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PRESERVATION OF FAST COMPLETENESS UNDER INDUCTIVE LIMITS

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A topological vector space V is said to be fast complete if each bounded set is bounded in some continuously embedded Banach space. Here we are concerned with the problem of when an inductive limit of fast complete locally convex spaces is fast complete.

Let E be a Hausdorff locally convex inductive limit of a nested sequence of Hausdorff locally convex spaces E_n. Then E is said to be regular provided that each bounded subset of E is contained and bounded in one of the constituent spaces E_n. For instance it is known that an inductive limit of Hilbert spaces must always be regular [1], while there are inductive limits of reflexive Fréchet spaces which are not [2].

Our present results are concerned with inductive limits which satisfy the following property

BP: each space E_n has at least one open subset which is bounded in some subsequent space E_{n+k}.

We note that in particular inductive limits of Banach spaces satisfy the property BP.

Before commencing we recall that a set S is said to be nowhere dense in another set T provided the closure of $S \cap T$ in T has void interior in T—otherwise S is somewhere dense in T.

Lemma. *Let E satisfy the property BP. Let N be a Banach space continuously embedded in E. Then there exists at least one $m \in \mathbb{N}$ such that each non-void open subset of E_m is somewhere dense in N.*

Proof. Assume false. In view of the property BP, the space E_1 possesses a balanced, convex, neighborhood V_1 of the origin which is bounded in some later space $E_{S(1)}$—by making V_1 smaller if necessary, we may assume that it is nowhere dense in N.

Suppose that an increasing sequence $\{S(j)\}_{j=1}^{n}$ of natural numbers has been chosen along with a sequence $\{V_j\}_{j=1}^{n}$ of subsets such that each V_j is a balanced, convex, neighborhood of the origin in $E_{S(j-1)}$ (where $S(0) \equiv 1$), V_j is nowhere dense in N, and V_j is bounded in $E_{S(j)}$. We then apply the property BP to select a balanced, convex, neighborhood V_{n+1} of the origin on $E_{S(n)}$ and a natural number $S(n+1)$ greated than $S(n)$ such that V_{n+1} is bounded in $E_{S(n+1)}$, being careful to make our selection so that V_{n+1} is nowhere dense in N.

Having chosen our sequence $\{V_n\}_{n=1}^{\infty}$ inductively, we let W be the convex hull of the set theoretic union $\bigcup_{n=1}^{\infty} V_n$. Evidently W is a neighborhood of the origin in E.

For each $m \in \mathbf{N}$ we construct a point p_m in N as follows. Since V_1 is nowhere dense in N, there exists a ball $B(x_1)$ of radius less than $\frac{1}{2}$ contained in the unit ball of N such that $B(x_1)$ does not intersect the closure of mV_1 in N. Suppose that a sequence $\{x_j\}_{j=1}^{n}$ has been chosen along with a sequence $\{B(x_j)\}_{j=1}^{n}$ of balls such that, for each $j = 1, \ldots, n$, $B(x_j)$ does not intersect $m(V_1 + \ldots + V_j)$, the radius of $B(x_j)$ is less than 2^{-j}, and (if $j > 1$) the closure of $B(x_j)$ is contained in $B(x_{j-1})$.

Note that V_1 is bounded in $E_{S(1)}$ and so $V_1 + V_2$ is contained in a scalar multiple of V_2. Similarly $V_1 + V_2$ is bounded in V_3 and so $V_1 + V_2 + V_3$ is contained a scalar multiple of V_3. Evidently $V_1 + \ldots + V_n$ is contained in a scalar multiple of V_n, and so $m(V_1 + \ldots + V_n)$ is as well. Since V_n is nowhere dense in N, it follows that $m(V_1 + \ldots + V_n)$ is as well. Hence there exists an open ball $B(x_{n+1})$ contained in $B(x_n)$ which does not intersect $m(V_1 + \ldots + V_n)$. By reducing the radius if necessary, we may select $B(x_{n+1})$ such that its closure is in $B(x_n)$ and its radius is less than $2^{-(n+1)}$.

The sequence x_n is Cauchy and so has a limit p_m which lies in the intersection of all the balls $B(x_n)$. This point p_m does not lie in any of the sets $m(V_1 + \ldots + V_n)$. Consequently p_m is not in the neighborhood mW.

Since p_m is in the unit ball of N for each $m \in \mathbf{N}$, it follows that W does not absorb the unit ball of N. By hypothesis N is continuously embedded and so its unit ball is bounded in E: an absurdity. □

Theorem. *Let E be a Hausdorff inductive limit of fast complete spaces satisfying property BP. Then the following are equivalent:*

(i) *E is fast complete;*

(ii) *E is a regular inductive limit.*

Proof. We first suppose that (ii) holds and show that (i) follows. Let D be any bounded subset of E. Regularity implies that D is a bounded subset of E_n for some $n \in \mathbf{N}$. Since E_n is fast complete, D is a bounded subset of some Banach space N continuously embedded in E_n. Since E_n is continuously embedded in E, N is as well. Thus E is fast complete.

Now we show that (i) implies (ii) by assuming that (i) holds and (ii) does not hold, and deriving absurdity. Since (ii) does not hold there exists a bounded subset

D of E which is not bounded in any of the constituent spaces E_n. Since (i) holds, we may assume that D is in the closed unit ball of some Banach space N which is continuously embedded in E. From the preceding lemma follows that there exists some $m \in \mathbb{N}$ such that each open subset of E_m is somewhere dense in N. Because of the property BP we may select a balanced, convex neighborhood V of the origin in E_m and an integer $k > m$ such that V is bounded in E_k.

Since V is somewhere dense in N, the closure ∇ of V in N has non-void interior ∇° in N. Since V is balanced, ∇° is also balanced and so must be a neighborhood of the origin in N. Hence there exists $r > 0$ such that $D \subseteq r\nabla^\circ$.

Let $p \in D$ be arbitrary. Then there exists $x_1 \in rV$ such that

$$\|x_1 - p\| < \frac{1}{2}.$$

Suppose that the sequence $\{x_j\}_{j=1}^n$ has been chosen such that

$$x_j \in 2^{-j+1} rV \quad \text{and} \quad \Big\| \sum_{i=1}^{j} x_i - p \Big\| < 2^{-j}$$

for each $j = 1, \ldots, n$. Then $\left(p - \sum_{j=1}^{n} x_j\right) \in 2^{-n} D \subseteq 2^{-n} r\nabla^\circ$ and so there exists $x_{n+1} \in 2^{-n} rV$ such that $\left\| x_{n+1} - \left(p - \sum_{j=1}^{n} x_j\right) \right\| < 2^{-(n+1)}$.

Evidently we have $p = \sum_{n=1}^{\infty} x_n$ in the Banach space N. Let $|\!|\!|\ |\!|\!|$ denote the Minkowski functional on E_m genereted by the neighborhood rV. For each $n \in \mathbb{N}$ we have $|\!|\!| x_n |\!|\!| \leqslant 2^{-n+1}$, whence follows that the sequence $\sum_{j=1}^{n} x_j$ of partial sums is $|\!|\!|\ |\!|\!|$-Cauchy. Since rV is bounded subset of E_k, it follows that the sequence $\sum_{j=1}^{n} x_j$ is Cauchy in E_k. Since E_k is fast-complete and the Cauchy sequence $\sum_{j=1}^{n} x_j$ is bounded, there exists a Banach space M continuously embedded in E_k such that rV is in the unit ball of M. Since $|\!|\!|\ |\!|\!|$ dominates the norm on M, it follows that $\sum_{j=1}^{n} x_j$ is absolutely convergent in M and has a limit in the unit ball of M. Thus $\sum_{j=1}^{n} x_j$ converges to q in E_k and so to q in E as well. But $\sum_{j=1}^{n} x_j$ converges to p in N (and thus in E). Since E is Hausdorff, we have $p = q$.

Since p was arbitrary, it follows that D is a subset of the unit ball of M, and thus D is bounded in E_k. This implies that E is regular. □

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