_m(t)f \geqslant f$.*

P r o o f. The function $f - ae_1$, being convex and bounded on $[0, +\infty[$ because of our assumption and (5.2), is necessarily decreasing and therefore the limit l in (5.5) exists and, specifically, $b \leqslant l \leqslant c$. Moreover, $y = ax + l$ is the asymptote of f as $x \to +\infty$, and

$$le_0 \leqslant f - ae_1 \leqslant f(0)e_0,$$

which easily implies (a) (by virtue of Theorem 3.1) and (b). To see that the left-hand side inequality in (5.6) is sharp it suffices to let $x \to +\infty$; for the right-hand side inequality, take $x = 0$. The proof is now accomplished, since (c) restates Proposition 5.1. □

6. The function $\varphi(x) := (x - K)^+$

Let $K > 0$, $\varphi(x) := (x - K)^+$ $(x \geqslant 0)$ and assume $m \geqslant 2$, so that $\varphi \in E_m^0$.

Thus φ is a positive, increasing and convex function in $\mathrm{Lip}_1(1)$ with $e_1 - Ke_0 \leqslant \varphi \leqslant e_1$; moreover, according to (5.5), $l = -K$ and, of course, $\varphi(0) = 0$. We are therefore in a position to apply Corollary 5.1 (see also Theorem 4.1).

Corollary 6.1. *The following assertions are true:*

(i) *$(S_m(t)\varphi)_{t\geqslant 0}$ is an increasing (with respect to t) family of positive, increasing, convex functions in* $\mathrm{Lip}_1(1)$ *such that for any $t \geqslant 0$, $\varphi \leqslant S_m(t)\varphi \leqslant e_1$ (whence $S_m(t)\varphi(0) = 0$).*

(ii) *Each $S_m(t)\varphi$ has the asymptote $y = x - K\mathrm{e}^{-rt}$ as $x \to +\infty$.*

(iii) *$(e_1 - K\mathrm{e}^{-rt}e_0)^+ \leqslant S_m(t)\varphi \leqslant e_1$ for all $t \geqslant 0$.*

(iv) *$\lim_{t\to+\infty} S_m(t)\varphi = e_1$ uniformly on $[0, +\infty[$.*

Consider now the function $h(p) := (p-1)p^{p/(1-p)}$ $(p > 1)$. It may be readily shown that h is strictly increasing on $]1, +\infty[$ and

$$\lim_{p\to 1^+} h(p) = 0, \qquad \lim_{p\to +\infty} h(p) = \sup_{p>1} h(p) = 1. \tag{6.1}$$

The next result provides information about the behaviour of $S_m(t)\varphi(x)$ when $x \to 0^+$.

Proposition 6.1. *If the positive constant K appearing in φ fulfils $K \geqslant h(p)$ for some $p \in]1, m[$, then*

$$0 \leqslant S_m(t)\varphi \leqslant e_p \cdot \exp\Big((p-1)\Big(\frac{p\sigma^2}{2} + r\Big)t\Big) \tag{6.2}$$

for all $t \geqslant 0$ whence $(\mathrm{d}/\mathrm{d}x)S_m(t)\varphi(x)|_{x=0} = 0$. In particular, if $K \geqslant 1$, then (6.2) holds true for all $p \in]1, m[$.

Proof. By direct computation one may easily check that in the case $K = h(p)$ for a certain $p \in]1, m[$, the graphs of φ and e_p are tangent at a point with the abscissa $p^{1/(1-p)}$ and therefore $0 \leqslant \varphi \leqslant e_p$, which in turn implies (6.2) on account of (2.5). Of course, the same happens *a fortiori* if $K > h(p)$. The last part of the assertion is a consequence of (6.1), because $1 \geqslant h(p)$ for all $p \in]1, m[$. □

Remarks.

(1) By using Theorem 2.3 and [11, Exercise 8.6, p. 154], we deduce that

$$u(x,t) := S_m(t)f(x) = \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \Big(x \cdot \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma u\Big) - K\Big)^+ \mathrm{e}^{-\frac{1}{2}u^2/t}\,\mathrm{d}u$$

is the solution of (2.6) with $f(x) = (x-K)^+$ $(x \geqslant 0)$.

(2) The inequalities (iii) in Corollary 6.1 may be found, with different notation, in [9, p. 124, Prop. 3.29, (3.39)], where they are proved by using financial arguments.

(3) The function $\varphi(x) = (x-K)^+$ is important in connection with *European call options*. When discussing *European put options*, a similar role is played by the function $\psi(x) := (K-x)^+$ (see, for instance, [9, Chap. 3]). The methods presented above can be used in order to investigate the properties of the functions $S_m(t)\psi$, too. In particular, since $\varphi - \psi = e_1 - Ke_0$, for all $t \geqslant 0$ we have

$$S_m(t)\psi = S_m(t)\varphi - e_1 + K\mathrm{e}^{-rt}e_0.$$

(4) We finally point out that more realistic and, consequently, more sophisticated models in option pricing theory lead to a multidimensional (non autonomous) Black-Scholes operator; the study of this case, however, lying beyond the limits of the present paper, will be the object of our future investigations.

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