Czechoslovak Mathematical Journal

Vy Khoi Le

On a sub-supersolution method for the prescribed mean curvature problem

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 541–560

Persistent URL: `http://dml.cz/dmlcz/128276`

Czechoslovak Mathematical Journal, 58 (133) (2008), 541–560

ON A SUB-SUPERSOLUTION METHOD FOR THE PRESCRIBED MEAN CURVATURE PROBLEM

Vy khoi Le, Rolla

(Received May 8, 2006)

Abstract. The paper is about a sub-supersolution method for the prescribed mean curvature problem. We formulate the problem as a variational inequality and propose appropriate concepts of sub- and supersolutions for such inequality. Existence and enclosure results for solutions and extremal solutions between sub- and supersolutions are established.

Keywords: variational inequality, sub-supersolution, enclosure, extremal solution, prescribed mean curvature problem

MSC 2000: 35J85, 53A10, 47J20

1. Introduction-problem setting

In this paper, we consider the following quasilinear equation describing a prescribed mean curvature problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(x,u) \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

The prescribed mean curvature problem is an important problem in the geometrical theory of partial differential equations and has been studied extensively by different methods. Classical existence theorems for this problem (and in particular for the minimal surface problem) are presented in [23] with references to the original papers by Finn, Bombieri/De Giorgi/Miranda, Jenkins, Serrin, etc. (see e.g. [5], [6], [18], [16], [24], [36] and the references therein). Here, we formulate (1.1) as a problem in the space of functions of bounded variation. This approach was developed in e.g. [5], [36], [20], [21], [22]. However, the existence theorems established in most of those papers are concerned with solutions of the prescribed mean curvature problem as global minimizers of the corresponding energy functionals. We also refer to [37], [38], [40], [4], [27], [13], [39], [34], [28], [14], [26], [31] and the references therein for recent discussions concerning solutions of the prescribed mean curvature problem. In [31], we proposed a variational (min-max) approach to study an eigenvalue problem related to (1.1) by formulating it as a variational inequality in a space of functions of bounded variation, which allows us to consider solutions other than global minimizers.

In our work here, we shall use the weak formulation given in [31]. As presented in that paper, the weak formulation of (1.1) is the following variational inequality:

$$\begin{cases} J(v) - J(u) - \int_\Omega f(x,u)(v-u)\,\mathrm{d}x \geqslant 0, \quad \forall v \in X, \\ u \in X. \end{cases} \tag{1.2}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with Lipschitz boundary, $\mathscr{B}$ is an open ball in $\mathbb{R}^N$ containing $\overline{\Omega}$, and

$$X = \{u \in BV(\mathscr{B})\colon\ u = 0 \ \text{ a.e. in } \ \mathscr{B} \setminus \Omega\}.$$

X is a (Banach) subspace of $BV(\mathscr{B})$ with the norm:

$$\|u\| = \|u\|_X = \int_{\mathscr{B}} |\nabla u|, \ \forall u \in X,$$

which is equivalent to the usual norm on $BV(\mathscr{B})$, defined by

$$\|u\|_{BV(\mathscr{B})} = \int_{\mathscr{B}} |u|\,\mathrm{d}x + \int_{\mathscr{B}} |\nabla u|, \ u \in BV(\mathscr{B}),$$

restricted to X. Here,

$$\int_{\mathscr{B}} |\nabla u| := \sup\Big\{ \int_{\mathscr{B}} u \operatorname{div} g\,\mathrm{d}x\colon\ g = (g_1,\dots,g_N) \in C_0^1(\mathscr{B},\mathbb{R}^N) \text{ and } \max_{x\in\mathscr{B}} |g(x)| \leqslant 1 \Big\},$$

$(\operatorname{div} g = \sum_{i=1}^{N} \partial_i g_i)$. The functional $J\colon\ BV(\mathscr{B}) \to \mathbb{R}$ is given by

$$J(v) = \int_{\mathscr{B}} [\sqrt{1+|\nabla v|^2} - 1] = \int_{\mathscr{B}} \sqrt{1+|\nabla v|^2} - |\mathscr{B}|, \tag{1.3}$$

where (cf. e.g. [24], [1])

$$\int_{\mathscr{B}} \sqrt{1+|\nabla u|^2} = \sup\Big\{ \int_{\mathscr{B}} (g_{n+1} + u \operatorname{div} g)\,\mathrm{d}x\colon\ g = (g_1, g_2, \dots, g_{n+1}) \in C_0^1(\mathscr{B},\mathbb{R}^{n+1}), \max_{x\in\mathscr{B}} |g(x)| \leqslant 1 \Big\}.$$

It is known that J is convex on $BV(\mathscr{B})$ and lower semicontinuous with respect to the $L^1(\mathscr{B})$-topology (cf. [31]). We refer to [1], [2], [6], [11], [12], [13], [16], [17], [24], [25], [41] and their references for the related properties of function of bounded variation, the BV space, and their relations to the prescribed mean curvature problem that we discuss here.

The goal of this paper is to start a systematic study of the boundary value problem (1.1), formulated as (1.2), by a sub-supersolution method. This method could, in many cases, give useful information not only on the existence of solutions of the problem but also on the structure of solution sets, such as their compactness or the existence of extremal (i.e., maximal and/or minimal) solutions. However, it seems that this powerful method, which has been applied widely to equations that contain Leray-Lions operators in Sobolev spaces, has not been employed so far to problems whose principal operators have linear growth, such as the prescribed mean curvature problem. We note that a sub-supersolution method for variational inequalities has been proposed recently in [30], [29] and extended to other types of inequalities in first-order Sobolev spaces $W^{1,p}$ (see for example [7], [9], [8], [10], [32] and the references therein). This approach has not been elaborated so far for equations or inequalities in nonreflexive Banach spaces such as the space of functions of bounded variation and for inequalities in which the potential functionals of the principal operators are nonsmooth. A new sub-supersolution approach is therefore needed for our present problem (1.1)–(1.2).

We are interested here in the existence and properties of solutions of the variational inequality (1.1) in the case where the lower order term $f(x,u)$ also depends on u. In this general case, the problem may no longer be coercive and thus may not have solutions. In our approach, in order to study the solution set by the sub-supersolution method, we need certain appropriate existence theorems for the variational inequality (1.2) in the case where the inequality is coercive (in some sense). However, it seems that such existence results have not been established in the literature for our problem here, even in coercive cases. Therefore, in a preparatory section, we consider the problem under various coercivity conditions and prove the corresponding existence theorems. Although those theorems are auxiliary results for our main discussion on sub-supersolution arguments for noncoercive problems, they seem new and have their own interest. In our main section, we define suitable concepts of sub- and supersolutions for (1.2) and next consider the existence together with other properties of solutions of (1.1)–(1.2) between sub- and supersolutions.

The paper is organized as follows. In the second section, we establish existence theorems for (1.2) under a number of coercivity conditions. Existence and enclosure properties of solutions of (1.1) between sub- and supersolutions are shown in Section 3. We also consider the existence of extremal solutions, that is, of smallest and greatest solutions of (1.1) between sub- and supersolutions.

2. Existence of solutions in coercive prescribed mean curvature problems

In this section, we study (1.1) in the coercive case. We consider some conditions that guarantee the existence of solutions of (1.2). The first condition is on the growth of $f(x,u)$ or on its anti-derivative $F(x,u)$ (in u), which is defined by

$$F(x,u)=\int_0^u f(x,t)\,\mathrm{d}t,\qquad x\in\mathscr{B}, u\in\mathbb{R}.$$

Assume that f is a Carathéodory function with the growth condition:

$$|f(x,u)|\leqslant A_1+B_1|u|^{q-1},\qquad x\in\mathscr{B}, u\in\mathbb{R}, \tag{2.1}$$

with $q\in(1,N/(N-1))$ and $A_1\in L^{q'}(\mathscr{B})$. It follows that F satisfies the following growth condition:

$$|F(x,u)|\leqslant A_2+B_2|u|^{q},\qquad x\in\mathscr{B}, u\in\mathbb{R}. \tag{2.2}$$

Also, we suppose that

$$f(x,u)=0,\qquad \text{for a.e. } x\in\mathscr{B}\setminus\Omega,\qquad \text{all } u\in\mathbb{R}. \tag{2.3}$$

This implies that $F(x,u)$ has the same property. From the continuous (in fact, compact) embedding

$$BV(\mathscr{B})\hookrightarrow L^q(\mathscr{B}), \tag{2.4}$$

we see that the functional $\mathscr{F}\colon BV(\mathscr{B})\to\mathbb{R}$ defined by

$$\mathscr{F}(u)=\int_{\mathscr{B}}F(x,u(x))\,\mathrm{d}x,$$

is of class C^1 on $BV(\mathscr{B})$ and

$$\langle\mathscr{F}'(u),v\rangle=\int_{\mathscr{B}}f(x,u)v\,\mathrm{d}x,\qquad \forall u,v\in BV(\mathscr{B}). \tag{2.5}$$

If some growth conditions stronger than (2.2) are imposed on $F(x,u)$ then the functional $I-\mathscr{F}$ is coercive and thus (1.2) is solvable. In fact, we have the following existence result.

Theorem 2.1. *Assume there exist* $\alpha < 1$ *and* $A_3, B_3 \geqslant 0$ *such that*

$$|F(x,u)| \leqslant A_3 + B_3|u|^\alpha, \quad \text{for a.e. } x \in \mathscr{B}, \forall u \in \mathbb{R}, \tag{2.6}$$

then (1.2) *has a solution.*

Proof. For $R > 0$, we denote $B_R = \{x \in X\colon \|x\|_X \leqslant R\}$. From the lower semicontinuity of the norm $\|\cdot\|_X$ with respect to the $L^1(\mathscr{B})$-topology in $BV(\mathscr{B})$ restricted to X, we immediately see that B_R is closed with respect to the weak*-topology in X. Moreover, J is lower semicontinuous with respect to the $L^1(\mathscr{B})$-topology (and thus the weak*-topology) of X. Let us prove that for each $R > 0$, the functional $J - \mathscr{F}$ has a minimizer in B_R. In fact, let $\{u_n\}$ be a sequence in B_R such that

$$\lim_{n\to\infty} (J - \mathscr{F})(u_n) = \inf_{v \in B_R} (J - \mathscr{F})(v). \tag{2.7}$$

Since $\{u_n\}$ is bounded in X, there is a subsequence, still denoted by $\{u_n\}$ for simplicity, such that

$$u_n \rightharpoonup^* u$$

in the weak*-topology, and in particular in the $L^1(\mathscr{B})$-topology, of X. As noted above, we have

$$u \in B_R. \tag{2.8}$$

From the compact embedding (2.4), we see that the set $\{u_n\colon n \in \mathbb{N}\}$ is relatively compact in $L^q(\mathscr{B})$. From the growth condition (2.2) we obtain

$$\mathscr{F}(u_n) \to \mathscr{F}(u).$$

Hence $\mathscr{F}$ is continuous in X with respect to the weak*-topology. This implies that $J - \mathscr{F}$ is lower semicontinuous in $BV(\mathscr{B})$ with respect to the weak* topology and, in particular,

$$(J - \mathscr{F})(u) \leqslant \liminf (J - \mathscr{F})(u_n).$$

Combining this limit with (2.7) and (2.8), we see that u is a minimizer of $J - \mathscr{F}$ on B_R.

For each $R > 0$, let $u_R \in B_R$ be any minimizer of $J - \mathscr{F}$ on B_R, that is,

$$(J - \mathscr{F})(u_R) \leqslant (J - \mathscr{F})(v), \quad \forall v \in B_R.$$

Let us show that there is $R > 0$ such that

$$\|u_R\| < R. \tag{2.9}$$

In fact, assume otherwise that

$$\|u_R\| = R, \ \forall R > 0.$$

Consequently,

$$\lim_{R\to\infty} \|u_R\| = \infty. \tag{2.10}$$

On the other hand, it follows from (2.6) that

$$\begin{aligned} |\mathscr{F}(u)| &\leqslant \int_{\mathscr{B}} |F(x,u)|\,\mathrm{d}x \leqslant A_2|\mathscr{B}| + B_2 \int_{\mathscr{B}} |u|^\alpha \\ &\leqslant C_3(1 + \|u\|^\alpha_{L^1(\mathscr{B})}) \leqslant C_4(1 + \|u\|^\alpha), \end{aligned}$$

for all $u \in X$ for some constant $C_4 > 0$. Moreover, note that

$$J(u) \geqslant \int_{\mathscr{B}} |\nabla u| = \|u\|, \ \forall u \in X.$$

Combining these estimates, we get

$$(J - \mathscr{F})(u) \geqslant \|u\|(1 - C_4\|u\|^{\alpha-1}) - C_4, \ \forall u \in X. \tag{2.11}$$

Thus, $J - \mathscr{F}$ is coercive on X in the sense that

$$\lim_{\|u\|\to\infty} (J - \mathscr{F})(u) = \infty. \tag{2.12}$$

This limit, together with (2.10), shows that

$$\lim_{R\to\infty} (J - \mathscr{F})(u_R) = \infty,$$

contradicting the fact that

$$(J - \mathscr{F})(u_R) \leqslant (J - \mathscr{F})(0) = 0, \ \forall R > 0.$$

This contradiction proves (2.9). Now, let us show that u_R also satisfies (1.2). Let $v \in X$ and put $w = u_R + t(v - u_R)$ with $t \in (0,1)$. For t sufficiently small, we have $w \in B_R$ and thus

$$J(w) - \mathscr{F}(w) \geqslant J(u_R) - \mathscr{F}(u_R).$$

However, since $J(w) \leqslant tJ(v) + (1-t)J(u_R)$, we get

$$J(v) - J(u_R) \geqslant \frac{1}{t}[\mathscr{F}(u_R + t(v-u_R)) - \mathscr{F}(u_R)].$$

Letting $t \to 0^+$ in this estimate and using (2.5), one obtains

$$J(v) - J(u_R) \geqslant \int_{\mathscr{B}} f(x, u_R)(v - u_R).$$

Since this holds for every $v \in X$, u_R is a solution of (1.2). $\square$

Remark 2.2. Assume $f(x,u)$ is locally bounded with respect to u and satisfies the following growth condition: There are $M > 0$ and $\beta \in (0,1)$ such that

$$|f(x,u)| \leqslant \frac{B_4}{|u|^\beta} \text{ for a.e. } x \in \mathscr{B}, \text{ all } |u| \geqslant M.$$

Then F satisfies (2.6).

In fact, for $x \in \mathscr{B}$ and $u \in [-M, M]$, we have

$$|F(x,u)| \leqslant \int_0^M |f(x,u)|\,\mathrm{d}x \leqslant M \sup_{u \in [-M,M]} |f(x,u)| \;:=\; A_3\,(<\infty).$$

If $|u| > M$ then

$$|F(x,u)| \leqslant \int_0^M |f(x,u)|\,\mathrm{d}x + \int_M^{|u|} \frac{B_4}{t^\beta}\,\mathrm{d}t \leqslant A_3 + \frac{B_4}{1-\beta}|u|^{1-\beta}.$$

One obtains (2.6) with $\alpha = 1 - \beta \in (0,1)$.

Let us consider another coercivity condition based on the norm $\|f(u)\|_*$ of $f(u)$, which is valid also in the case where the lower order term is not given by an integral. Let $\tilde{f}\colon X \to X^*$ be the Niemytskii operator associated with f:

$$\langle \tilde{f}(u), v\rangle = \int_\Omega f(x,u)v\,\mathrm{d}x, \quad \forall u, v \in X.$$

It follows from (2.1) that $\tilde{f}$ is well defined. We prove next the following non variational existence result.

Theorem 2.3. *If there is $R > 0$ such that*

$$J(u) - \langle \tilde{f}(u), u\rangle > 0, \tag{2.13}$$

for all $u \in X$ with $\|u\| = R$, then (1.2) *has a solution.*

P r o o f. For each $R > 0$, let us consider the following variational inequality on B_R:

$$\begin{cases} J(v) - J(u) \geqslant \langle \tilde{f}(u), v - u\rangle, & \forall v \in B_R, \\ u \in B_R. \end{cases} \tag{2.14}$$

Note that $\tilde{f}$ is pseudomonotone on X (with respect to the weak*-topology) in the following sense: If

$$u_n \rightharpoonup^* u \quad \text{weak}^* \text{ in } X, \tag{2.15}$$

and

$$\limsup \langle \tilde{f}(u_n), u_n - u\rangle \leqslant 0,$$

then

$$\liminf \langle \tilde{f}(u_n), u_n - v\rangle \geqslant \langle \tilde{f}(u), u - v\rangle, \quad \forall v \in X. \tag{2.16}$$

This is a direct consequence of the compact embedding (2.4) and the growth condition (2.1). In fact, it follows from (2.15) that $u_n \to u$ in $L^q(\Omega)$. Hence, from (2.1) and the Dominated Convergence Theorem,

$$f(\cdot, u_n) \to f(\cdot, u) \quad \text{in} \quad L^{q'}(\Omega),$$

(q' is the Hölder conjugate of q). Because

$$\|\tilde{f}(u_n) - \tilde{f}(u)\|_* \leqslant C \|f(\cdot, u_n) - f(\cdot, u)\|_{L^{q'}(\Omega)},$$

we have $\tilde{f}(u_n) \to \tilde{f}(u)$ in X^*. Moreover, $\langle \tilde{f}(u_n), u_n - v\rangle \to \langle \tilde{f}(u), u - v\rangle$ and (2.16) immediately follows.

Now, since B_R is bounded in X and closed with respect to the weak*-topology, the existence of solutions of (2.14) is well known in the theory of variational inequalities (cf. e.g. [33]). This existence result is usually stated for reflexive Banach spaces, but the adaptation to our case of a dual space with the weak*-topology is straightforward. One can also use the existence result in [19] for this purpose. In fact, as noted above, conditions (1.1)–(1.3) in [19] are fulfilled (see also Remark 2.11 in [19]). Condition (2.1) is trivially satisfied since the recession cone B_R^∞ of B_R is $\{0\}$. Also, if $\{w_n\}$ and $\{t_n\}$ are as in Definition 2.5 of [19], then we must have $w_n \to 0$ because $t_n w_n \in B_R$, $\forall n$ and $t_n \to \infty$.

Let us show that there exists $R > 0$ such that

$$\|u_R\| < R. \tag{2.17}$$

Assume (2.17) does not hold, i.e., $\|u_R\| = R$, $\forall R > 0$. By letting $v = 0$ in (2.14) and noting that $J(0) = 0$, one has

$$0 \geqslant J(u_R) - \langle f(u_R), u_R\rangle, \ \forall R > 0. \tag{2.18}$$

However, this contradicts (2.13) and therefore proves (2.17). The verification that u_R is in fact a solution of (1.2) is similar to that in Theorem 2.1 and is therefore omitted. □

Remark 2.4. (a) In our case, since the principal functional has linear growth, the usual coercivity condition (superlinear at infinity) does not hold.

(b) If J and $\tilde{f}$ satisfy the following coercivity condition at infinity

$$\lim_{\|u\|\to\infty} [J(u) - \langle \tilde{f}(u), v\rangle] = \infty, \tag{2.19}$$

then (2.13) is clearly satisfied. Hence, we have existence of solutions in this particular case.

(c) Because

$$J(u) \geqslant \|u\|_X - |\mathscr{B}|, \tag{2.20}$$

we see that if $\tilde{f}$ satisfies the following growth condition:

$$\|\tilde{f}(u)\|_* \leqslant \gamma, \quad \text{if } \|u\| \geqslant M, \tag{2.21}$$

with $\gamma \in (0, 1)$, then (2.19) is fulfilled. This follows directly from the following estimate:

$$J(u) - \langle \tilde{f}(u), u\rangle \geqslant (1-\gamma)\|u\|_* - |\mathscr{B}|,$$

if $\|u\| \geqslant M$. A particular case of (2.21) is when

$$\|\tilde{f}(u)\|_* \leqslant \frac{\alpha}{\|u\|^\beta}, \tag{2.22}$$

for $\|u\| \geqslant M$, with $\alpha, \beta > 0$.

(d) If $f(x,u)$ satisfies the growth condition

$$|f(x,u)| \leqslant \frac{B_4}{|u|^{\beta}}, \tag{2.23}$$

for a.e. $x \in \mathscr{B}$, all $u \in \mathbb{R}$ with $|u| \geqslant M$ for some $M > 0$, $\beta \in (0,1)$, then (2.19) is satisfied. In fact, for $u \in X$, it follows from (2.23) and the growth condtion (2.1) that

$$|\langle \tilde{f}(u), v\rangle| \leqslant \left(\int_{\{x\colon |u(x)|\leqslant M\}} + \int_{\{x\colon |u(x)|>M\}}\right)|f(x,u)u|\,\mathrm{d}x \leqslant A_5 + B_5\|u\|_X^{1-\beta},$$

$(A_5, B_5 \in (0,\infty))$. Together with (2.20), this gives

$$J(u) - \langle \tilde{f}(u), u\rangle \geqslant \|u\|_X - B_5\|u\|_X^{1-\beta} - A_5 - |\mathscr{B}|.$$

Since $1-\beta < 1$, we immediately have (2.19).

(e) Although simple and following directly from the theory of pseudomonotone operators and variational inequalities, the above results present a new point of view for the formulation of the prescribed mean curvature problem in the space of functions of bounded variation, which is different from the classical setting in [36], [24], [20], [22], where the solutions were sought as minimizers of the associated potential functional. The variational inequality formulation here allows us to study other types of critical points as well. As a result, for example, the assumption that $f(x,u)$ is increasing with respect to u, which was usually required in the above classical papers, is not required in our results here.

3. Existence of solutions in noncoercive cases—Sub-supersolution method

In this main part of our paper, we study the case of noncoercive problems. For this purpose, we use a sub-supersolution method. By defining appropriate sub- and supersolutions for the variational inequality (1.2) and making use of the lattice structures of the spaces $BV(\mathscr{B})$ and X, we shall show the solvability and enclosure properties of solutions of (1.2). As usual, for $u, v \in L^1(\mathscr{B})$ and $A, B \subset L^1(\mathscr{B})$, we denote

$$\begin{aligned} &u \vee v = \max\{u,v\},\ u \wedge v = \min\{u,v\},\\ &A * B = \{u * v\colon\ u \in A, v \in B\},\ u * A = \{u\} * A,\quad \text{with } * \in \{\vee, \wedge\}. \end{aligned} \tag{3.1}$$

It is known that $BV(\mathscr{B})$ is closed with respect to the operations $\vee$ and $\wedge$, that is,

$$u, v \in BV(\mathscr{B}) \Rightarrow u \wedge v, u \vee v \in BV(\mathscr{B})$$

(cf. e.g. [3], [2]). As a consequence, X is also closed with respect to $\vee$ and $\wedge$. We propose the following concepts of sub- and supersolutions for the inequality (1.2).

Definition 3.1. A function $\underline{u} \in BV(\mathscr{B})$ is called a subsolution of (1.2) if

$$\underline{u} \leqslant 0 \text{ a.e. on } \mathscr{B} \setminus \Omega \tag{3.2 (i)}$$

$$f(\cdot, \underline{u}) \in L^{q'}(\mathscr{B}), \text{ where } q \in [1, N(N-1)^{-1}) \tag{3.3 (ii)}$$

(q' is the Hölder conjugate of q) and

$$J(v) - J(\underline{u}) - \int_{\mathscr{B}} f(\cdot, \underline{u})(v - \underline{u})\, dx \geqslant 0, \ \forall v \in \underline{u} \wedge X. \tag{3.4 (iii)}$$

Similarly, a function $\overline{u} \in BV(\mathscr{B})$ is called a supersolution of (1.2) if

$$\overline{u} \geqslant 0 \text{ a.e. on } \mathscr{B} \setminus \Omega \tag{3.5 (i)}$$

$$f(\cdot, \overline{u}) \in L^{q'}(\mathscr{B}), \quad \text{and} \tag{3.6 (ii)}$$

$$J(v) - J(\overline{u}) - \int_{\mathscr{B}} f(\cdot, \overline{u})(v - \overline{u})\, dx \geqslant 0, \ \forall v \in \overline{u} \vee X. \tag{3.7 (iii)}$$

We have the following existence and enclosure result for solutions of (1.2).

Theorem 3.2. *Assume there is a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (1.2) such that*

$$\underline{u} \leqslant \overline{u} \quad \text{a.e. on } \Omega \tag{3.8}$$

and that f satisfies the following growth condition between $\underline{u}$ and $\overline{u}$: There exists a function $h \in L^{q'}(\mathscr{B})$ such that

$$|f(x,u)| \leqslant h(x) \tag{3.9}$$

for a.e. $x \in \mathscr{B}$, for all $u \in [\underline{u}(x), \overline{u}(x)]$.

Then, there exists a solution u of (1.2) such that

$$\underline{u} \leqslant u \leqslant \overline{u} \quad \text{on} \quad \mathscr{B}. \tag{3.10}$$

P r o o f. Let us define, for $x \in \mathscr{B}$ and $u \in \mathbb{R}$,

$$b(x,u) = \begin{cases} [u - \overline{u}(x)]^{q-1} & \text{if } u > \overline{u}(x), \\ 0 & \text{if } \underline{u}(x) \leqslant u \leqslant \overline{u}(x), \\ -[\underline{u}(x) - u]^{q-1} & \text{if } u < \underline{u}(x), \end{cases} \tag{3.11}$$

where $q, \underline{u}, \overline{u}$ are as in (3.2)–(3.7). b is a Carathéodory function and since $\underline{u}, \overline{u} \in BV(\mathscr{B}) \hookrightarrow L^q(\Omega)$, b satisfies the growth condition

$$|b(x,u)| \leqslant D_1(x) + D_2|u|^{q-1}, \text{ a.e. } x \in \mathscr{B}, \text{ all } u \in \mathbb{R}, \tag{3.12}$$

with $D_1 \in L^{q'}(\mathscr{B})$, $D_2 > 0$. For $u \in BV(\mathscr{B})$, we have

$$\begin{aligned} \int_{\mathscr{B}} b(x,u)u \, \mathrm{d}x &= \int_{\{x\in\mathscr{B}:\ u(x)>\overline{u}(x)\}} [u(x) - \overline{u}(x)]^{q-1} u(x) \, \mathrm{d}x \\ &\quad - \int_{\{x\in\mathscr{B}:\ u(x)<\underline{u}(x)\}} [\underline{u}(x) - u(x)]^{q-1} u(x) \, \mathrm{d}x \\ &\geqslant D_3 \|u\|^q_{L^q(\mathscr{B})} - D_4, \end{aligned} \tag{3.13}$$

for some $D_3, D_4 > 0$. On the other hand, for $u \in BV(\mathscr{B})$, we define the truncated function Tu by

$$Tu = (u \vee \underline{u}) \wedge \overline{u} \quad (= (u \wedge \overline{u}) \vee \underline{u}). \tag{3.14}$$

Since $\underline{u}, \overline{u} \in BV(\mathscr{B})$, we have $Tu \in BV(\mathscr{B})$. Moreover, if $u \in X$ then $Tu \in X$.

Let us consider the following auxiliary variational inequality in X:

$$\begin{cases} J(v) - J(u) + \langle \beta\tilde{b}(u) - \tilde{f}(Tu), v - u\rangle \geqslant 0, \quad \forall v \in X, \\ u \in X, \end{cases} \tag{3.15}$$

where $\beta > 0$ sufficiently large to be determined later, $\tilde{f}$ is the Niemytskii operator associated with f and $\tilde{b}$ is the Niemytskii operator associated with b.

For $u, v \in BV(\mathscr{B})$, one gets from (3.3), (3.6), and (3.9), the following estimates:

$$\begin{aligned} |\langle \tilde{f}(Tu), v\rangle| &= \left| \int_{\mathscr{B}} f(\cdot, Tu) v \, \mathrm{d}x \right| \\ &\leqslant \int_{\{x\in\mathscr{B}:\ u(x)<\underline{u}(x)\}} |f(\cdot,\underline{u})|\,|v| \, \mathrm{d}x + \int_{\{x\in\mathscr{B}:\ u(x)>\overline{u}(x)\}} |f(\cdot,\overline{u})|\,|v| \, \mathrm{d}x \\ &\quad + \int_{\{x\in\mathscr{B}:\ \underline{u}(x)\leqslant u(x)\leqslant\overline{u}(x)\}} h\,|v| \, \mathrm{d}x \\ &\leqslant (\|f(\cdot,\underline{u})\|_{L^{q'}(\mathscr{B})} + \|f(\cdot,\overline{u})\|_{L^{q'}(\mathscr{B})} + \|h\|_{L^{q'}(\mathscr{B})})\|v\|_{L^q(\mathscr{B})}. \end{aligned}$$

As above, from the compact embedding (2.4), we see that if

$$u_n \rightharpoonup^* u \quad \text{in } BV(\mathscr{B})\text{-weak}^*,$$

then $u_n \to u$ in $L^q(\mathscr{B})$ and thus $Tu_n \to Tu$ in $L^q(\mathscr{B})$. Therefore,

$$\tilde{b}(u_n) \to \tilde{b}(u), \quad \tilde{f}(Tu_n) \to \tilde{f}(Tu) \quad \text{in } L^{q'}(\mathscr{B}). \tag{3.17}$$

The complete continuity properties of $\tilde{b}$ and of $\tilde{f} \circ T$ show that $\beta\tilde{b} - \tilde{f} \circ T$ is a pseudomonotone operator from X to X^*.

Let us verify now that (3.15) is coercive on X in the sense of (2.19), that is,

$$\lim_{\|u\| \to \infty, u \in X} J(u) + \langle \beta\tilde{b}(u) - \tilde{f}(Tu), u \rangle = \infty, \tag{3.18}$$

whenever $\beta > 0$ is chosen sufficiently large. In fact, from (2.20), (3.13), and (3.16), we have

$$\begin{aligned} J(u) + \langle \beta\tilde{b}(u) - \tilde{f}(Tu), u \rangle \geqslant \|u\|_X - |\mathscr{B}| + \beta D_3 \|u\|^q_{L^q(\mathscr{B})} - \beta D_4 \\ - (\|f(\cdot, \underline{u})\|_{L^{q'}(\mathscr{B})} + \|f(\cdot, \overline{u})\|_{L^{q'}(\mathscr{B})} + \|h\|_{L^{q'}(\mathscr{B})}) \|u\|_{L^q(\mathscr{B})}. \end{aligned}$$

Since $q \geqslant 1$, by choosing

$$\beta > D_3^{-1}(\|f(\cdot, \underline{u})\|_{L^{q'}(\mathscr{B})} + \|f(\cdot, \overline{u})\|_{L^{q'}(\mathscr{B})} + \|h\|_{L^{q'}(\mathscr{B})}),$$

we obtain (3.18).

According to Theorem 2.3, (3.15) has solutions. In the next part, we show that any solution of u of (3.15) is in fact a solution of (1.2) and satisfies (3.10) as well. First, let us prove that $u \geqslant \underline{u}$. Because $u \in X$, by letting $v = \underline{u} \wedge u$ ($\in \underline{u} \wedge X$) in (3.4), one obtains

$$J(\underline{u} \wedge u) - J(\underline{u}) - \int_{\mathscr{B}} f(\cdot, \underline{u})(\underline{u} \wedge u - \underline{u}) \, \mathrm{d}x \geqslant 0.$$

Since $\underline{u} \wedge u - \underline{u} = -(\underline{u} - u)^+$, we have

$$J(\underline{u} \wedge u) - J(\underline{u}) - \int_{\mathscr{B}} f(\cdot, \underline{u})(\underline{u} - u)^+ \, \mathrm{d}x \geqslant 0. \tag{3.19}$$

It follows from (3.2) that $\underline{u} \vee u \in BV(\mathscr{B})$ and that $\underline{u} \vee u = 0$ a.e. on $\mathscr{B} \setminus \Omega$. Hence, $\underline{u} \vee u \in X$. Letting $v = \underline{u} \vee u$ in (3.15) and noting that $\underline{u} \vee u = u + (\underline{u} - u)^+$, one gets

$$J(\underline{u} \vee u) - J(u) + \langle \beta\tilde{b}(u) - \tilde{f}(Tu), (\underline{u} - u)^+ \rangle \geqslant 0. \tag{3.20}$$

Adding (3.19) and (3.20) yields

$$\begin{aligned} 0 \leqslant J(\underline{u} \wedge u) + J(\underline{u} \vee u) - J(\underline{u}) - J(u) + \int_{\mathscr{B}} f(\cdot, \underline{u})(\underline{u} - u)^+ \, \mathrm{d}x \\ + \beta \int_{\mathscr{B}} b(\cdot, u)(\underline{u} - u)^+ \, \mathrm{d}x - \int_{\mathscr{B}} f(\cdot, Tu)(\underline{u} - u)^+ \, \mathrm{d}x. \end{aligned} \tag{3.21}$$

Next, let us show that

$$J(u \wedge v) + J(u \vee v) \leqslant J(u) + J(v), \ \forall u, v \in BV(\mathscr{B}). \tag{3.22}$$

In fact, from classical results in relaxation and BV-functions (cf. e.g. Theorem 3.3 and Definition 1.1, [12] or [6], [15]), there are sequences $\{u_n\}$ and $\{v_n\}$ in $W^{1,1}(\mathscr{B})$ such that

$$u_n \to u, \ v_n \to v \ \text{ in } \ L^1(\mathscr{B}), \tag{3.23}$$

and

$$J(u_n) \to J(u), \ J(v_n) \to J(v). \tag{3.24}$$

It follows from (3.23) that

$$u_n \wedge v_n \to u \wedge v, \ u_n \vee v_n \to u \vee v \ \text{ in } \ L^1(\mathscr{B}).$$

From the lower semicontinuity of J with respect to the $L^1(\mathscr{B})$-topology, we obtain

$$\begin{cases} J(u \wedge v) \leqslant \liminf J(u_n \wedge v_n), & \text{and} \\ J(u \vee v) \leqslant \liminf J(u_n \vee v_n). \end{cases}$$

On the other hand, since $u_n, v_n \in W^{1,1}(\mathscr{B})$, we have (cf. e.g. [23])

$$\nabla(u_n \wedge v_n) = \begin{cases} \nabla u_n & \text{a.e. on } \{x\colon\ u_n(x) < v_n(x)\}, \\ \nabla u_n = \nabla v_n & \text{a.e. on } \{x\colon\ u_n(x) = v_n(x)\}, \\ \nabla v_n & \text{a.e. on } \{x\colon\ u_n(x) > v_n(x)\}, \end{cases}$$

$$\nabla(u_n \vee v_n) = \begin{cases} \nabla u_n & \text{a.e. on } \{x\colon\ u_n(x) < v_n(x)\}, \\ \nabla u_n = \nabla v_n & \text{a.e. on } \{x\colon\ u_n(x) = v_n(x)\}, \\ \nabla v_n & \text{a.e. on } \{x\colon\ u_n(x) < v_n(x)\}, \end{cases}$$

and thus

$$\begin{aligned} J(u_n \wedge v_n) + J(u_n \vee v_n) &= \int_{\mathscr{B}} \sqrt{1 + |\nabla(u_n \vee v_n)|^2} + \int_{\mathscr{B}} \sqrt{1 + |\nabla(u_n \wedge v_n)|^2} - 2|\mathscr{B}| \\ &= \int_{\mathscr{B}} \sqrt{1 + |\nabla u_n|^2} + \int_{\mathscr{B}} \sqrt{1 + |\nabla v_n|^2} - 2|\mathscr{B}| \\ &= J(u_n) + J(v_n), \quad \forall n. \end{aligned}$$

Combining this identity with (3.24) and (3.25), one obtains

$$J(u \wedge v) + J(u \vee v) \leqslant \liminf[J(u_n \wedge v_n) + J(u_n \vee v_n)] \\ = \liminf[J(u_n) + J(v_n)] = J(u) + J(v).$$

We have proved (3.22). Using (3.22) with $v = \underline{u}$ in (3.21) yields

$$\begin{aligned}(3.26)\qquad 0 &\leqslant \beta \int_{\mathscr{B}} b(\cdot, u)(\underline{u} - u)^+ \,\mathrm{d}x + \int_{\mathscr{B}} [f(\cdot, \underline{u}) - f(\cdot, Tu)](\underline{u} - u)^+ \,\mathrm{d}x \\ &= \beta \int_{\{x:\, \underline{u}(x) > u(x)\}} b(\cdot, u)(\underline{u} - u) \,\mathrm{d}x \\ &\quad + \int_{\{x:\, \underline{u}(x) > u(x)\}} [f(\cdot, \underline{u}) - f(\cdot, Tu)](\underline{u} - u) \,\mathrm{d}x \\ &= -\beta \int_{\{x:\, \underline{u}(x) > u(x)\}} (\underline{u} - u)^{q-1}(\underline{u} - u) \,\mathrm{d}x.\end{aligned}$$

This shows that $\int_{\{x:\, \underline{u}(x)>u(x)\}} (\underline{u} - u)^q \,\mathrm{d}x = 0$ and thus $\underline{u} \leqslant u$ a.e. on $\mathscr{B}$. Analogous arguments show that $u \leqslant \overline{u}$ a.e. on $\mathscr{B}$, which completes our proof of (3.10).

From (3.10) and the definitions of b and T in (3.11) and (3.14), it is immediate that $\tilde{b} = 0$ and $Tu = u$. Therefore, the inequality in (3.15) coincides with that in (1.2) in our case. Hence, u is also a solution of (1.2). □

Remark 3.3. (a) We can extend the above existence result to the case where only subsolutions (or supersolutions) exist and f satisfies a one-sided sub-constant growth condition as in (2.23). The proof in this situation is similar to and, in fact, simpler than that of Theorem 3.2 and is omitted.

(b) Theorem 3.2 can also be generalized to the enclosure of solutions of (1.2) between several subsolutions and supersolutions. We have the following result:

Theorem 3.4. *Assume $\underline{u}_1, \ldots, \underline{u}_k$ are subsolutions and $\overline{u}_1, \ldots, \overline{u}_m$ are supersolutions of* (1.2) *such that*

$$\underline{u} := \max\{\underline{u}_1, \ldots, \underline{u}_k\} \leqslant \overline{u} := \min\{\overline{u}_1, \ldots, \overline{u}_m\},$$

a.e. on Ω and f satisfies the growth condition (3.9) for a.e. $x \in \mathscr{B}$ and all

$$u \in [\min\{\underline{u}_1(x), \ldots, \underline{u}_k(x)\}, \max\{\overline{u}_1(x), \ldots, \overline{u}_m(x)\}].$$

Then, there exists a solution u of (1.2) *that satisfies* (3.10).

The proof of this more general theorem follows the same lines as that of Theorem 3.2 with the following modifications. The auxiliary inequality (3.15) is replaced by the inequality

$$
\begin{cases} J(v) - J(u) + \langle \tilde{\beta}(u) - C(u), v - u \rangle \geqslant 0, \quad \forall v \in X, \\ u \in X. \end{cases} \tag{3.27}
$$

The operator C is given by

$$
\begin{aligned}
\langle C(u), v \rangle = \int_\Omega \Bigg[& f(\cdot, Tu) + \sum_{i=1}^{k} |f(\cdot, T_{i0}(u)) - f(\cdot, Tu)| \\
& - \sum_{j=1}^{m} |f(\cdot, T_{0j}(u)) - f(\cdot, Tu)| \Bigg] v \, \mathrm{d}x,
\end{aligned}
$$

for all $u, v \in X$, where $\underline{u}_0 = \min\{\underline{u}_i \colon 1 \leqslant i \leqslant k\}$, $\overline{u}_0 = \max\{\overline{u}_j \colon 1 \leqslant j \leqslant m\}$, and

$$
T_{ij} u = (u \vee \underline{u}_i) \wedge \overline{u}_j (= (u \wedge \overline{u}_j) \vee \underline{u}_i)
$$

for $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant m$. Using arguments analogous to those in the proof of Theorem 3.2, we see that (3.27) has a solution u such that

$$
\underline{u}_i \leqslant u \leqslant \overline{u}_j, \ \forall i \in \{1, \ldots, k\}, j \in \{1, \ldots, m\}.
$$

This implies that $\tilde{b}(u) = 0$ and $Tu = T_{ij} u = 0, \ \forall i, j$ and thus $C(u) = \tilde{f}(u)$. Hence, u is a solution of (1.2).

(c) Under the assumptions of either Theorem 3.2 or 3.4, any solution u of (1.2) between $\underline{u}$ and $\overline{u}$ is both a subsolution and a supersolution of (1.2) in the sense of Definition 3.1.

In this next part, we show the existence of extremal solutions between sub- and supersolutions. Suppose that (1.2) has a pair of sub- and supersolutions and that the assumptions of Theorem 3.2 are satisfied. We consider on $BV(\mathscr{B})$ (and thus on X) the usual partial ordering:

$$
u \leqslant v \ \text{ if and only if } \ u(x) \leqslant v(x) \quad \text{for a.e. } x \in \mathscr{B}.
$$

Let x be the set of solutions of (1.2) within the interval $[\underline{u}, \overline{u}]$, where $[\underline{u}, \overline{u}] = \{u \in X : \underline{u} \leqslant u \leqslant \overline{u}\}$. We have the following theorem.

Theorem 3.5. *x has the greatest and the smallest element with respect to the partial ordering "$\leqslant$" on X.*

Proof. We note that $BV(\mathscr{B})$ is a separable metric space with the metric generated by the L^q-norm (q is given in (2.1). Therefore, X and thus x are also separable with respect to the metric generated by $\|\cdot\|_{L^q(\mathscr{B})}$ (with respect to $\|\cdot\|_{L^q(\mathscr{B})}$ for short). Hence, there exists a sequence $\{v_n\} \subset x$ such that the set $\{v_n \colon n \in \mathbb{N}\}$ is dense in x with respect to $\|\cdot\|_{L^q(\mathscr{B})}$.

We construct a sequence $\{u_n\}$ in x iteratively as follows. Choose $u_1 = v_1 \in x$. Assume u_n is constructed. As in Remark 3.3 (c), v_n and u_n are subsolutions of (1.2) with

$$(\underline{u} \leqslant) \max\{v_{n+1}, u_n\} \leqslant \overline{u}.$$

By Theorem 3.4, there is a solution $u = u_{n+1}$ of (1.2) such that

$$\underline{u} \leqslant \max\{u_n, v_{n+1}\} \leqslant u_{n+1} \leqslant \overline{u}. \tag{3.28}$$

Therefore, $u_{n+1} \in x$ and $u_{n+1} \geqslant v_{n+1}$. From (3.28), we see that $\{u_n\}$ is an increasing sequence, and

$$u_n \geqslant v_n, \ \forall n. \tag{3.29}$$

Also,

$$u_n \leqslant \underline{u}, \ \forall n. \tag{3.30}$$

Let $u := \sup\limits_{n \in \mathbb{N}} u_n$. Thus,

$$u_n \to u \text{ a.e. in } \mathscr{B}. \tag{3.31}$$

We show that u is a solution of (1.2). In fact, since $\underline{u} \leqslant u \leqslant \overline{u}$, we have $u \in L^q(\mathscr{B})$. Also, since $\underline{u}, \overline{u} \in L^q(\mathscr{B})$, by using (3.31) and the Dominated Convergence Theorem, we get

$$u_n \to u \ \text{ in } \ L^q(\mathscr{B}). \tag{3.32}$$

Applying once more the Dominated Convergence Theorem and using the growth condition (3.9), one obtains

$$\tilde{f}(u_n) \to \tilde{f}(u) \ \text{ in } \ L^{q'}(\mathscr{B}). \tag{3.33}$$

Now, since $u_n \in x$, we have

$$J(v) - J(u_n) \geqslant \langle \tilde{f}(u_n), v - u_n \rangle, \quad \forall v \in X. \tag{3.34}$$

Letting $v = 0$ in this inequality yields

$$\int_{\mathscr{B}} |\nabla u_n| \leqslant J(u_n) \leqslant \langle \tilde{f}(u_n), u_n \rangle. \tag{3.35}$$

From the lower semicontinuity of the functional $u \mapsto \int_{\mathscr{B}} |\nabla u|$ with respect to the $\| \cdot \|_{L^1(\mathscr{B})}$-topology (cf. [24] or [17]), we have

$$\int_{\mathscr{B}} |\nabla u| \leqslant \liminf \int_{\mathscr{B}} |\nabla u_n| \leqslant \lim \langle \tilde{f}(u_n), u_n \rangle = \langle \tilde{f}(u), u \rangle \ < \infty.$$

Hence, $u \in BV(\mathscr{B})$. Also, it follows from (3.31) that

$$u = 0 \quad \text{a.e. on } \mathscr{B} \setminus \Omega,$$

which shows that $u \in X$.

As a consequence of (3.32)–(3.34) and the lower semicontinuity of J with respect to the $L^1(\mathscr{B})$-topology (cf. [24]), we have $J(u) < \infty$ and

$$J(v) - J(u) \geqslant \liminf [J(v) - J(u_n)] \geqslant \lim \langle \tilde{f}(u_n), v - u_n \rangle = \langle \tilde{f}(u), v - u \rangle.$$

Since this holds for all $v \in X$, u is a solution of (1.2), i.e. $u \in x$.

From (3.29), we have

$$u \geqslant v_n \quad \text{a.e. on } \mathscr{B}, \ \forall n \in \mathbb{N}. \tag{3.36}$$

Let $v \in x$. By the density of $\{v_n \colon n \in \mathbb{N}\}$ in x, there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \to v$ in $L^q(\mathscr{B})$ and also $v_{n_k} \to v$ a.e. in $\mathscr{B}$. From (3.36), one also has $u \geqslant v$. We have shown that u is the greatest element of x with respect to the ordering "$\leqslant$". The existence of the smallest element of x is carried out analogously. □

We conclude this section with a simple example of sub- and supersolutions of (1.2) as constants. Further examples of sub-supersolutions in some particular problems will be the subject of a future work. We have the following simple criteria for constant sub-supersolutions.

Proposition 3.6. *Let $D \in \mathbb{R}$. If $D \leqslant 0$ (resp. $D \geqslant 0$), $f(\cdot, D) \in L^{q'}(\mathscr{B})$, and $f(x, D) \geqslant 0$ (resp. $f(x, D) \leqslant 0$) for a.e. $x \in \mathscr{B}$, then D is a subsolution (resp. supersolution) of* (1.2).

References

[1] *R. Acar and C. Vogel*: Analysis of bounded variation penalty methods for ill-posed problems. Inverse Problems *10* (1994), 1217–1229. zbl

[2] *L. Ambrosio, S. Mortola and V. Tortorelli*: Functionals with linear growth defined on vector valued BV functions. J. Math. Pures Appl. *70* (1991), 269–323. zbl

[3] *G. Anzellotti and M. Giaquinta*: *BV* functions and traces. Rend. Sem. Mat. Univ. Padova *60* (1978), 1–21.

[4] *F. Atkinson, L. Peletier and J. Serrin*: Ground states for the prescribed mean curvature equation: the supercritical case. Nonlinear Diffusion Equations and Their Equilibrium States, Math. Sci. Res. Inst. Publ., vol. 12, 1988, pp. 51–74. zbl

[5] *E. Bombieri and E. Giusti*: Local estimates for the gradient of non-parametric surfaces of prescribed mean curvature. Comm. Pure Appl. Math. *26* (1973), 381–394. zbl

[6] *G. Buttazzo*: Semicontinuity, relaxation and integral representation in the calculus of variations. Pitman Research Notes in Mathematics, vol. 207, Longman Scientific & Technical, Harlow, 1989. zbl

[7] *S. Carl and V. K. Le*: Sub-supersolution method for quasilinear parabolic variational inequalities. J. Math. Anal. Appl. *293* (2004), 269–284. zbl

[8] *S. Carl, V. K. Le and D. Motreanu*: Existence and comparison results for quasilinear evolution hemivariational inequalities. Electron. J. Differential Equations *57* (2004), 1–17. zbl

[9] *S. Carl, V. K. Le and D. Motreanu*: The sub-supersolution method and extremal solutions for quasilinear hemivariational inequalities. Differential Integral Equations *17* (2004), 165–178. zbl

[10] *S. Carl, V. K. Le and D. Motreanu*: Existence and comparison principles for general quasilinear variational-hemivariational inequalities. J. Math. Anal. Appl. *302* (2005), 65–83.

[11] *M. Carriero, A. Leaci and E. Pascali*: On the semicontinuity and the relaxation for integrals with respect to the Lebesgue measure added to integrals with respect to a Radon measure. Ann. Mat. Pura Appl. *149* (1987), 1–21. zbl

[12] *M. Carriero, Dal Maso, A. Leaci and E. Pascali*: Relaxation of the nonparametric Plateau problem with an obstacle. J. Math. Pures Appl. *67* (1988), 359–396.

[13] *C. V. Coffman and W. K. Ziemer*: A prescribed mean curvature problem on domains without radial symmetry. SIAM J. Math. Anal. *22* (1991), 982–990. zbl

[14] *M. Conti and F. Gazzola*: Existence of ground states and free-boundary problems for the prescribed mean curvature equation. Adv. Differential Equations *7* (2002), 667–694. zbl

[15] *G. Dal-Maso*: An introduction to Γ-convergence. Birkhäuser, 1993. zbl

[16] *I. Ekeland and R. Temam*: Analyse convexe et problèmes variationnels. Dunod, 1974. zbl

[17] *L. C. Evans and R. F. Gariepy*: Measure theory and fine properties of functions. CRC Press, Boca Raton, 1992. zbl

[18] *R. Finn*: Equilibrium capillary surfaces. Springer, New York, 1986. zbl

[19] *F. Gastaldi and F. Tomarelli*: Some remarks on nonlinear noncoercive variational inequalities. Boll. Un. Math. Ital. *7* (1987), 143–165. zbl

[20] *C. Gerhardt*: Existence, regularity, and boundary behaviour of generalized surfaces of prescribed mean curvature. Math. Z. *139* (1974), 173–198. zbl

[21] *C. Gerhardt*: On the regularity of solutions to variational problems in $BV(\Omega)$. Math. Z. *149* (1976), 281–286. zbl

[22] *C. Gerhardt*: Boundary value problems for surfaces of prescribed mean curvature. J. Math. Pures Appl. *58* (1979), 75–109. zbl

[23] *D. Gilbarg and N. Trudinger*: Elliptic partial differential equations of second order. Springer, Berlin, 1983. zbl

[24] *E. Giusti*: Minimal surfaces and functions of bounded variations. Birkhäuser, Basel, 1984.

[25] *C. Goffman and J. Serrin*: Sublinear functions of measures and variational integrals. Duke Math. J. *31* (1964), 159–178. zbl

[26] *P. Habets and P. Omari*: Positive solutions of an indefinite prescribed mean curvature problem on a general domain. Adv. Nonlinear Studies *4* (2004), 1–13. zbl

[27] *N. Ishimura*: Nonlinear eigenvalue problem associated with the generalized capillarity equation. J. Fac. Sci. Univ. Tokyo *37* (1990), 457–466. zbl

[28] *T. Kusahara and H. Usami*: A barrier method for quasilinear ordinary differential equations of the curvature type. Czech. Math. J. *50* (2000), 185–196. zbl

[29] *V. K. Le*: Existence of positive solutions of variational inequalities by a subsolution–supersolution approach. J. Math. Anal. Appl. *252* (2000), 65–90. zbl

[30] *V. K. Le*: Subsolution-supersolution method in variational inequalities. Nonlinear Analysis *45* (2001), 775–800. zbl

[31] *V. K. Le*: Some existence results on nontrivial solutions of the prescribed mean curvature equation. Adv. Nonlinear Studies *5* (2005), 133–161. zbl

[32] *V. K. Le and K. Schmitt*: Sub-supersolution theorems for quasilinear elliptic problems: A variational approach. Electron. J. Differential Equations (2004), 1–7. zbl

[33] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris, 1969. zbl

[34] *M. Marzocchi*: Multiple solutions of quasilinear equations involving an area-type term. J. Math. Anal. Appl. *196* (1995), 1093–1104. zbl

[35] *J. Mawhin and M. Willem*: Critical point theory and Hamiltonian systems. Springer Verlag, New York, 1989. zbl

[36] *M. Miranda*: Dirichlet problem with L^1 data for the non-homogeneous minimal surface equation. Indiana Univ. Math. J. *24* (1974), 227–241. zbl

[37] *W. M. Ni and J. Serrin*: Existence and non-existence theorems for quasilinear partial differential equations the anomalous case. Accad. Naz. Lincei, Atti dei Convegni *77* (1985), 231–257.

[38] *W. M. Ni and J. Serrin*: Non-existence theorems for quasilinear partial differential equations. Rend. Circ. Math. Palermo *2* (1985), 171–185. zbl

[39] *E. S. Noussair, C. A. Swanson and Y. Jianfu*: A barrier method for mean curvature problems. Nonlinear Anal. *21* (1993), 631–641. zbl

[40] *L. Peletier and J. Serrin*: Ground states for the prescribed mean curvature equation. Proc. Amer. Math. Soc. *100* (1987), 694–700. zbl

[41] *W. Ziemer*: Weakly differentiable functions. Springer, New York, 1989. zbl

Author's address: Vy Khoi Le, Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409, USA, e-mail: `vy@mst.edu`.