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Several  $q$ -series identities from the Euler expansions of  $(a; q)_\infty$  and  $\frac{1}{(a; q)_\infty}$

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**SEVERAL  $q$ -SERIES IDENTITIES FROM THE EULER  
EXPANSIONS OF  $(a; q)_\infty$  AND  $\frac{1}{(a; q)_\infty}$**

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ABSTRACT. In this paper, we first give several operator identities which extend the results of Chen and Liu, then make use of them to two  $q$ -series identities obtained by the Euler expansions of  $(a; q)_\infty$  and  $\frac{1}{(a; q)_\infty}$ . Several  $q$ -series identities are obtained involving a  $q$ -series identity in Ramanujan's Lost Notebook.

1. INTRODUCTION

Throughout this paper, assume that  $0 < q < 1$  and adopt the customary notation in [6] for  $q$ -series. Let

$$(1) \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

For any integer  $n$ , the  $q$ -shifted factorial  $(a; q)_n$  is given by

$$(2) \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

The  $q$ -binomial coefficient is defined by

$$(3) \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

We will also use frequently the following equations:

$$(4) \quad (q/a; q)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(q^{-n}a; q)_\infty}{(a; q)_\infty}$$

or

$$(5) \quad (q^{-n}a; q)_\infty = (-a)^n q^{-\binom{n+1}{2}} (q/a; q)_n (a; q)_\infty.$$

For convenience, we employ the following notation for multiple  $q$ -shifted factorial:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n,$$

where  $n$  is an integer or  $\infty$ .

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The  $q$ -differential operator  $D_q$  and the  $q$ -shifted operator are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and

$$\eta\{f(a)\} = f(aq),$$

respectively. These two operators have been introduced in [9, 10, 11] due to Rogers, who applied them for proving Roger-Ramanujan identities. Built on the  $D_q$  and  $\eta$ , the operator appeared in the work of Roman [12] and will be denoted by

$$\theta = \eta^{-1}D_q.$$

Two operators introduced in the papers [4] and [5] by Chen and Liu are the exponential operators constructed from  $D_q$  and  $\theta$ :

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}$$

and

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q; q)_n}.$$

Then the following operator identities were obtained.

**Theorem 1.1** (Chen and Liu, [4] and [5]). *Let  $T(bD_q)$  and  $E(b\theta)$  be respectively defined as above. Then*

$$(6) \quad T(bD_q)\left\{\frac{1}{(at; q)_{\infty}}\right\} = \frac{1}{(at, bt; q)_{\infty}},$$

$$(7) \quad T(bD_q)\left\{\frac{1}{(as, at; q)_{\infty}}\right\} = \frac{(abst; q)_{\infty}}{(as, at, bs, bt; q)_{\infty}},$$

$$(8) \quad E(b\theta)\{(at; q)_{\infty}\} = (at, bt; q)_{\infty},$$

$$(9) \quad E(b\theta)\{(as, at; q)_{\infty}\} = \frac{(as, at, bs, bt; q)_{\infty}}{(abst/q; q)_{\infty}}.$$

By these operator identities, in [4, 5, 7, 8, 13], a lot of  $q$ -series identities can be derived. In [14], Theorem 1.1 was extended. Applying generalized operator identities to some terminating basic hypergeometric series and  $q$ -integrals, we obtain some summation formulas involving  ${}_3\phi_2$  summation and some  $q$ -integrals involving  ${}_3\phi_2$  summation which extend some famous  $q$ -integrals, such as the Ismail-Stanton-Viennot integral, Gasper integral formula.

In this paper, we first give several operator identities which extend the results of Chen and Liu. Then making use of them to two  $q$ -series identities obtained by the Euler expansions of  $(a; q)_{\infty}$  and  $\frac{1}{(a; q)_{\infty}}$ , several  $q$ -series identities are obtained involving a  $q$ -series identity in Ramanujan's Lost Notebook.

Throughout this paper,  $n$  and  $k$  are two nonnegative integers.

## 2. SEVERAL OPERATOR IDENTITIES

First of all, by induction we give a result which is to be used later.

**Lemma 2.1.** *We have*

$$(10) \quad \theta^n \{a^k\} = \begin{cases} 0, & \text{if } n > k, \\ (-1)^n q^n (q^{-k}; q)_n a^{k-n}, & \text{if } n \leq k, \end{cases}$$

$$(11) \quad D_q^n \{a^k\} = \begin{cases} 0, & \text{if } n > k, \\ (q^{k-n+1}; q)_n a^{k-n}, & \text{if } n \leq k, \end{cases}$$

$$(12) \quad \theta^n \{a^{-k}\} = (-q)^n (q^k; q)_n a^{-(k+n)},$$

$$(13) \quad D_q^n \{a^{-k}\} = q^{-\binom{n}{2} - kn} (q^k; q)_n a^{-(k+n)}.$$

In [13], we obtained the following operator identity.

$$(14) \quad \begin{aligned} & E(d\theta) \{a^k(at, as; q)_\infty\} \\ &= a^k \frac{(as, at, ds, dt; q)_\infty}{(adst/q)_\infty} {}_3\phi_2 \left( \begin{matrix} q^{-k}, & q/as, & q/at \\ & 0, & q^2/adst; q, q \end{matrix} \right). \end{aligned}$$

This identity can be rewritten by

$$(15) \quad \begin{aligned} & E(d\theta) \{a^k(at, as; q)_\infty\} = a^k \frac{(at, as, dt, ds; q)_\infty}{(adst/q; q)_\infty} \\ & \times \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q/at, q/as; q)_j}{(q^2/adst; q)_j} q^{\binom{j+1}{2} - kj}. \end{aligned}$$

Next we give three operator identities which are extensions of Theorem 1.1. They specialize to Theorem 1.1 by setting  $k = 0$ .

**Theorem 2.2.** *We have*

$$(16) \quad T(dD_q) \left\{ \frac{a^k}{(at, as; q)_\infty} \right\} = a^k \frac{(adst; q)_\infty}{(at, as, dt, ds; q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(at, as; q)_j}{(adst; q)_j} \left( \frac{d}{a} \right)^j.$$

$$(17) \quad \begin{aligned} & E(d\theta) \{a^{-k}(at, as; q)_\infty\} \\ &= a^{-k} \frac{(at, as, dtq^k, dsq^k; q)_\infty}{(adstq^{k-1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k, adstq^{k-1}; q)_n}{(q, dtq^k, dsq^k; q)_n} q^{\binom{n+1}{2}} \left( -\frac{d}{a} \right)^n \\ &= a^{-k} \frac{(at, as, dt, ds; q)_\infty}{(adst/q; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k; q)_n}{(q; q)_n} \frac{(adst/q; q)_{n+k}}{(dt, ds; q)_{n+k}} q^{\binom{n+1}{2}} \left( -\frac{d}{a} \right)^n \end{aligned}$$

where  $|d/a| < 1$ , and

$$\begin{aligned}
 & T(dD_q) \left\{ \frac{a^{-k}}{(at, as; q)_\infty} \right\} \\
 &= a^{-k} \frac{(adtsq^{-k}; q)_\infty}{(at, as, dtq^{-k}, dsq^{-k}; q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(q^k, q^{1+k}/adst; q)_n}{(q, q^{1+k}/dt, q^{1+k}/ds; q)_n} q^n \\
 (18) \quad &= \left(-\frac{1}{d}\right)^k q^{\binom{k+1}{2}} \frac{(adts; q)_\infty}{(at, as, dt, ds; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k; q)_n}{(q; q)_n} \frac{(q/adst; q)_{n+k}}{(q/dt, q/ds; q)_{n+k}} (-q)^n.
 \end{aligned}$$

**Proof.** We note that the rule of Leibniz for  $D_q^n$  (see [4])

$$(19) \quad D_q^n \{f(a)g(a)\} = \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \{f(a)\} D_q^{n-k} \{g(aq^k)\}.$$

Therefore, we have

$$\begin{aligned}
 & T(dD_q) \left\{ \frac{a^k}{(at, as; q)_\infty} \right\} = \sum_{n=0}^{\infty} \frac{d^n}{(q; q)_n} D_q^n \left\{ \frac{a^k}{(at, as; q)_\infty} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{d^n}{(q; q)_n} \sum_{j=0}^n q^{j(j-n)} \begin{bmatrix} n \\ j \end{bmatrix} D_q^j \{a^k\} D_q^{n-j} \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{d^n}{(q; q)_n} \sum_{j=0}^n q^{j(j-n)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(q; q)_k}{(q; q)_{k-j}} a^{k-j} D_q^{n-j} \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{d^n}{(q; q)_n} q^{j(j-n)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(q; q)_k}{(q; q)_{k-j}} a^{k-j} D_q^{n-j} \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= \sum_{j=0}^{\infty} \frac{d^j (q; q)_k}{(q; q)_j (q; q)_{k-j}} a^{k-j} \sum_{n=0}^{\infty} \frac{d^n}{(q; q)_n} q^{-nj} D_q^n \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= a^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \left(\frac{d}{a}\right)^j \sum_{n=0}^{\infty} \frac{(dq^{-j} D_q)^n}{(q; q)_n} \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= a^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \left(\frac{d}{a}\right)^j \cdot T(dq^{-j} D_q) \left\{ \frac{1}{(atq^j, asq^j; q)_\infty} \right\} \\
 &= a^k \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \left(\frac{d}{a}\right)^j \frac{(adstq^j; q)_\infty}{(atq^j, asq^j, dt, ds; q)_\infty} \\
 &= a^k \frac{(adst; q)_\infty}{(at, as, dt, ds; q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(at, as; q)_j}{(adst; q)_j} \left(\frac{d}{a}\right)^j.
 \end{aligned}$$

We obtain (16). (17) and (18) can be proved similarly. The proof is completed.  $\square$

Taking  $s = 0$  in Theorem 2.2, we have

**Corollary 2.3.**

$$(20) \quad T(dD_q)\left\{\frac{a^k}{(at; q)_\infty}\right\} = a^k \frac{1}{(at, dt; q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (at; q)_j \left(\frac{d}{a}\right)^j.$$

$$(21) \quad E(d\theta)\{a^{-k}(at; q)_\infty\} = a^{-k}(at, dtq^k; q)_\infty \\ \times \sum_{n=0}^{\infty} \frac{(q^k; q)_n}{(q; q)_n} \frac{1}{(dt; q)_{n+k}} q^{\binom{n+1}{2}} \left(-\frac{d}{a}\right)^n$$

where  $|d/a| < 1$ , and

$$(22) \quad T(dD_q)\left\{\frac{a^{-k}}{(at; q)_\infty}\right\} = \left(-\frac{1}{adt}\right)^k q^{\binom{k+1}{2}} \frac{1}{(at, dt; q)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{(q^k; q)_n}{(q; q)_n} \frac{1}{(q/dt; q)_{n+k}} \left(-\frac{q}{at}\right)^n,$$

where  $|q/at| < 1$ .

Further, setting  $k = 1$  or  $t = 0$  in (21), we get the following results of Chen and Liu.

**Corollary 2.4** (Chen and Liu, [5]).

$$(23) \quad E(b\theta)\{a^{-1}\} = a^{-1} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} b^n a^{-n},$$

$$(24) \quad E(b\theta)\{a^{-1}(-a; q)_\infty\} = a^{-1}(-a; q)_\infty \sum_{m=0}^{\infty} (-b/a)^m q^{\binom{m+1}{2}} (-bq^{m+1}; q)_\infty.$$

### 3. SEVERAL $q$ -SERIES IDENTITIES FROM EXPANSION OF $(a; q)_\infty$

From the Euler expansion of  $(a; q)_\infty$  (see [3]):

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} a^n}{(q; q)_n} = (a; q)_\infty,$$

the following identity can be verified.

$$(25) \quad \sum_{n=0}^{\infty} (-aq^{n+1}; q)_\infty q^n = -a^{-1} + a^{-1}(-a; q)_\infty.$$

**Theorem 3.1.** *We have*

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n \sum_{j=0}^{k+1} \frac{(q^{-(k+1)}, -q^{-n}/a, q/at; q)_j}{(q, -q^{1-n}/adt; q)_j} q^j \\
 &= -a^{-1} \frac{(-adt; q)_{\infty}}{(-aq, -dq; q)_{\infty}} \sum_{j=0}^k \frac{(q^{-k}, q/at; q)_j}{(q; q)_j} (dt)^j \\
 (26) \quad &+ a^{-1} \frac{(1+a)(1+d)}{1+adt/q} \sum_{j=0}^k \frac{(q^{-k}, -q/a, q/at; q)_j}{(q, -q^2/adt; q)_j} q^j.
 \end{aligned}$$

**Proof.** Multiply both sides in equation (25) by  $a^{k+1}(at; q)_{\infty}$ . Then

$$(27) \quad \sum_{n=0}^{\infty} a^{k+1}(-aq^{n+1}, at; q)_{\infty} q^n = -a^k(at; q)_{\infty} + a^k(-a, at; q)_{\infty}.$$

Apply the operator  $E(d\theta)$  to its both sides with respect to the variable  $a$ . Then

$$\begin{aligned}
 (28) \quad & \sum_{n=0}^{\infty} q^n E(d\theta)\{a^{k+1}(-aq^{n+1}, at; q)_{\infty}\} = -E(d\theta)\{a^k(at; q)_{\infty}\} \\
 &+ E(d\theta)\{a^k(-a, at; q)_{\infty}\}.
 \end{aligned}$$

Applying Theorem 2.2 and Corollary 2.3, we have

$$\begin{aligned}
 E(d\theta)\{a^{k+1}(-aq^{n+1}, at; q)_{\infty}\} &= a^{k+1} \frac{(-aq^{n+1}, at, -dq^{n+1}, dt; q)_{\infty}}{(-adtq^n; q)_{\infty}} \\
 &\times \sum_{j=0}^{k+1} \frac{(q^{-(k+1)}, -q^{-n}/a, q/at; q)_j}{(q, -q^{1-n}/adt; q)_j} q^j, \\
 E(d\theta)\{a^k(at; q)_{\infty}\} &= a^k(at, dt; q)_{\infty} \sum_{j=0}^k \frac{(q^{-k}, q/at; q)_j}{(q; q)_j} (dt)^j, \\
 E(d\theta)\{a^k(-a, at; q)_{\infty}\} &= a^k \frac{(-a, at, -d, dt; q)_{\infty}}{(-adt/q; q)_{\infty}} \sum_{j=0}^k \frac{(q^{-k}, -q/a, q/at; q)_j}{(q, -q^2/adt; q)_j} q^j.
 \end{aligned}$$

Substituting these three identities into (27) and then using (4), we obtain (3.1) after simplifying. This completes the proof.  $\square$

Taking  $k = 0$  in Theorem 3.1, we have

**Corollary 3.2.**

$$\begin{aligned}
 (29) \quad & a \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n + d(1-at/q) \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq; q)_n(-dq; q)_{n+1}} q^{n+1} \\
 &= -\frac{(-adt; q)_{\infty}}{(-aq, -dq; q)_{\infty}} + 1 + a.
 \end{aligned}$$

Taking  $t = 0$  in Corollary 3.2 we have

**Corollary 3.3.**

$$(30) \quad a \sum_{n=0}^{\infty} \frac{1}{(-aq, -dq; q)_n} q^n + d \sum_{n=0}^{\infty} \frac{1}{(-aq; q)_n (-dq; q)_{n+1}} q^{n+1} \\ = - \frac{1}{(-aq, -dq; q)_{\infty}} + 1 + a.$$

Taking  $k = 1$  in Theorem 3.1, we have

**Corollary 3.4.**

$$(31) \quad a^2 q \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n - ad(1+q)(at-q) \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq; q)_n (-dq; q)_{n+1}} q^n \\ + d^2(at-q)(at-q^2) \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq; q)_n (-dq; q)_{n+2}} q^n \\ = \frac{(1+a)(aq+dq+adq^2-adt)}{1+dq} - (aq+dq-adt) \frac{(-adt; q)_{\infty}}{(-aq, -dq; q)_{\infty}}.$$

Taking  $t = 0$  in Corollary 3.4, we have

**Corollary 3.5.**

$$(32) \quad a^2 \sum_{n=0}^{\infty} \frac{1}{(-aq, -dq; q)_n} q^n + ad(1+q) \sum_{n=0}^{\infty} \frac{1}{(-aq; q)_n (-dq; q)_{n+1}} q^n \\ + d^2 q^2 \sum_{n=0}^{\infty} \frac{1}{(-aq; q)_n (-dq; q)_{n+2}} q^n \\ = \frac{(1+a)(a+d+adq)}{1+dq} - (a+d) \frac{1}{(-aq, -dq; q)_{\infty}}.$$

**Theorem 3.6.** *We have*

$$(33) \quad (1-dtq^k) \sum_{n=0}^{\infty} \frac{(-adtq^k; q)_n}{(-aq, -dq^{k+1}; q)_n} q^n \sum_{i=0}^{\infty} \frac{(q^k, -adtq^{n+k}; q)_i}{(q, -dq^{n+k+1}, dtq^k; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i \\ = -a^{-1} \frac{(-adtq^k; q)_{\infty}}{(-aq, -dq^{k+1}; q)_{\infty}} \sum_{i=0}^{\infty} \frac{(q^{k+1}; q)_i}{(q, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i \\ + (1+a^{-1}) \sum_{i=0}^{\infty} \frac{(q^{k+1}, -adtq^k; q)_i}{(q, -dq^{k+1}, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i,$$

where  $|d/a| < 1$ .

**Proof.** (25) can be rewritten by

$$(34) \quad \sum_{n=0}^{\infty} q^n a^{-k} (-aq^{n+1}, at; q)_{\infty} = -a^{-(k+1)} (at; q)_{\infty} + a^{-(k+1)} (-a, at; q)_{\infty}$$



Apply the operator  $E(d\theta)$  to its both sides with respect to the variable  $a$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} q^n E(d\theta) \{ a^{-k} (-aq^{n+1}, at; q)_{\infty} \} &= -E(d\theta) \{ a^{-(k+1)} (at; q)_{\infty} \} \\ &\quad + E(d\theta) \{ a^{-(k+1)} (-a, at; q)_{\infty} \}. \end{aligned}$$

By Theorem 2.2 and Corollary 2.3, we have

$$\begin{aligned} E(d\theta) \{ a^{-k} (-aq^{n+1}, at; q)_{\infty} \} &= a^{-k} \frac{(-aq^{n+1}, at, -dq^{n+k+1}, dtq^k; q)_{\infty}}{(-adtq^{n+k}; q)_{\infty}} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(q^k, -adtq^{n+k}; q)_i}{(q, -dq^{n+k+1}, dtq^k; q)_i} q^{\binom{i+1}{2}} \left( -\frac{d}{a} \right)^i, \\ E(d\theta) \{ a^{-(k+1)} (at; q)_{\infty} \} &= a^{-(k+1)} (at, dtq^{k+1}; q)_{\infty} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(q^{k+1}; q)_i}{(q, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left( -\frac{d}{a} \right)^i, \\ E(d\theta) \{ a^{-(k+1)} (-a, at; q)_{\infty} \} &= a^{-(k+1)} \frac{(-a, at, -dq^{k+1}, dtq^{k+1}; q)_{\infty}}{(-adtq^k; q)_{\infty}} \\ &\quad \times \sum_{i=0}^k \frac{(q^{k+1}, -adtq^k; q)_i}{(q, -dq^{k+1}, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left( -\frac{d}{a} \right)^i. \end{aligned}$$

Substituting these three identities into (34) and then using (4), we obtain the theorem.  $\square$

Taking  $k = 0$  in Theorem 3.6 we have

**Corollary 3.7.**

$$\begin{aligned} (1-dt) \sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n &= -a^{-1} \frac{(-adt; q)_{\infty}}{(-aq, -dq; q)_{\infty}} \sum_{i=0}^{\infty} \frac{1}{(dtq; q)_i} q^{\binom{i+1}{2}} \left( -\frac{d}{a} \right)^i \\ (35) \quad &\quad + (1+a^{-1}) \sum_{i=0}^{\infty} \frac{(-adt; q)_i}{(-dq, dtq; q)_i} q^{\binom{i+1}{2}} \left( -\frac{d}{a} \right)^i, \end{aligned}$$

where  $|d/a| < 1$ .

Taking  $t = 0$  in Corollary 3.7 we obtain the following Ramanujan's identity.

**Corollary 3.8.**

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(-aq, -dq; q)_n} q^n &= -a^{-1} \frac{1}{(-aq, -dq; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} (d/a)^n \\ (36) \quad &\quad + (1+a^{-1}) \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} (d/a)^n}{(-dq; q)_n}, \end{aligned}$$

where  $|d/a| < 1$ .

**Note.** This identity is a formula in Ramanujan's Lost Notebook, and its proofs are given by Andrews [1, 2]. In [5], Chen and Liu gave also a simple proof by using the method of the operator identity.

#### 4. SEVERAL $q$ -SERIES IDENTITIES FROM EXPANSION OF $\frac{1}{(a;q)_\infty}$

From the Euler expansion of  $\frac{1}{(a;q)_\infty}$  (see [3]):

$$\sum_{n=0}^{\infty} \frac{1}{(q;q)_n} a^n = \frac{1}{(a;q)_\infty},$$

the following identity can be verified.

$$(37) \quad \sum_{n=0}^{\infty} \frac{1}{(aq^n;q)_\infty} q^n = a^{-1} \frac{1}{(a;q)_\infty} - a^{-1}.$$

From this identity, by the operator  $T(dD_q)$  we can obtain the following results.

**Theorem 4.1.** *We have*

$$(38) \quad \sum_{n=0}^{\infty} q^n (d;q)_n \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (at;q)_j \frac{(a;q)_{n+j}}{(adt;q)_{n+j}} \left(\frac{d}{a}\right)^j = a^{-1} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a, at;q)_j}{(adt;q)_j} \left(\frac{d}{a}\right)^j - a^{-1} \frac{(a, d;q)_\infty}{(adt;q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (at;q)_j \left(\frac{d}{a}\right)^j.$$

**Proof.** By (37), we have

$$(39) \quad \sum_{n=0}^{\infty} \frac{a^{k+1}}{(aq^n, at;q)_\infty} q^n = \frac{a^k}{(a, at;q)_\infty} - \frac{a^k}{(at;q)_\infty}.$$

Apply the operator  $T(dD_q)$  to its both sides with respect to the variable  $a$ . Then

$$(40) \quad \sum_{n=0}^{\infty} q^n \cdot T(dD_q) \left\{ \frac{a^{k+1}}{(aq^n, at;q)_\infty} \right\} = T(dD_q) \left\{ \frac{a^k}{(a, at;q)_\infty} \right\} - T(dD_q) \left\{ \frac{a^k}{(at;q)_\infty} \right\}.$$

By Theorem 2.2 and Corollary 2.3, we have

$$(41) \quad T(dD_q) \left\{ \frac{a^{k+1}}{(aq^n, at;q)_\infty} \right\} = a^{k+1} \frac{(adtq^n;q)_\infty}{(aq^n, at, dq^n, dt;q)_\infty} \times \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} \frac{(aq^n, at;q)_j}{(adtq^n;q)_j} \left(\frac{d}{a}\right)^j,$$

$$(42) \quad T(dD_q) \left\{ \frac{a^k}{(a, at;q)_\infty} \right\} = a^k \frac{(adt;q)_\infty}{(a, at, d, dt;q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a, at;q)_j}{(adt;q)_j} \left(\frac{d}{a}\right)^j$$

and

$$(43) \quad T(dD_q) \left\{ \frac{a^k}{(at; q)_\infty} \right\} = a^k \frac{1}{(at, dt; q)_\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} (at; q)_j \left( \frac{d}{a} \right)^j.$$

Substituting these three identities into (40) and then using (4), we obtain the theorem.  $\square$

Taking  $k = 0$  in Theorem 4.1 we have

**Corollary 4.2.**

$$(44) \quad a \sum_{n=0}^{\infty} \frac{(a, d; q)_n}{(adt; q)_n} q^n + d(1 - at) \sum_{n=0}^{\infty} (d; q)_n \frac{(a; q)_{n+1}}{(adt; q)_{n+1}} q^n = 1 - \frac{(a, d; q)_\infty}{(adt; q)_\infty}.$$

Specially, taking  $t = 0$  we have

**Corollary 4.3.**

$$(45) \quad a \sum_{n=0}^{\infty} (a, d; q)_n q^n + d \sum_{n=0}^{\infty} (d; q)_n (a; q)_{n+1} q^n = 1 - (a, d; q)_\infty.$$

Taking  $k = 1$  in Theorem 4.1 we have

**Corollary 4.4.**

$$(46) \quad \begin{aligned} & a^2 \sum_{n=0}^{\infty} \frac{(a, d; q)_n}{(adt; q)_n} q^n + ad(1 + q)(1 - at) \sum_{n=0}^{\infty} \frac{(a; q)_{n+1}(d; q)_n}{(adt; q)_{n+1}} q^n \\ & \quad + d^2(1 - at)(1 - atq) \sum_{n=0}^{\infty} \frac{(a; q)_{n+2}(d; q)_n}{(adt; q)_{n+2}} q^n \\ & = a + d \frac{(1 - a)(1 - at)}{1 - adt} - \frac{(a, d; q)_\infty}{(adt; q)_\infty} (a + d - adt). \end{aligned}$$

Specially, taking  $t = 0$  we have

**Corollary 4.5.**

$$(47) \quad \begin{aligned} & a^2 \sum_{n=0}^{\infty} (a, d; q)_n q^n + ad(1 + q) \sum_{n=0}^{\infty} (a; q)_{n+1} (d; q)_n q^n + d^2 \sum_{n=0}^{\infty} (a; q)_{n+2} (d; q)_n q^n \\ & = (a + d)(1 - (a, d; q)_\infty) - ad. \end{aligned}$$

**Theorem 4.6.** *We have*

$$(48) \quad \begin{aligned} & \sum_{n=0}^{\infty} q^n \frac{(a, d; q)_n}{(adt; q)_n} \sum_{j=0}^{\infty} \frac{(q^k; q)_j}{(q; q)_j} \frac{(q^{1-n}/adt; q)_{k+j}}{(q/dt, q^{1-n}/d; q)_{k+j}} (-q)^j \\ & = - \frac{q^{k+1}}{d} \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{(q/adt; q)_{k+1+j}}{(q/d, q/dt; q)_{k+1+j}} (-q)^j \\ & \quad + \frac{1}{d} \frac{(a, d; q)_\infty}{(adt; q)_\infty} \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{(-1)^j}{(q/dt; q)_{k+1+j}} (q/at)^{k+1+j}, \end{aligned}$$

where  $|q/at| < 1$ .

**Proof.** We rewrite (37) into the following form:

$$(49) \quad \sum_{n=0}^{\infty} \frac{a^{-k}}{(aq^n, at; q)_{\infty}} q^n = \frac{a^{-(k+1)}}{(a, at; q)_{\infty}} - \frac{a^{-(k+1)}}{(at; q)_{\infty}}.$$

Apply the operator  $T(dD_q)$  to its both sides with respect to the variable  $a$ . Then

$$(50) \quad \sum_{n=0}^{\infty} q^n \cdot T(dD_q) \left\{ \frac{a^{-k}}{(aq^n, at; q)_{\infty}} \right\} = T(dD_q) \left\{ \frac{a^{-(k+1)}}{(a, at; q)_{\infty}} \right\} \\ - T(dD_q) \left\{ \frac{a^{-(k+1)}}{(at; q)_{\infty}} \right\}.$$

By Theorem 2.2 and Corollary 2.3, we have

$$(51) \quad T(dD_q) \left\{ \frac{a^{-k}}{(aq^n, at; q)_{\infty}} \right\} = (-d)^{-k} q^{\binom{k+1}{2}} \frac{(adtq^n; q)_{\infty}}{(aq^n, at, dq^n, dt; q)_{\infty}} \\ \times \sum_{j=0}^{\infty} \frac{(q^k; q)_j}{(q; q)_j} \frac{(q^{1-n}/adt; q)_{k+j}}{(q/dt, q^{1-n}/d; q)_{k+j}} (-q)^j,$$

$$(52) \quad T(dD_q) \left\{ \frac{a^{-(k+1)}}{(a, at; q)_{\infty}} \right\} = (-d)^{-(k+1)} q^{\binom{k+2}{2}} \frac{(adt; q)_{\infty}}{(a, at, d, dt; q)_{\infty}} \\ \times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{(q/adt; q)_{k+j+1}}{(q/d, q/dt; q)_{k+j+1}} (-q)^j$$

and

$$(53) \quad T(dD_q) \left\{ \frac{a^{-(k+1)}}{(at; q)_{\infty}} \right\} = \left( -\frac{1}{adt} \right)^{-(k+1)} q^{\binom{k+2}{2}} \frac{1}{(at, dt; q)_{\infty}} \\ \times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{1}{(q/dt; q)_{k+j+1}} \left( -\frac{q}{at} \right)^j.$$

Substituting these three identities into (50) and then using (4), we obtain the theorem.  $\square$

Taking  $k = 0$  in Theorem 4.6 we have

**Corollary 4.7.**

$$(54) \quad \sum_{n=0}^{\infty} q^n \frac{(a, d; q)_n}{(adt; q)_n} = -\frac{q}{d} \sum_{j=0}^{\infty} \frac{(q/adt; q)_{j+1}}{(q/d, q/dt; q)_{j+1}} (-q)^j \\ + \frac{1}{d} \frac{(a, d; q)_{\infty}}{(adt; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(q/dt; q)_{j+1}} \left( \frac{q}{at} \right)^{j+1},$$

where  $|q/at| < 1$ .

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