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ON THE PRIME GRAPHS OF THE AUTOMORPHISM GROUPS OF SPORADIC SIMPLE GROUPS

Behrooz Khosravi

Abstract. In this paper as the main result, we determine finite groups with the same prime graph as the automorphism group of a sporadic simple group, except J_2.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then the set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of order elements of G is denoted by $\pi_e(G)$. We construct the prime graph of G as follows: *The prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G)$, $\pi_2(G)$, …, $\pi_{t(G)}(G)$ be the connected components of $\Gamma(G)$. We use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. The author in [11] determined finite groups with the same prime graph as $PSL(2, q)$. For some simple groups S, finite groups with the same prime graph as $\Gamma(S)$ are determined (see the references of [11]). Hagie in [7] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. As the main result of this paper, we determine finite groups with the same prime graph as the automorphism group of a sporadic simple group, except J_2. The structure of the automorphism groups of sporadic simple groups are described in [3].

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [3].

2. Preliminary results

First we give an easy remark:

Remark 2.1. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

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Definition 2.1 ([6]). A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.1 ([2, Lemma 5]). *Let G be a finite group with disconnected prime graph. Then we have two possibilities.*

(i) *G is a Frobenius group or a 2-Frobenius group;*

(ii) *G has a chain $G \supseteq M \supseteq N \supseteq 1$ of normal subgroups such that N is a nilpotent π-group, M/N is a non-abelian simple group and G/M is a solvable π-group where π is the connected component of $\Gamma(G)$ containing 2.*

By the above lemma it follows that if G is a solvable group with $t(G) \geq 2$, then G is a Frobenius group or a 2-Frobenius group, and $t(G) = 2$.

Lemma 2.2 ([12]). *Let G be a finite group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $\big(|F|, |N|\big) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Lemma 2.3 ([14]). *Let G be a finite group and N a nontrivial normal p-subgroup, for some prime p, and set $K = G/N$. Suppose that K contains an element x of order m coprime to p such that $\langle \varphi|_{\langle x \rangle}, 1|_{\langle x \rangle} \rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) K in characteristic p. Then G contains elements of order pm.*

Definition 2.2. Let p be a prime number. A group G is called a C_{pp} group if the centralizers in G of its elements of order p are p-groups.

Lemma 2.4 ([1]).

(i) *The $C_{13,13}$-simple groups are: A_{13}, A_{14}, A_{15};* Suz, Fi_{22}*; $L_2(q)$, $q = 3^3$, 5^2, 13^n or $2 \times 13^n - 1$ which is a prime, $n \geq 1$; $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q = 2^2$, 3; $F_4(2)$,$U_3(q)$, $q = 2^2$, 23; $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.*

(ii) *The $C_{19,19}$-simple groups are: A_{19}, A_{20}, A_{21}; J_1, J_3, $O'N$, Th, HN; $L_2(q)$, $q = 19^n$, $2 \times 19^n - 1$ which is a prime, $(n \geq 1)$; $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$.*

Lemma 2.5 (Zsigmondy's Theorem [16]). *Let p be a prime and n be a positive integer. Then one of the following holds:*

(i) *p is a Mersenne prime and $n = 2$;*

(ii) *$p = 2$, $n = 1$ or 6;*

(iii) *there is a primitive prime p' for $p^n - 1$, that is, $p'|(p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$.*

Lemma 2.6 ([4]).

(i) *With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation $p^m - 2q^n = \pm 1$; where p and q are prime and $m, n > 1$; has exponents $m = n = 2$.*

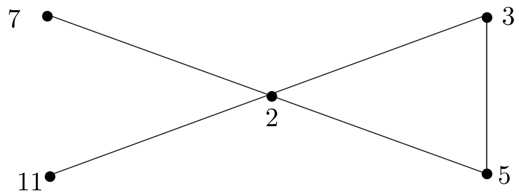
(ii) *The only solution of the equation* $p^m - q^n = 1$*;* p*,* q *prime; and* $m, n > 1$ *is* $3^2 - 2^3 = 1$.

3. Main results

We note that for some of the sporadic simple groups we have $\mathrm{Aut}(S) = S$. Also if S is one of the following groups: M_{12}, He, Fi_{22} or HN, then $\mathrm{Aut}(S) \neq S$ but $\Gamma(S) = \Gamma(\mathrm{Aut}(S))$. These cases were considered by Hagie [7]. Therefore we consider the case $A = \mathrm{Aut}(S)$, where S is one of the following groups: M_{22}, J_3, HS, Suz, $O'N$, Fi'_{24} or McL. First we consider $\mathrm{Aut}(McL)$, since its prime graph is connected.

Theorem 3.1. *Let* G *be a finite group such that* $\Gamma(G) = \Gamma(\mathrm{Aut}(McL))$*. Then* $G/O_2(G)$ *is isomorphic to* HS*,* $\mathrm{Aut}(HS)$*,* McL*,* $\mathrm{Aut}(McL)$*,* $U_6(2)$ *or* $U_6(2):2$*,*

Proof. We note that the prime graph of $\mathrm{Aut}(McL)$ is connected and $\Gamma(G)$ is as follows:



If G is a solvable group, then consider a Hall $\{5, 7, 11\}$-subgroup H of G. Then H is solvable and $t(H) = 3$, which is a contradiction. Therefore G is a non-solvable group.

Let N be a maximal normal solvable subgroup of G. It is obvious that $N \neq G$. Let $\overline{G} = G/N$ and $S = \mathrm{Socle}(\overline{G})$. We know that $C_{\overline{G}}(S) = 1$ and $N_{\overline{G}}(S) = \overline{G}$, which implies that $S \leq \overline{G} \leq \mathrm{Aut}(S)$. The socle of a group is a direct product of minimal normal subgroups and so $S = M_1 \times M_2 \times \cdots \times M_r$, where M_i, $1 \leq i \leq r$, are minimal normal subgroups. Also every minimal normal subgroup is characteristically simple and so is a product of isomorphic simple groups. Hence $S = P_1 \times \cdots \times P_k$, where P_i, $1 \leq i \leq k$, are non-abelian simple groups.

Step 1. *If* $A = \pi(N) \cap \{5, 7, 11\}$*, then* A *has at most one element.*

If $|A| = 3$, then similar to the above argument we get a contradiction. If $|A| = 2$, then let $A = \{p_1, p_2\}$, $p \in \{5, 7, 11\} \setminus A$ and H be a Hall A-subgroup of N. Now N is a normal subgroup of G and H is a Hall subgroup of N. Therefore $G = NN_G(H)$, by the Frattini argument. Since $p \notin \pi(N)$, it follows that $p \in \pi\big(N_G(H)\big)$ and so there is an element $y \in N_G(H)$ of order p. It is obvious that y acts fixed point freely on H and $o(y) = p$. Therefore H is nilpotent by Thompson's Theorem [5, Theorem 10.2.1], which implies that $p_1 \sim p_2$, a contradiction. Similarly we can prove that $\pi(N) \cap \{3, 7, 11\}$ has at most one element.

As a consequence of this result we conclude that $\pi(\overline{G}) \cap A$ has at least two elements and so there exists $p \in \{7, 11\}$ such that $p \in \pi(\overline{G})$.

Step 2. *The subgroup* S *is a nonabelian simple group.*

As we mentioned above, $S = P_1 \times \cdots \times P_k$, where every P_i, $1 \le i \le k$, is a non-abelian simple group. Also note that $\pi(S) \subseteq \pi(G) = \{2, 3, 5, 7, 11\}$ and so $\pi(P_i) \subseteq \{2, 3, 5, 7, 11\}$, for every $1 \le i \le k$. There exist only finitely many nonabelian simple groups P such that $\pi(P) \subseteq \{2, 3, 5, 7, 11\}$ and if P is a nonabelian simple group such that $\pi(P) \subseteq \{2, 3, 5, 7, 11\}$, then we can see that $2, 3 \in \pi(P)$ and $\pi\big(\mathrm{Out}(P)\big) \subseteq \{2, 3\}$ (see [13]).

We claim that $k = 1$. Let $k \ge 2$. Then $7, 11 \notin \pi(S)$, since $3 \in \pi(P_i)$, for every $1 \le i \le k$, and $3 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$. Hence $\pi(P_i) \subseteq \{2, 3, 5\}$ and by using [13] we see that for every $1 \le i \le k$, P_i is isomorphic to A_5, A_6 or $U_4(2)$. On the other hand, $7, 11 \in \pi\big(\mathrm{Out}(S)\big)$, since $Z(S) = 1$. We note that $\{7, 11\} \cap \pi(N)$ has at most one element. So let $p \in \{7, 11\} \cap \pi(\overline{G})$ and let $\varphi \in \overline{G}$ be an element of order p. Obviously $\varphi \in \mathrm{Aut}(S)$. Let $Q = P_1^{\varphi}$ and $f_i \colon Q \to P_i$, $1 \le i \le k$, be the natural projection of Q to P_i. Also P_1 is a normal subgroup of S and so Q is a normal subgroup of S. Therefore $Im\ f_i \trianglelefteq P_i$ and P_i is a simple group, which implies that $Im\ f_i = 1$ or $Im\ f_i = P_i$, for every $1 \le i \le k$. On the other hand, P_1 is a simple group, and so Q is a simple group. Therefore $\ker f_i = 1$ or $\ker f_i = Q$. If $\ker f_i = 1$, then $\mathrm{Im}\ f_i = P_i$, which implies that $Q \cong P_i$. Also if $\ker f_i = Q$, then $\mathrm{Im}\ f_i = 1$. Hence there exists a unique j, $1 \le j \le k$, such that $P_1^{\varphi} = P_j$. Now if $j \ne 1$, then there exists a φ-orbit of length p. Without loss of generality let $\{P_1, \dots, P_p\}$ be a φ-orbit. As we mentioned above $3 \in \pi(P_1)$. Let $g_1 \in P_1$ be an element of order 3 and let $g_{i+1} = g_i^{\varphi}$, where $1 \le i \le p-1$. Now let x be the element of S whose projections x_i to P_i are defined as follows: $x_i = g_i$ for $i = 1, \dots, p$ and $x_i = 1$ otherwise. Obviously x is of order 3 and so $x\varphi \in \overline{G}$ is of order $3p$, which is a contradiction since $3 \nsim p$ in $\Gamma(G)$. Therefore for every $1 \le i \le k$, we have $P_i^{\varphi} = P_i$. Since $\varphi \ne 1$, there exists $1 \le i \le k$ such that φ acts nontrivially on P_i. Therefore φ induces an outer automorphism of P_i of order p. Hence p is a divisor of $\big|\mathrm{Out}(P_i)\big|$, which is a contradiction. Therefore $k = 1$ and S is a nonabelian simple group.

Step 3. *The subgroup S is isomorphic to McL, HS or $U_6(2)$.*

Up to now we prove that there is a nonabelian simple group S such that $S \le G/N \le \mathrm{Aut}(S)$. Also we know that $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Now we consider each possibility for S, separately.

If $S \cong A_5$, then $\pi(S) = \pi\big(\mathrm{Aut}(S)\big) = \{2, 3, 5\}$ and so $\{7, 11\} \subseteq \pi(N)$, which is a contradiction by Step 1. Similarly it follows that S is not isomorphic to $L_2(7)$, $L_2(8)$, $A_6 \cong L_2(9)$, $U_3(3)$, $U_4(2)$.

If $S \cong L_2(11)$, then $\pi(S) = \{2, 3, 5, 11\}$ and so $7 \in \pi(N)$. Also $S \le G/N$ contains a Frobenius subgroup $11 : 5$ of order 55. Now by using Lemma 2.2, G contains an element of order 35, which is a contradiction. Similarly if $S \cong M_{11}$, M_{12}, $U_5(2)$, then $L_2(11) < S$ and $7 \in \pi(N)$. Therefore similarly follows that $5 \sim 7$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_7$, $A_8 \cong L_4(2)$, $L_3(4)$, $L_2(49)$, $U_3(5)$, A_9, J_2, $S_6(2)$, $U_4(3)$, $O_8^+(2)$, then $L_2(7) < S$ and $\pi(S) = \{2, 3, 5, 7\}$. Therefore $11 \in \pi(N)$ and also $L_2(7)$ contains a Frobenius subgroup $7 : 3$ of order 21. Now Lemma 2.4 implies that G contains an element of order 33 and so $3 \sim 11$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_{10}$, A_{11}, A_{12}, $S_4(7)$, then $3 \sim 7$ in $\Gamma(S)$, which is a contradiction by Remark 2.1, since $3 \nsim 7$ in $\Gamma(G)$.

If $S \cong M_{22}$, then since $3 \nsim 5$ in $\Gamma(S)$ it follows that $3 \in \pi(N)$ or $5 \in \pi(N)$.

Let $5 \in \pi(N)$. Let $x \in G/N$, $X = \langle x\rangle$ and $o(x) = 11$. Now by using [9] about the irreducible characters of $M_{22} \pmod 5$, we can see that

$$\begin{aligned}
\langle 1_G|_X, 1|_X\rangle &= 1;\\
\langle 21|_X, 1|_X\rangle &= \frac{1}{11}(21 + (-1)\times 10) = 1;\\
\langle 45_1|_X, 1|_X\rangle = \langle 45_2|_X, 1|_X\rangle &= \frac{1}{11}(45 + 10) = 5;\\
\langle 55|_X, 1|_X\rangle &= \frac{1}{11}(55 + 0) = 5;\\
\langle 98|_X, 1|_X\rangle &= \frac{1}{11}(98 + (-1)\times 10) = 8;\\
\langle 133|_X, 1|_X\rangle &= \frac{1}{11}(133 + 10) = 13;\\
\langle 210|_X, 1|_X\rangle &= \frac{1}{11}(210 + 10) = 20;\\
\langle 385|_X, 1|_X\rangle &= \frac{1}{11}(385 + 0) = 35;
\end{aligned}$$

$$\begin{aligned}
\langle 280_1|_X, 1|_X\rangle = \langle 280_2|_X, 1|_X\rangle &= \frac{1}{11}(280 + 5(b_{11} + \overline{b_{11}}))\\
&= \frac{1}{11}(280 + 5(\frac{-1+i\sqrt{11}}{2} + \frac{-1-i\sqrt{11}}{2})) = 25\,.
\end{aligned}$$

Therefore for every irreducible character φ of $M_{22} \pmod 5$ we show that

$$\langle \varphi|_X, 1|_X\rangle = \frac{1}{|X|}\sum_{x\in X}\varphi(x) > 0\,.$$

Now by using Lemma 2.3, it follows that $55 \in \pi_e(G)$, which is a contradiction. Therefore $5 \notin \pi(N)$. Similarly we can prove that $3 \notin \pi(N)$ and so $S \ncong M_{22}$.

If $S \cong HS$, then $HS \leq G/N \leq \mathrm{Aut}(HS)$. Therefore $G/N \cong HS$ or $G/N \cong \mathrm{Aut}(HS)$. In each case there exists a subgroup H of G such that $H/N \cong HS$. If $\{3, 5, 11\} \cap \pi(N) \neq \emptyset$, then let $p \in \{3, 5, 11\} \cap \pi(N)$, x be an element of order 7 in H/N and $X = \langle x\rangle$. Similar to the last case by using [9] we can see that for every irreducible character φ of $HS \pmod p$ we have

$$\langle \varphi|_X, 1|_X\rangle = \frac{1}{|X|}\sum_{x\in X}\varphi(x) > 0\,,$$

and so G has an element of order $7p$, by Lemma 2.3, which is a contradiction. Similarly it follows that $7 \notin \pi(N)$. Therefore N is a 2-group.

Similar to the above discussion it follows that $G/O_2(G) \cong McL$. With the same method we conclude that $G/O_2(G) \cong \mathrm{Aut}(McL)$, $G/O_2(G) \cong U_6(2)$ or $G/O_2(G) \cong U_6(2) : 2$. We omit the details of the proof for convenience. Now the proof of this theorem is completed. □

We note that if k is a natural number, then obviously

$$\begin{aligned}\Gamma\big(\,\mathrm{Aut}(McL)\big) &= \Gamma(\mathbb{Z}_{2^k}\times \mathrm{Aut}(HS)) = \Gamma(\mathbb{Z}_{2^k}\times HS) = \Gamma(\mathbb{Z}_{2^k}\times McL)\\ &= \Gamma\big(\mathbb{Z}_{2^k}\times \mathrm{Aut}(McL)\big) = \Gamma\big(\mathbb{Z}_{2^k}\times U_6(2)\big) = \Gamma\big(\mathbb{Z}_{2^k}\times U_6(2):2\big)\,.\end{aligned}$$

Now we discuss about the automorphism group of M_{22}, J_3, HS, Suz, $O'N$ and Fi_{24}'. We note that the prime graphs of the automorphism groups of these groups are disconnected. Now by using Lemma 2.1 we have the following result.

Lemma 3.1. *Let G be a finite group and let A be the automorphism group of M_{22}, J_3, HS,* Suz*, $O'N$ or Fi_{24}'. If $\Gamma(G)=\Gamma(A)$, then one of the following holds:*
(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group with $t(K/H)\geq 2$ and $G/K\leq \mathrm{Out}(K/H)$. Also $\pi_2(A)=\pi_i(K/H)$ for some $i\geq 2$ and $\pi_2(A)\subseteq \pi(K/H)\subseteq\pi(S)$.*

Lemma 3.2. *Let M be a simple group of Lie type over $GF(q)$, where $q=p_0^{\alpha}$ and p_0 is a prime number.*

- **(a)** *If $p_0\in\{2,3,5,7\}$, and M is a $C_{11,11}$-group, then M is one of the following simple groups: $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O_{10}^+(3)$.*
- **(b)** *If $p_0\in\{2,3,5,7,11,13,17,19,23\}$ and M is a $C_{29,29}$-group, then $M=L_2(29)$.*
- **(c)** *If $p_0\in\{2,3,5,7,11,19\}$ and M is a $C_{31,31}$-group, then M is $L_5(2)$, $L_3(5)$, $L_6(2)$, $L_4(5)$, $O_{10}^+(2)$, $O_{12}^+(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or* $\mathrm{Sz}(32)$.

Proof. The odd order components of finite non-abelian simple groups are listed in Table 1 in [8]. The odd order components of some non-abelian simple groups of Lie type are of the form $(q^p\pm 1)/\big((q\pm 1)(p,q\pm 1)\big)$. Therefore we consider the following diophantine equations:

$$\text{(i)}\quad \frac{q^p-1}{q-1}=y^n\,,\qquad \text{(ii)}\quad \frac{q^p-1}{(q-1)(p,q-1)}=y^n\,,$$
$$\text{(iii)}\quad \frac{q^p+1}{q+1}=y^n\,,\qquad \text{(iv)}\quad \frac{q^p+1}{(q+1)(p,q+1)}=y^n\,,$$

where $p\geq 3$ is a prime number. Now by solving these diophantine equations we get the result.
(a) If M is a $C_{11,11}$ simple group and the odd order component of M is of the form (i)–(iv), then in the corresponding diophantine equation we have $y=11$. We will show that $(p,q,n)=(5,3,2)$ is the only solution of (i) and (ii). If $(q^p-1)/(q-1)=11^n$ or $(q^p-1)/((q-1)(p,q-1))=11^n$, then 11 is a primitive prime for $p_0^{\alpha p}-1$. Therefore $\mathrm{ord}_{11}(p_0)=\alpha p$, by the definition of primitive prime (see Lemma 2.5). Now by using the Fermat theorem, αp is a divisor of 10. Hence $p=5$ and so $1\leq\alpha\leq 2$. Now by checking the possibilities for q it follows that $(p,q,n)=(5,3,2)$ is the only solution of the diophantine equations (i) and (ii). Similar to the above discussion, by considering the diophantine equations (iii) and (iv) for $y=11$, we

conclude that 11 is a divisor of $p_0^{2\alpha p} - 1$ and in a similar manner it follows that $p = 5$ and $\alpha = 1$. Therefore the only solution of these diophantine equations is $(p, q, n) = (5, 2, 1)$. Now by using this result and by using Table 1 in [8], we can determine $C_{11,11}$ simple groups. We omit the details of the proof for convenience.

For the proof of (b) and (c), similarly we can prove that if $p_0 \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ and $y = 29$, then the diophantine equations (i)–(iv) have no solution. Also we can show that if $y = 31$ and $p_0 \in \{2, 3, 5, 7, 11, 19\}$, then $(p, q, n) = (5, 2, 1)$ and $(3, 5, 1)$ are the only solutions of (i) and (ii). Also (iii) and (iv) have no solution in this case. For convenience we omit the proof. □

We recall a definition from graph theory. A nonempty subset I of $\pi(G)$ is called an *independent subset* if there exists no edge between elements of I in $\Gamma(G)$.

Lemma 3.3. *Let G be a finite group such that G has an independent subset I such that $|I| = 3$. Also let there exist two nonadjacent primes p_1 and p_2 such that $J = \{p_1, p_2\} \subseteq \pi(G) \setminus \{2, 3, 5\}$ and each p_i $(1 \le i \le 2)$ is nonadjacent to at least one element of $\{2, 3, 5\}$ in $\Gamma(G)$. Then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. First we prove that G is not solvable. If G is a solvable group, then let H be a Hall I-subgroup of G. Since H is solvable it follows that $t(H) \le 2$, which is a contradiction, since there exists no edge between elements I in $\Gamma(G)$. Thus G is not solvable, and so G is not a 2-Frobenius group.

If G is a non-solvable Frobenius group, then G has a Frobenius kernel K and a Frobenius complement H. By using Lemma 2.3 in [11], it follows that H has a normal subgroup $H_0 = SL(2, 5) \times Z$, where $|H : H_0| \le 2$ and $\big(|Z|, 30\big) = 1$. Since each p_i $(1 \le i \le 2)$ is not adjacent to at least one element of $\{2, 3, 5\}$ in $\Gamma(G)$, we conclude that $\{p_1, p_2\} \subseteq \pi(K)$. Now since the kernel of every Frobenius group is nilpotent, it follows that $p_1 \sim p_2$ in $\Gamma(G)$, which is a contradiction. Therefore G is not a Frobenius group or a 2-Frobenius group. □

Theorem 3.2. *Let G be a finite group.*

(a) *If $\Gamma(G) = \Gamma(\mathrm{Aut}(M_{22}))$, then $G/O_2(G) \cong M_{22}$ and $O_2(G) \ne 1$ or $G/O_2(G) \cong \mathrm{Aut}(M_{22})$.*

(b) *If $\Gamma(G) = \Gamma\big(\mathrm{Aut}(\mathrm{Fi}'_{24})\big)$, then $G/O_\pi(G) \cong \mathrm{Fi}'_{24}$, where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \ne 1$ or $G/O_{\{2,3\}}(G) \cong \mathrm{Aut}(\mathrm{Fi}'_{24})$.*

Proof. (a) We can see that $\{3, 5, 7\}$ is an independent subset of $\Gamma(G)$. Also $5 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$ and since $7 \nsim 11$ in $\Gamma(G)$, by using Lemma 3.3, we conclude that G is not a Frobenius group nor a 2-Frobenius group. Now by using Lemma 3.1 it follows that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{11,11}$-simple group. If K/H is an alternating group or a sporadic simple group which is a $C_{11,11}$-group, then K/H is: A_{11}, A_{12}, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, HS, Sz, $O'N$, Co_2 or J_1. Also $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$, by Remark 2.1. Therefore $3 \nsim 5$ in $\Gamma(K/H)$ and $\pi(K/H) \subseteq \{2, 3, 5, 7, 11\}$, which implies that the only possibilities for K/H are $L_2(11)$, M_{11}, M_{12} and M_{22}. If $K/H \cong M_{11}$, M_{12} or $L_2(11)$, then K/H has a $11 : 5$ subgroup by [3]. Also in these cases $7 \notin \pi(K/H)$ and hence $7 \in \pi(H)$. Now consider the $\{5, 7, 11\}$ subgroup T of G which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore $K/H \cong M_{22}$ and since $Out(M_{22}) \cong \mathbb{Z}_2$ it follows that $G/H \cong M_{22}$ or $M_{22} \cdot 2$. Also H is a nilpotent π_1-group and so $\pi(H) \subseteq \{2, 3, 5, 7\}$. By using [3] we know that M_{22} has a $11 : 5$ subgroup. If $3 \in \pi(H)$, then let T be a $\{3, 5, 11\}$ subgroup of G which is solvable and hence $t(T) \leq 2$, which is a contradiction, since there exists any edge between 3, 5 and 11 in $\Gamma(G)$. Therefore $3 \notin \pi(H)$. Similarly it follows that $7 \notin \pi(H)$. Let $5 \in \pi(H)$ and $Q \in \mathrm{Syl}_5(H)$. Also let $P \in Syl_3(K)$. We know that H is nilpotent and hence Q char H. Since $H \lhd K$ it follows that $Q \lhd K$. Therefore P acts by conjugation on Q and since $3 \nsim 5$ in $\Gamma(G)$ it follows that P acts fixed point freely on Q. Hence QP is a Frobenius group with Frobenius kernel Q and Frobenius complement P. Now by using Lemma 2.9 it follows that P is a cyclic group which implies that a Sylow 3-subgroup of M_{22} is cyclic. But this is a contradiction since a 3-Sylow subgroup of M_{22} are elementary abelian by [3]. Therefore H is a 2-group. Then $G/O_2(G) \cong M_{22}$ where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong \mathrm{Aut}(M_{22})$, where $\pi \subseteq \{2\}$.

(b) Let $I = \{7, 17, 23\}$ and $J = \{11, 13\}$. Now using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \unlhd H \unlhd K \unlhd G$, where K/H is a $C_{29,29}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $L_2(29)$, Ru or Fi'_{24}. If $K/H \cong L_2(29)$ or Ru, then $\{17, 23\} \subseteq \pi(H)$, which is a contradiction, since H is nilpotent and $17 \nsim 23$ in $\Gamma(G)$. Therefore $K/H \cong \mathrm{Fi}'_{24}$ and so $G/H \cong \mathrm{Fi}'_{24}$ or $\mathrm{Aut}(\mathrm{Fi}'_{24})$. By using [3], we know that Fi'_{24} has a $23 : 11$ subgroup. Therefore $\pi(H) \cap \{5, 7, 13, 17\} = \emptyset$. Also Fi'_{24} has a $29 : 7$ subgroup, and hence $\pi(H) \cap \{11, 13\} = \emptyset$. Therefore $\pi(H) \subseteq \{2, 3\}$ and so $G/O_\pi(G) \cong \mathrm{Fi}'_{24}$ where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong \mathrm{Aut}(\mathrm{Fi}'_{24})$ where $\pi \subseteq \{2, 3\}$. □

Theorem 3.3. *Let G be a finite group satisfying $\Gamma(G) = \Gamma\big(\mathrm{Aut}(J_3)\big)$. Then $G/O_\pi(G) \cong J_3$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(J_3)$.*

Proof. Let $I = \{5, 17, 19\}$, $J = \{17, 19\}$. Now by using Lemmas 3.1 and 3.3, G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that K/H is a $C_{19,19}$ simple group. By using Lemma 2.4, K/H is A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN, $L_3(7)$, $U_3(8)$, $R(27)$, ${}^2E_6(2)$, $L_2(q)$ where $q = 19^n$ or $L_2(q)$ where $q = 2 \times 19^n - 1$ ($n \geq 1$) is a prime number. But $\pi(K/H) \subseteq \pi(J_3)$ and $\pi(J_3) \cap \{7, 11, 13, 31\} = \emptyset$. Also $q = 2 \times 19^n - 1 > 19$ and hence the only possibilities for K/H are J_3 and $L_2(19^n)$, where $n \geq 1$. The orders of maximal tori of $A_m(q) = \mathrm{PSL}(m + 1, q)$ are

$$\frac{\prod_{i=1}^{k}(q^{r_i} - 1)}{(q - 1)(m + 1, q - 1)}; \qquad (r_1, \ldots, r_k) \in \mathrm{Par}(m + 1).$$

Therefore every element of $\pi_e(\mathrm{PSL}(2, q))$ is a divisor of q, $(q + 1)/d$ or $(q - 1)/d$ where $d = (2, q - 1)$. If $q = 19^n$, then $3 \mid (19^n - 1)/2$ and since $3 \sim 5$ and $3 \nsim 17$ in $\Gamma(G)$, it follows that if 5 divides $|G|$, then $5 \mid (19^n - 1)$ and if 17 is a divisor of $|G|$, then $17 \mid (19^n + 1)$. Note that $\pi(19 - 1) = \{2, 3\}$, $\pi(19^2 - 1) = \{2, 3, 5\}$ and $17 \mid (19^4 + 1)$. Now by using the Zsigmondy's Theorem, Lemma 2.9 it follows that the only possibility is $n = 1$.

Now we consider these possibilities for K/H, separately. First let $K/H \cong J_3$. We note that $\mathrm{Out}(J_3) \cong \mathbb{Z}_2$ and hence G/H is isomorphic to J_3 or $J_3 \cdot 2$. Also H is a nilpotent π_1-group. Hence $\pi(H) \subseteq \{2,3,5,17\}$. If $17 \in \pi(H)$, then let T be a $\{3,17,19\}$ subgroup of G, since J_3 has a $19:9$ subgroup. Obviously T is solvable and hence $t(T) \leq 2$, which is a contradiction. Therefore $\pi = \pi(H) \subseteq \{2,3,5\}$ and $G/O_\pi(G) \cong J_3$ or $G/O_\pi(G) \cong \mathrm{Aut}(J_3)$. If $G/O_\pi(G) \cong J_3$, then $O_\pi(G) \neq 1$ and $2 \in \pi$, since $2 \nsim 17$ in $\Gamma(J_3)$.

Now let $K/H \cong L_2(19)$. Since $\mathrm{Out}(L_2(19)) \cong \mathbb{Z}_2$, it follows that $G/H \cong L_2(19)$ or $L_2(19) \cdot 2$. But in this case $\pi(K/H) = \{2,3,5,19\}$ and so $17 \mid |H|$. We know that $L_2(19)$ contains a $19:9$ subgroup and hence G has a $\{3,17,19\}$-subgroup T which is solvable and so $t(T) \leq 2$. But this is a contradiction, since $t(T) = 3$. Therefore $K/H \ncong L_2(19)$. □

Theorem 3.4. *Let G be a finite group satisfying $\Gamma(G) = \Gamma\big(\mathrm{Aut}(HS)\big)$. Then $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(HS)$, $U_6(2) \cdot 2$ or McL.*

Proof. Let $I = \{3,7,11\}$ and $J = \{7,11\}$. Now by using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is one of the following groups: M_{11}, M_{12}, M_{22}, McL, HS, $U_5(2)$, $U_6(2)$ and $L_2(11)$.

Case 1. Let $K/H \cong M_{11}$, M_{12}, $U_5(2)$ or $L_2(11)$.
By using [3] we know that $|Out(K/H)|$ is a divisor of 2. Therefore $7 \notin \pi(G/H)$, and hence $7 \in \pi(H)$. Since in each case, K/H has a $11:5$ subgroup it follows that G has a $\{5,7,11\}$ subgroup T, which is solvable and hence $t(T) \leq 2$. But this is a contradiction and so this case is impossible.

Case 2. Let $K/H \cong M_{22}$.
We note that $\mathrm{out}(M_{22}) \cong \mathbb{Z}_2$. Hence $G/H \cong M_{22}$ or $\mathrm{Aut}(M_{22})$. First let $G/H \cong M_{22}$, where H is a π_1-group and $\pi_1 = \{2,3,5,7\}$. We know that M_{22} has a $11:5$ subgroup (see [3]). If $2 \in \pi(H)$, then G has a $\{2,5,11\}$ subgroup T which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore $2 \notin \pi(H)$. If $3 \in \pi(H)$ or $7 \in \pi(H)$, then let T be a $\{3,5,11\}$ or $\{5,7,11\}$ subgroup of G, respectively. Then $t(T) \leq 2$, which is a contradiction. If $5 \in \pi(H)$, then let P be a Sylow 5-subgroup of H. If $Q \in \mathrm{Syl}_3(G)$, then Q acts fixed point freely on P, since $3 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group which implies that Q be a cyclic group and it is a contradiction. Hence $H = 1$ and so $G = M_{22}$. But $\Gamma(M_{22}) \neq \Gamma(\mathrm{Aut}(HS))$, since $2 \nsim 5$ in $\Gamma(M_{22})$. Therefore this case is impossible.

Now let $G/H \cong \mathrm{Aut}(M_{22})$. By using [3], M_{22} has a $11:5$ subgroup. Similar to the above discussion we conclude that $\{3,5,7\} \cap \pi(H) = \emptyset$, and hence H is a 2-group. But in this case 3 and 5 are not joined which is a contradiction. Therefore Case 2 is impossible, too.

Case 3. Let $K/H \cong U_6(2)$.
By using [3], it follows that $\mathrm{Out}(K/H) \cong S_3$. We know that $U_6(2) \cdot 3$ has an element of order 21. Therefore $G/H \cong U_6(2)$ or $U_6(2) \cdot 2$. Also $7 \notin \pi(H)$, since $U_6(2)$ has a $11:5$ subgroup. Therefore if $G/H \cong U_6(2)$, then $2 \in \pi$, $\pi \subseteq \{2,3,5\}$

and $G/O_\pi(G) \cong U_6(2)$, where $O_\pi(G) \neq 1$. Similarly if $G/H \cong U_6(2) \cdot 2$, then $G/O_\pi(G) \cong U_6(2) \cdot 2$ where $\pi \subseteq \{2,3,5\}$.

Case 4. Let $K/H \cong McL$.
Note that $\mathrm{Out}(McL) = 2$, but $G/H \ncong \mathrm{Aut}(McL)$, since $\mathrm{Aut}(McL)$ has an element of order 22. Similar to the above proof it follows that $G/O_\pi(G) \cong McL$ and $\pi \subseteq \{2,3,5\}$, since McL has a $11:5$ subgroup.

Case 5. Let $K/H \cong HS$.
There exists a $11:5$ subgroup in HS. Similar to Case 3, it follows that $G/O_\pi(G) \cong HS$, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$, or $G/O_\pi(G) \cong \mathrm{Aut}(HS)$ where $\pi \subseteq \{2,3,5\}$. □

Theorem 3.5. *Let G be a finite group.*

(a) *If $\Gamma(G) = \Gamma\big(\mathrm{Aut}(O'N)\big)$, then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$ or $G/O_2(G) \cong \mathrm{Aut}(O'N)$.*

(b) *If $\Gamma(G) = \Gamma\big(\mathrm{Aut}(\mathrm{Suz})\big)$, then $G/O_\pi(G) \cong \mathrm{Suz}$, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(\mathrm{Suz})$.*

Proof. (a) Let $I = \{3,11,31\}$ and $J = \{7,11\}$. Now by using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{31,31}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Hence K/H is $L_3(5)$, $L_5(2)$, $L_6(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $O'N$. If $K/H \cong L_2(5)$, $L_6(2)$, $L_2(31)$ or $G_2(5)$, then $11, 19 \in \pi(H)$, which is a contradiction, since $209 \notin \pi_e(G)$ and H is nilpotent. If $K/H \cong L_3(5)$ or $L_2(32)$, then $\{7,19\} \subseteq \pi(H)$, which is a contradiction, since $7 \nsim 19$ in $\Gamma(G)$. Therefore $K/H \cong O'N$ and $\mathrm{Out}(O'N) = 2$, which implies that $G/H \cong O'N$ or $O'N.2$. We know that $O'N$ has a $11:5$ subgroup by [3] and if we consider $\{5,11,p\}$-subgroup of G, where $p \in \{7,19,31\}$, it follows that $\pi(H) \cap \{7,19,31\} = \emptyset$. Therefore $\pi(H) \subseteq \{2,3,5,11\}$. Also $O'N$ has a $19:3$ subgroup, which implies that $\pi(H) \cap \{11\} = \emptyset$. Let $p \in \{3,5\}$. If $p \in \pi(H)$, then let P be the p-Sylow subgroup of H. If $Q \in \mathrm{Syl}_7(G)$, then Q acts fixed point freely on P, since $7 \nsim 3$ and $7 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group and hence Q is a cyclic group. But this is a contradiction since Sylow 7-subgroups of $O'N$ are elementary abelian by [3]. Therefore $\pi(H) \cap \{3,5\} = \emptyset$. Hence $\pi(H)$ is a 2-group. Then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$; or $G/O_\pi(G) \cong \mathrm{Aut}(O'N)$ where $\pi \subseteq \{2\}$.

(b) Let $I = \{7,11,13\}$ and $J = \{11,13\}$. Now by using Lemmas 3.1 and 3.3 we conclude that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a $C_{13,13}$ simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $\mathrm{Sz}(8)$, $U_3(4)$, ${}^3D_4(2)$, Suz, Fi_{22}, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $S_6(3)$, $G_2(4)$, $S_4(5)$ or $G_2(3)$.

If $K/H \cong {}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, $L_4(3)$, $S_4(5)$ or $G_2(3)$, then $\{7,11\} \subseteq \pi(H)$, which implies that $7 \sim 11$, since H is nilpotent. But this is a contradiction. If $K/H \cong {}^3D_4(2)$, $L_2(27)$, $L_2(13)$ or $L_3(3)$, then $\{5,11\} \subseteq \pi(H)$ and we get a contradiction similarly, since $5 \nsim 11$.

If $K/H \cong G_2(4)$, $S_6(3)$, $O_7(3)$ or $O_8^+(3)$, then $11 \in \pi(H)$ and K/H has a $13:3$ subgroup by [3]. Let T be a $\{3,11,13\}$-subgroup of G. It follows that $t(T)=3$, which is a contradiction since T is solvable.

If $K/H \cong \mathrm{Fi}_{22}$, then $G/H \cong \mathrm{Fi}_{22}$ or $\mathrm{Fi}_{22}\cdot 2$, where $\pi(H) \subseteq \{2,3,5,7,11\}$. Since Fi_{22} has $11:5$ and $13:3$ subgroups it follows that $\{7,11\} \cap \pi(H) = \emptyset$. Therefore $G/O_\pi(G) \cong \mathrm{Fi}_{22}$ or $\mathrm{Aut}(\mathrm{Fi}_{22})$, where $\pi \subseteq \{2,3,5\}$.

Let $K/H \cong Sz(8)$. It is known that $\mathrm{Out}(\mathrm{Sz}(8)) \cong \mathbb{Z}_3$ and so $G/H \cong \mathrm{Sz}(8)$ or $\mathrm{Sz}(8)\cdot 3$. If $G/H \cong \mathrm{Sz}(8)$, then $\{3,11\} \subseteq \pi(H)$ which is a contradiction, since $3 \nsim 11$. If $G/H \cong \mathrm{Sz}(8)\cdot 3$, then let T be $\{3,7,11\}$-subgroup of G, since $Sz(8)$ has a $7:6$ subgroup. Then $t(T)=3$, which is a contradiction.

If $K/H \cong \mathrm{Suz}$, then $G/H \cong \mathrm{Suz}$ or $\mathrm{Aut}(\mathrm{Suz})$. If $G/K \cong \mathrm{Suz}$, then $\pi(H) \subseteq \{2,3,5,7,11\}$. Since Suz has a $11:5$ and $13:3$ subgroups it follows that $7, 11 \notin \pi(H)$. Therefore $G/O_\pi(G) \cong \mathrm{Suz}$, where $2 \in \pi$ and $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$. If $G/H \cong \mathrm{Suz}\cdot 2$, then it follows that $G/O_\pi(G) \cong \mathrm{Aut}(\mathrm{Suz})$, where $\pi \subseteq \{2,3,5\}$. □

Remark. W. Shi and J. Bi in [15] put forward the following conjecture:
Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$, (ii) $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for sporadic simple groups, alternating groups and some simple groups of Lie type. As a consequence of the main results, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.6. *Let G be a finite group and A be the automorphism group of a sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$, then $G \cong A$.*

We note that Theorem 3.6 was proved in [10] by using the characterization of almost sporadic simple groups with their order components. Now we give a new proof for this theorem. In fact we prove the following result which is a generalization of Shi-Bi Conjecture and so Theorem 3.6 is an immediate consequence of Theorem 3.7.

Theorem 3.7. *Let A be the automorphism group of a sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If G is a finite group satisfying $|G| = |A|$ and $\Gamma(G) = \Gamma(A)$, then $G \cong A$.*

Proof. First let $A = \mathrm{Aut}(M_{22})$. By using Theorem 3.3, it follows that $G/O_2(G) \cong M_{22}$ or $G/O_\pi(G) \cong \mathrm{Aut}(M_{22})$, where $\pi \subseteq \{2\}$. If $G/O_2(G) \cong M_{22}$, then $|O_2(G)| = 2$ and hence $O_2(G) \subseteq Z(G)$ which is a contradiction, since G has more than one component and hence $Z(G)=1$. Therefore $G/O_\pi(G) \cong M_{22}\cdot 2(M_{22})$, where $2 \in \pi$, which implies that $O_\pi(G)=1$ and hence $G \cong \mathrm{Aut}(M_{22})$

Let $A = \mathrm{Aut}(HS)$. By using Theorem 3.4, it follows that $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong U_6(2)\cdot 2$, McL or $\mathrm{Aut}(HS)$, where $\pi \subseteq \{2,3,5\}$.

By using [3], it follows that 3^6 divides the orders of $U_6(2)$, $U_6(2)\cdot 2$ and McL, but $3^6 \nmid |G|$.

Therefore $G/O_\pi(G) \cong HS$ or $\mathrm{Aut}(HS)$. Now we get the result similarly to the last case.

For convenience we omit the details of the proof of other cases. □

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