Bohdan Zelinka
Double covers and logics of graphs. II.


Persistent URL: http://dml.cz/dmlcz/128282

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1983

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
DOUBLE COVERS AND LOGICS OF GRAPHS II

BOHDAN ZELINKA

This paper is a continuation of results from [4]. The considered graphs are undirected graphs without loops and multiple edges.

The main concepts of this topic are the logic of a graph (introduced in [2] and based on a more general concept from [1]) and the double cover of a graph [3].

Let \( V(G) \) be the vertex set of a graph \( G \). If \( A \) is a subset of \( V(G) \), then by \( A^\perp \) we denote the set of all vertices of \( V(G) \) which are adjacent to all vertices of \( A \) in \( G \). Further we denote \( A^{\perp \perp} = (A^\perp)^\perp \) and for a one-element subset \( \{u\} \) of \( V(G) \) we write \( u^\perp \) and \( u^{\perp \perp} \) instead of \( \{u\}^\perp \) and \( \{u\}^{\perp \perp} \).

Obviously \( A \subseteq A^{\perp \perp} \) for each subset \( A \) of \( V(G) \) and \( A \subseteq B \) implies \( B^\perp \subseteq A^\perp \) for any two subsets \( A, B \) of \( V(G) \). For each subset \( A \) of \( V(G) \) we have \((A^{\perp \perp})^\perp = (A^\perp)^{\perp \perp} = A^\perp \). If \( A = \emptyset \), then \( A^\perp = V(G) \), \( A^{\perp \perp} = \emptyset \). If \( A = A^{\perp \perp} \), we say that \( A \) is \( \perp \) closed. The \( \perp \) closed subsets of \( V(G) \) form a complete lattice with respect to the set inclusion. This lattice together with the unary operation assigning \( A^\perp \) to \( A \) (this operation is an operation of complementation on this lattice) is called the logic of the graph \( G \) and denoted by \( \mathcal{L}(G) \). The least element of \( \mathcal{L}(G) \) is the empty set, its greatest element is \( V(G) \). For each \( A \in \mathcal{L}(G) \) we have \( A = \bigcap_{a \in A^\perp} a^\perp = \bigcup_{a \in A} a^{\perp \perp} \).

Also the following two assertions are evident. For any \( A \subseteq V(G) \) we have \( A^\perp = \bigcap_{a \in A} a^\perp \). For any system \( \{A_i\}_{i \in I} \) of subsets of \( V(G) \), where \( I \) is a subscript set, we have

\[
(\bigcup_{i \in I} A_i)^\perp = \bigcap_{i \in I} A_i^\perp.
\]

We shall not reproduce the general definition of the double cover of a graph. We shall study only a particular case of double covers — the bipartite double covers.

If \( G \) is a graph with the vertex set \( V(G) \), then the bipartite double cover \( B(G) \) of \( G \) is the bipartite graph on the (disjoint) sets \( V = V(G) \) and \( V' = \{v' \mid v \in V(G)\} \) such that if \( u \) is adjacent to \( v \) in \( G \), then \( u \) is adjacent to \( v' \) and \( u' \) is adjacent to \( v \) in \( B(G) \) and no other edges in \( B(G) \) exist.

We shall consider some properties of graphs concerning the sets \( A^\perp \).
**Property P1.** A graph $G$ has no vertices of the degree 0 or 1 and $|u^+ \cap v^+| \leq 1$ for any two distinct vertices $u, v$ of $G$.

**Property P2.** For any two vertices $u, v$ of $G$ the inclusion $u^+ \subseteq v^+$ implies $u = v$.

**Property P3.** For any two vertices $u, v$ of $G$ the inclusion $u^+ \subseteq v^+$ implies $u^+ = v^+$.

**Property P4.** For any two vertices $u, v$ of $G$ the equality $u^+ = v^+$ implies $u = v$.

**Property P5.** For each vertex $x \in V(G)$ and each subset $Y \subseteq V(G)$ the equality $x^+ = Y^+$ implies $x \in Y$.

**Property P6.** For each vertex $x \in V(G)$ the element $x^+$ is completely meet-irreducible in $\mathcal{L}(G)$.

Evidently $P1 \Rightarrow P2 \Rightarrow P3$, but not conversely, $P2 \Rightarrow P4$, but not conversely, and $P2 \Leftrightarrow P3 \& P4$.

**Proposition 1.** A graph $G$ has the property $P5$ if and only if it has the properties $P4$ and $P6$.

**Proof.** Let $G$ have the properties $P4$ and $P6$. If $x \in V(G)$ and $Y \subseteq V(G)$ are such that $x^+ = Y^+$, then $x^+ = \bigcap_{y \in Y} y^+$. As $G$ has the property $P6$, we have $x^+ = y^+$ for an element $y \in Y$. According to the property $P4$ this implies $x = y \in Y$.

Conversely, let $G$ have the property $P5$. Evidently it has also the property $P4$. If $x \in V(G)$ and $x^+ = \bigcap_{i \in I} A_i$ for a family $\{A_i\}_{i \in I}$ of elements of $\mathcal{L}(G)$, then $x^+ = \bigcap_{i \in I} A_i^{\perp} = (\bigcup_{i \in I} A_i^{\perp})^\perp$. According to the property $P5$ this implies $x \in A_i^+$ for some $i \in I$ and therefore $A_i \subseteq x^+$, which together with $x^+ \subseteq A_i$ implies $x^+ = A_i$. Hence $x^+$ is completely meet-irreducible in $\mathcal{L}(G)$.

**Proposition 2.** If $G, H$ are graphs with the property $P5$ and $\mathcal{L}(G) \cong \mathcal{L}(H)$, then $G \cong H$.

**Proof.** Let $\varphi: \mathcal{L}(G) \to \mathcal{L}(H)$ be an isomorphism. According to the property $P6$ for each $x \in V(G)$ there exists $y \in V(H)$ such that $\varphi(x^+) = y^+$. According to the property $P4$ such an element $y$ is unique. Define the mapping $\psi: V(G) \to V(H)$, $x \mapsto y$ in such a way that $\varphi(x^+) = y^+$. Evidently $\psi$ is a bijection. If $x \in V(G)$, $y \in V(G)$, then $(x, y) \in E(G) \iff x \in y^+ \iff x^{\perp} \subseteq y^{\perp} \iff \varphi(x^{\perp}) \subseteq \varphi(y^{\perp}) \iff \varphi(x^+) \subseteq \varphi(y^+) \iff \psi(x)^{\perp} \subseteq \psi(y)^{\perp} \iff \psi(x)^{\perp} \subseteq \psi(y)^{\perp} \iff (\psi(x), \psi(y)) \in E(H)$. Hence $\psi$ is an isomorphism of the graphs $G$ and $H$.

Now let $G$ be a graph and let $A \in \mathcal{L}(G)$. If $A$ is an atom in $\mathcal{L}(G)$, then $A = x^+$ for an element $x \in V(G)$. If $A$ is a dual atom in $\mathcal{L}(G)$, then $A = x^+$ for an element $x \in V(G)$. These assertions are evident.

330
Proposition 3. Let $G$ be a graph. Then the following three assertions are equivalent:

(i) $G$ has the property $P_2$.

(ii) For any vertex $u$ of $G$, $u^{++} = \{ u \}$.

(iii) The set of atoms of $\mathcal{L}(G)$ is equal to the set of all one-element subsets of $V(G)$.

Proof. (i) $\Rightarrow$ (ii). If $u \in V(G)$ and $v \in u^{++}$, then $u^+ = u^{++} \subseteq v^+$ and according to the property $P_2$ this implies $u = v$.

(ii) $\Rightarrow$ (i). If $u \in V(G)$, $v \in V(G)$ and $u^+ \subseteq v^+$, then by (ii) we have $\{ v \} = v^+ \subseteq u^{++} = \{ u \}$, hence $u = v$.

(ii) $\Leftrightarrow$ (iii). This is now evident. ■

Proposition 4. Let $G$ be a graph. Then the following three assertions are equivalent:

(i) $G$ has the property $P_3$.

(ii) The set of atoms of $\mathcal{L}(G)$ is equal to the set of all sets $u^{++}$ for $u \in V(G)$.

(iii) The set of dual atoms of $\mathcal{L}(G)$ is equal to the set of all sets $u^+$ for $u \in V(G)$.

Proof. (i) $\Rightarrow$ (ii). If $u \in V(G)$, $\emptyset \neq A \in \mathcal{L}(G)$ and $A \subseteq u^{++}$, then for each $a \in A$ we have $u^+ \subseteq a^+ \subseteq a$ and according to the property $P_3$ this implies $u^+ = A^+ = a^+$. Then $A = u^{++}$, because $A^{++} = A$.

(ii) $\Rightarrow$ (iii). If $u \in V(G)$, $A \in \mathcal{L}(G)$, $A \neq V(G)$ and $u^+ \subseteq A$, then $A^+ \subseteq u^{++}$ and according to (ii) this implies $A^+ = u^{++}$ and hence $A = u^+$.

(iii) $\Rightarrow$ (i). If $u \in V(G)$, $v \in V(G)$ and $u^+ \subseteq v^+$, then according to (iii) we have $u^+ = v^+$. Hence $G$ has $P_3$. ■

Proposition 5. Let $G$ be a graph, $|V(G)| \geq 2$. Then the following two assertions are equivalent:

(i) $G$ has the property $P_1$.

(ii) The logic $\mathcal{L}(G)$ of $G$ consists of the least element $\emptyset$, the set of atoms equal to the set of all one-element subsets of $V(G)$, the set of dual atoms equal to the set of all sets $u^+$ for $u \in V(G)$ and the greatest element $V(G)$ and no atom of $\mathcal{L}(G)$ is equal to a dual atom of $\mathcal{L}(G)$.

Proof. (i) $\Rightarrow$ (ii). Let $G$ have the property $P_1$. Then it has also the properties $P_2$ and $P_3$. Hence the set of atoms of $\mathcal{L}(G)$ is the set of all one-element subsets of $V(G)$ (by Proposition 3) and the set of dual atoms of $\mathcal{L}(G)$ is the set of all sets $u^+$ for $u \in V(G)$ (by Proposition 4). Let $A \in \mathcal{L}(G)$, $A \neq V(G)$. Since $A = \bigcap_{a \in A^+} a^+$, $A$ is either a dual atom of $\mathcal{L}(G)$, or the intersection of at least two dual atoms of $\mathcal{L}(G)$. The property $P_1$ implies that in the latter case $|A| \leq 1$, hence either $A = \emptyset$, or $A$ is a one-element subset of $V(G)$, i.e. an atom of $\mathcal{L}(G)$. As $G$ has no vertex of the degree 0 or 1, each dual atom of $\mathcal{L}(G)$ contains at least two vertices and it cannot be equal to an atom of $\mathcal{L}(G)$.
(ii) ⇒ (i). Let (ii) hold. Then the meet (i.e. the intersection) of any two dual atoms is either the least element (i.e. 0), or an atom (i.e. a one-element set). Hence \(|u^\uparrow \cap v^\uparrow| \leq 1\) for any two distinct vertices \(u, v\) of \(G\). As no dual atom is equal to an atom and as \(|V(G)| \geq 2\), we have \(|u^\uparrow| \geq 2\) for each \(u \in V(G)\) and there is no vertex of the degree 0 or 1.

**Theorem 1.** Let \(G\) be a graph. Let \(H\) be the ordered subset of \(\mathcal{L}(G)\) consisting of all \(x^\uparrow\) and all \(x^{\uparrow\downarrow}\) for \(x \in V(G)\) with the ordering induced by that of \(\mathcal{L}(G)\). Let \(\leq\) be the following ordering on the vertex set \(V(B(G))\) of the bipartite double cover \(B(G)\) of \(G\): for \(x, y \in V(B(G))\), \(x \leq y\) if and only if \(x \in V'\), \(y \in V\) and \(\{x, y\}\) is an edge in \(B(G)\). Let \(\varphi\) : \(V(B(G)) \rightarrow H\) be such that \(\varphi(x) = x^\uparrow\), \(\varphi(x') = x^{\uparrow\downarrow}\) for all \(x \in V(G)\). Then \(\varphi\) is an isomorphism of ordered sets if and only if \(G\) has the property P2 and has no vertices of the degree 0 or 1.

**Proof.** Evidently the mapping \(\varphi\) is a surjection and \(x \leq y\) implies \(\varphi(x) \leq \varphi(y)\) for any \(x, y\) from \(V(B(G))\). If \(\varphi\) is an isomorphism, \(x \in V(G)\), \(y \in V(G)\) and \(x^\uparrow \leq y^\uparrow\), then \(x \leq y\), because \(x^\uparrow = \varphi(x)\) and \(y^\uparrow = \varphi(y)\). As both \(x, y\) are in \(V\), there cannot be \(x < y\) and we have \(x = y\). We have the property P2. If \(x^\uparrow = \emptyset\) for an element \(x \in V\), then, according to the property P2, \(|V(G)| = 1\), which is a contradiction. If \(x^\uparrow = \{y\}\) for some \(x \in V(G)\), \(y \in V(G)\), then \(\varphi(x') = x^{\uparrow\downarrow} = y^\uparrow = \varphi(y)\), which is a contradiction, because \(x^\uparrow \neq y^\uparrow\). Conversely, let \(G\) have the property P2 and let it have no vertex of the degree 0 or 1. Let \(x, y\) be two vertices of \(B(G)\) and let \(\varphi(x) \leq \varphi(y)\). We shall consider all possible cases. If both \(x, y\) belong to \(V\), then \(x^\uparrow \leq y^\uparrow\), which implies \(x = y\) according to the property P2. If \(x, y\) belong to \(V'\), then \(x = z', y = t'\) for some vertices \(z, t\) of \(G\). Then \(\{z\} = z^{\uparrow\downarrow} \subseteq t^{\uparrow\downarrow} = \{t\}\) and hence \(x = y\). If \(x \in V'\), \(y \in V\), then \(x = z'\) for \(z \in V(G)\) and \(\{z\} = z^{\uparrow\downarrow} \subseteq y^\uparrow\), which implies \(x \leq y\). If \(x \in V\), \(y \in V'\), then \(y = t'\) for \(t \in V(G)\) and \(x^\uparrow \subseteq t^{\uparrow\downarrow} = \{t\}\), which is a contradiction. Thus we have proved that \(\varphi\) is an isomorphism.

**Corollary.** Let \(G\) be a graph with the property P1. Let \(H\) be the graph obtained from the Hasse diagram of \(\mathcal{L}(G)\) by deleting the vertices corresponding to \(V(G)\) and \(\emptyset\). Then \(H\) is isomorphic to \(B(G)\).

By the symbol \(\text{Aut} G\) the automorphism group of a graph \(G\) will be denoted. For each \(\alpha \in \text{Aut} G\) we define the mapping \(\alpha'\) such that \(\alpha'(A) = (\alpha(a)|a \in A\) for each subset \(A\) of \(V(G)\). Then evidently for each \(A \in \mathcal{L}(G)\) we have \(\alpha'(A) \in \mathcal{L}(G)\). Further \(\alpha'(A)^\downarrow = \alpha'(A^\downarrow)\) for each \(A \subseteq V(G)\). If \(A, B\) are two subsets of \(V(G)\), then \(A \subseteq B \iff \alpha'(A) \subseteq \alpha'(B)\). The restriction \(\alpha^*\) of \(\alpha'\) onto \(\mathcal{L}(G)\) belongs to the automorphism group \(\text{Aut} \mathcal{L}(G)\) of \(\mathcal{L}(G)\). The mapping \(\varphi\) : \(\text{Aut} G \rightarrow \text{Aut} \mathcal{L}(G)\), \(\alpha \mapsto \alpha^*\) is evidently a homomorphism of groups.

**Theorem 2.** Let \(G\) be a graph. Then \(\varphi\) is an imbedding if and only if \(G\) has the property P4. If \(G\) has the property P2, then \(\varphi\) is an isomorphism.

**Proof.** If \(\varphi\) is not a surjection, then there exist mappings \(\alpha, \beta\) from \(\text{Aut} G\) such that \(\alpha \neq \beta\) and \(\alpha^* = \beta^*\) and therefore there exists \(x \in V(G)\) such that
u = \alpha(x) \neq v = \beta(x). Then u^+ = \alpha(x)^+ = \alpha^*(x^+) = \beta^*(x^+) = \beta(x)^+ = v^+. Hence G has not the property P4. Conversely, let u, v be two vertices of G such that u \neq v and u^+ = v^+. Let \alpha be a mapping of V(G) onto V(G) such that \alpha(u) = v, \alpha(v) = u, \alpha(x) = x for any x distinct from u and v. Then \alpha \in \text{Aut} G. If \omega is the identity automorphism of G, then \alpha \neq \omega, \alpha^* = \omega^* and \varphi is not an injection.

Now suppose that the graph G has the property P2. Then it has also the property P4 and \varphi is an injection. Evidently \{u\} \in \mathcal{L}(G) for each vertex u \in V(G). Let \beta \in \text{Aut} \mathcal{L}(G). Define the mapping \alpha: V(G) \to V(G), x \mapsto y so that \beta(\{x\}) = \{y\}.

Evidently \alpha \in \text{Aut} G. If A \in \mathcal{L}(G), then A = \bigvee_{a \in A} \{a\}; this implies \beta(A) = \bigvee_{a \in \alpha(A)} \{\alpha(a)\} = \alpha^*(A). Hence \beta = \alpha^* and \varphi is a surjection. ■

Now consider a graph G with the property P2. If G has a vertex of the degree 0, then G consists only of this vertex. If G has a vertex of the degree 1, then there exists a connected component of G isomorphic to the complete graph K_2 with two vertices; other connected components of G are either isomorphic to K_2, or with the property P2 and without vertices of the degree 0 or 1.

**Theorem 3.** Let G be a graph with the property P2. Then the group of all automorphism of B(G) which map V onto V and V' onto V' is isomorphic to the group of all lattice automorphisms of \mathcal{L}(G).

**Remark.** By a lattice automorphism of \mathcal{L}(G) we mean a bijection of \mathcal{L}(G) onto itself which preserves the lattice operations, but need not preserve the mapping \mathcal{A} \mapsto \mathcal{A}^\bot.

**Proof.** First suppose that G has no vertices of the degree 0 or 1. Then we may take the mapping \varphi from Theorem 1 and consider its inverse \varphi^{-1}. This is an isomorphism of H onto V(B(G)) (as ordered set) which maps the set \mathcal{A} of atoms of \mathcal{L}(G) onto V and the set \mathcal{D} of dual atoms of \mathcal{L}(G) onto V'. Therefore it suffices to prove that each automorphism of H which maps \mathcal{A} onto \mathcal{A} and \mathcal{D} onto \mathcal{D} can be uniquely extended to an automorphism of \mathcal{L}(G). Let \alpha be an automorphism of H which maps \mathcal{A} onto \mathcal{A} and \mathcal{D} onto \mathcal{D}. Evidently this is not only an automorphism of H, but also an order automorphism of \mathcal{A} \cup \mathcal{D}. If u \in V(G), then let \alpha_0(u) be the vertex v of G such that \{v\} = \alpha(\{u\}). If A is a dual atom of \mathcal{L}(G), then evidently \alpha(A) = \{\alpha_0(u) | u \in A\}; therefore the images of dual atoms in \alpha are uniquely determined by the images of atoms. As each element of \mathcal{L}(G) distinct from V(G) is an intersection of dual atoms, evidently the unique possible extension of \alpha to a lattice automorphism of \mathcal{L}(G) is given by \alpha(A) = \{\alpha_0(u) | u \in A\} for each A \in \mathcal{L}(G). This extension is the image of \alpha in an isomorphism of the group of all automorphisms of H which map \mathcal{A} onto \mathcal{A} and \mathcal{D} onto \mathcal{D} onto the group of all lattice automorphisms of \mathcal{L}(G).

333
If $G$ has the vertices of the degree 0 or 1, the proof can be easily made using the assertions which were written above this theorem. If $G$ has a vertex of the degree 0, the proof is trivial. In the case when $G$ has vertices of the degree 1 we take into account that the logic of a disconnected graph is isomorphic to the algebra obtained from the logics of its connected components by identifying all least elements and all greatest elements.

REFERENCES


Received May 15, 1980

Katedra matematiky
Vysokej školy strojnej a textilnej
Felberova 2
460 01 Liberec

ДВОЙНЫЕ ПОКРЫТИЯ И ЛОГИКИ ГРАФОВ II

Bohdan Zelinka

резюме

Логика графа есть решетка определенных подмножеств множества вершин графа. Двойное покрытие графа есть определенный граф, соответствующий заданному графу. Исследуются соотношения между этими двумя понятиями.