J. L. Díaz-Barrero; J. J. Egozcue

A generalization of the Gauss-Lucas theorem

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 481–486

Persistent URL: `http://dml.cz/dmlcz/128271`

Czechoslovak Mathematical Journal, 58 (133) (2008), 481–486

A GENERALIZATION OF THE GAUSS-LUCAS THEOREM

J. L. Díaz-Barrero and J. J. Egozcue, Barcelona

(Received April 5, 2006)

Abstract. Given a set of points in the complex plane, an incomplete polynomial is defined as the one which has these points as zeros except one of them. The classical result known as Gauss-Lucas theorem on the location of zeros of polynomials and their derivatives is extended to convex linear combinations of incomplete polynomials. An integral representation of convex linear combinations of incomplete polynomials is also given.

Keywords: polynomials, location of zeros, convex hull of the zeros, Gauss-Lucas theorem

MSC 2000: 12D10, 26C05, 30C15

1. Introduction

The first to give a mechanical interpretation of the zeros of $A'(z)$ was Gauss in a note dated between 1836 and 1846 [1] found in a memorandum book devoted to astronomy. In 1870 Lucas [2] stated and proved the same theorem from which he got an immediate corollary known in the literature as the Gauss-Lucas theorem. It states that all the critical points of a nonconstant univariate polynomial $A(z)$ lie in the convex hull $H(z_1, z_2, \ldots, z_n)$ of the zeros of $A(z)$. Furthermore, if the zeros of $A(z)$ are not collinear and simple, then no critical point lies on the boundary of $H(z_1, z_2, \ldots, z_n)$.

In this note we are concerned with the study of the convex hull of the zeros of complex polynomials using incomplete polynomials [3]. As a result, a generalization of the classical and well known Gauss-Lucas theorem [4] is obtained.

2. Main results

The main result in this section is a theorem which generalizes Gauss-Lucas theorem on the location of zeros of polynomials and its derivatives. It is based on the concept of incomplete polynomial. A definition follows.

Definition 1 (Incomplete polynomials). Let $z_1, z_2, \dots, z_n$ be n, not necessarily distinct, complex numbers. The *incomplete polynomials* of degree $n-1$, associated with $z_1, z_2, \dots, z_n$, are the polynomials $g_k(z)$, $1 \leqslant k \leqslant n$, given by

$$g_k(z) = \prod_{\substack{j=1\\ j\neq k}}^{n} (z - z_j) \tag{1}$$

Let $A_n(z)$ be a monic polynomial whose zeroes are $z_1, z_2, \dots, z_n$. That is,

$$A_n(z) = \prod_{k=1}^{n} (z - z_k).$$

Notice that the derivative of $A_n(z)$, normalized to a monic polynomial, is a convex linear combination of incomplete polynomials. In fact, its derivative, reduced to monic, is

$$\frac{1}{n}\ A_n'(z) = \sum_{k=1}^{n} \frac{1}{n} g_k(z),$$

where all the coefficients of the convex linear combination are $1/n$. Convex linear combinations of incomplete polynomials are used throughout this development and, for simplicity, the following notation is introduced. Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ be nonnegative real numbers such that $\sum\limits_{k=1}^{n} \gamma_k = 1$. The corresponding convex linear combination of incomplete polynomials is denoted $A_n^{\gamma}(z) = \sum\limits_{k=1}^{n} \gamma_k g_k(z)$. Note that $A_n^{\gamma}(z)$ is a polynomial of degree $n-1$. As pointed out, the derivative, normalized to be monic is then one of such convex linear combinations.

The fact that the monic derivative is a convex linear combination of incomplete polynomials motivates the generalization of the Gauss-Lucas theorem to such kind of polynomials.

Theorem 2. *Let $z_1, z_2, \dots, z_n$ be n, not necessarily distinct, complex numbers. Then, the polynomial $A_n^{\gamma}(z) = \sum\limits_{k=1}^{n} \gamma_k g_k(z)$ has all its zeros in or on the convex hull $H(z_1, z_2, \dots, z_n)$ of the zeros of $A_n(z) = \prod\limits_{k=1}^{n} (z - z_k)$.*

Before giving the proof of the theorem we need the following lemma.

Lemma 1. *Let $z_1, z_2, \ldots, z_n$ be complex numbers, some of them being non-zero, and such that*

$$\omega \leqslant \arg z_k < \omega + \pi, \ 1 \leqslant k \leqslant n, \tag{2}$$

where $\omega \in \mathbb{R}$, then their sum $\sum\limits_{k=1}^{n} z_k$ cannot vanish.

Proof. Geometrically obvious. □

Notice that the preceding lemma also establishes that the point $z = \sum\limits_{k=1}^{n} z_k$ lies inside the convex sector consisting of the origin and all the points z for which $\omega \leqslant \arg z < \omega + \pi$, if all the points z_k, $1 \leqslant k \leqslant n$, lie in the same sector.

Proof of theorem 2. First, we write the polynomial $A_n^{\gamma}(z)$ as

$$A_n^{\gamma}(z) = A_n(z)\frac{A_n^{\gamma}(z)}{A_n(z)} = A_n(z)B_n(z),$$

where

$$B_n(z) = \frac{A_n^{\gamma}(z)}{A_n(z)} = \frac{1}{A_n(z)}\sum_{k=1}^{n}\gamma_k g_k(z) = \sum_{k=1}^{n}\frac{\gamma_k}{z - z_k},$$

and we proceed by contradiction. In fact, suppose that $z^{\sharp}$ is a zero of $A_n^{\gamma}(z)$ outside $H(z_1, z_2, \ldots, z_n)$ and let $\varphi(z^{\sharp})$ be the angle subtended at $z^{\sharp}$ by H. Then, by the convexity of H, we have $0 \leqslant \varphi(z^{\sharp}) < \pi$.

Now, we consider the vectors $v_k = \gamma_k/(\overline{z}_k - \overline{z}^{\sharp})$, $\gamma_k \in \mathbb{R}$ and $\gamma_k > 0$. We claim that the vectors v_k lie inside the convex sector $\varphi(z^{\sharp})$. Indeed, $w_k = z_k - z^{\sharp} \in \varphi(z^{\sharp})$. Its conjugate inverse is

$$\overline{w}_k^{-1} = \overline{\left[\frac{1}{z_k - z^{\sharp}}\right]} = \overline{\left[\frac{\overline{z}_k - \overline{z}^{\sharp}}{|z_k - z^{\sharp}|^2}\right]} = \frac{z_k - z^{\sharp}}{|z_k - z^{\sharp}|^2} \in \varphi(z^{\sharp})$$

and $v_k = \gamma_k/\overline{w}_k = \gamma_k\overline{w}_k^{-1} \in \varphi(z^{\sharp})$. Then, according to the previous lemma, every sum of the vectors v_k cannot vanish. Therefore, $B_n(z^{\sharp}) \neq 0$ and $A_n^{\gamma}(z^{\sharp}) = A_n(z^{\sharp})B(z^{\sharp}) \neq 0$. As this contradicts our hypothesis ($z^{\sharp}$ is a zero of $A_n^{\gamma}(z)$ outside H), no zero of $A_n^{\gamma}(z)$ can be outside $H(z_1, z_2, \ldots, z_n)$ and the theorem is proved.

An immediate consequence of the last theorem is the following

Corollary 1 (Gauss-Lucas Theorem)**.** *The convex hull of the zeros of a polynomial $A_n(z)$ contains all the zeros of its derivative $A_n'(z)$.*

Proof. Setting $\gamma_k = 1/n$, $1 \leqslant k \leqslant n$ in the preceding theorem the result immediately follows. $\square$

Note that in the case of multiple zeros of $A_n(z)$, the $\gamma_k{}'s$ are still equal to $1/n$, but repeated as many times as the multiplicity of the repeated zero.

Next, we consider convex linear combinations of incomplete polynomials and we obtain another generalization of Gauss-Lucas theorem. Let $A_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic complex polynomial whose zeros $z_1, z_2, \ldots, z_m$ have multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_m$, $\sum_{k=1}^{m} \alpha_k = n$. The incomplete polynomials associated with $A_n(z)$ can be expressed as $g_k(z) = \prod_{j=1}^{m} (z - z_j)^{\alpha_j - \delta_{jk}}$, $1 \leqslant k \leqslant m$, where δ_{jk} is the Kronecker delta. We denote by $A_n^{\boldsymbol{\gamma}}(z)$ the polynomial of degree $n-1$ defined by $A_n^{\boldsymbol{\gamma}}(z) = \sum_{k=1}^{m} \gamma_k g_k(z)$ where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, with $\gamma_k \in \mathbb{R}$ $(1 \leqslant k \leqslant m)$. The following theorem holds.

Theorem 3. *Let $A_n(z)$ be a monic complex polynomial whose zeros $z_1, z_2, \ldots, z_m$ have multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_m$, $\sum_{k=1}^{m} \alpha_k = n$. Then, a monic polynomial of degree $n-1$ is a convex linear combination of incomplete polynomials, with a vector of positive components $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, $\sum_{k=1}^{m} \gamma_k = 1$, namely $A_n^{\boldsymbol{\gamma}}(z) = \sum_{k=1}^{m} \gamma_k g_k(z)$, if and only if,*

$$
\begin{gathered}
\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{A_n^{\boldsymbol{\gamma}}(z)}{A_n(z)} \, \mathrm{d}z = 1, \\
\operatorname{Res}\Big(\frac{A_n^{\boldsymbol{\gamma}}(z)}{A_n(z)}, z = z_k \Big) \geqslant 0, \; k = 1, 2, \ldots, m,
\end{gathered}
$$

for a contour $\mathcal{C}$ containing all the zeros of $A_n(z)$.

Proof. $\Rightarrow$) Since $A_n^{\boldsymbol{\gamma}}(z) = \sum_{k=1}^{m} \gamma_k g_k(z)$ with $\gamma_k \geqslant 0$, $1 \leqslant k \leqslant m$, and $\sum_{k=1}^{m} \gamma_k = 1$, then

$$
\begin{aligned}
\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{A_n^{\boldsymbol{\gamma}}(z)}{A_n(z)} \, \mathrm{d}z &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \sum_{k=1}^{m} \frac{\gamma_k g_k(z)}{A_n(z)} \, \mathrm{d}z \\
&= \sum_{k=1}^{m} \gamma_k \, \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{g_k(z)}{A_n(z)} \, \mathrm{d}z = \sum_{k=1}^{m} \gamma_k = 1.
\end{aligned}
$$

Moreover, the residues satisfy

$$\begin{aligned}\operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{A_n(z)}, z=z_k\right) &= \operatorname{Res}\left(\sum_{j=1}^{m}\frac{\gamma_j g_j(z)}{A_n(z)}, z=z_k\right)\\ &= \operatorname{Res}\left(\sum_{j=1}^{m}\frac{\gamma_j}{z-z_j}, z=z_k\right) = \gamma_k \geqslant 0.\end{aligned}$$

$\Leftarrow$) We consider the integral

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{A_n^{\gamma}(z)}{A_n(z)}\frac{dz}{z-\zeta} \tag{3}$$

where $\mathcal{C}$ is a circle centered at the origin and of radius r with $r > \max\limits_{1\leqslant k\leqslant m}\{|z_k|\}$, and $|\zeta| < r$. First, evaluation of (3) outside of contour $\mathcal{C}$ is always zero provided that the degree of $A_n^{\gamma}(z)$ is at most $n-1$. Hence, assuming that the degree of $A_n^{\gamma}(z)$ is $n-1$, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{A_n^{\gamma}(z)}{A_n(z)}\frac{dz}{z-\zeta} &= \operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{(z-\zeta)A_n(z)}, z=\zeta\right) + \sum_{i=1}^{m}\operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{(z-\zeta)A_n(z)}, z=z_k\right)\\ &= \frac{A_n^{\gamma}(\zeta)}{A_n(\zeta)} + \sum_{i=1}^{m}\operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{(z-\zeta)A_n(z)}, z=z_k\right) = 0.\end{aligned}$$

Taking into account that

$$\operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{(z-\zeta)A_n(z)}, z=z_k\right) = -\frac{1}{\zeta-z_k}\operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{A_n(z)}, z=z_k\right)$$

and the preceding equality, we obtain

$$\frac{A_n^{\gamma}(\zeta)}{A_n(\zeta)} = \sum_{k=1}^{m}\frac{1}{\zeta-z_k}\ \operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{A_n(z)}, z=z_k\right)$$

from which we get

$$A_n^{\gamma}(\zeta) = \sum_{k=1}^{m}\gamma_k g_k(\zeta)$$

where

$$\gamma_k = \operatorname{Res}\left(\frac{A_n^{\gamma}(z)}{A_n(z)}, z=z_k\right).$$

Note that $\sum\limits_{k=1}^{m}\gamma_k = 1$ implies that $A_n^{\gamma}(z)$ is a monic polynomial and the fact that $\gamma_k \geqslant 0$ $(1 \leqslant k \leqslant n)$ guarantees that $A_n^{\gamma}(z) = \sum\limits_{k=1}^{m}\gamma_k g_k(z)$ is a convex linear combination. $\square$

An immediate consequence of the preceding result is

Theorem 4 (Gauss-Lucas Theorem generalized)**.** *Let $A_n(z)$ be a monic complex polynomial whose zeros $z_1, z_2, \ldots, z_m$ have multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_m$, $\sum_{k=1}^{m} \alpha_k = n$, and let $A_{n-1}(z)$ be a polynomial of degree $n-1$ such that, for a contour $\mathcal{C}$ containing all the zeros of $A_n(z)$,*

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{A_{n-1}(z)}{A_n(z)}\,\mathrm{d}z = 1,$$
$$\operatorname{Res}\Big(\frac{A_{n-1}(z)}{A_n(z)}, z = z_k\Big) \geqslant 0,\ \ k = 1, 2, \ldots, m.$$

Then, all the zeros of $A_{n-1}(z)$ lie inside of the convex hull $H(z_1, z_2, \ldots, z_m)$.

Observe that setting $A_{n-1}(z) = A_n^{\gamma}(z)$ with $\boldsymbol{\gamma} = (1/n, 1/n, \ldots, 1/n)$ into the preceding result immediately follows the Gauss-Lucas Theorem.

Acknowledgements. The authors would thank to the anonymous referee for his/her comments and suggestions that have improve the final version of the paper and to the Ministry of Education of Spain that has partially supported this research by grant MTM2006-03040.

References

[1] *C. F. Gauss*: Collected Works. Leipzig, Teubner, 1900–1903, vol. 3, p. 112, and vol. 8, p. 32, and vol. 9, p. 187.

[2] *F. Lucas*: Propriétés géométriques des fractions rationelles. C. R. Acad. Sci. Paris *77* (1874), 431–433; *78* (1874), 140–144; *78* (1874), 180–183; *78* (1874), 271–274. zbl

[3] *J. L. Díaz-Barrero*: Characterization of Polynomials by Reflection Coefficients. PhD. Disertation (Advisor J. J. Egozcue), Universitat Politècnica de Catalunya, Barcelona, 2000.

[4] *M. Marden*: The Geometry of the Zeros of a Polynomial in a Complex Variable. American Mathematical Society, Rhode Island, 1966. zbl

Authors' address: J. L. Díaz-Barrero, J. J. Egozcue, Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 080 34 Barcelona, Spain, e-mail: `jose.luis.diaz@upc.edu`, `juan.jose.egozcue@upc.edu`.