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ON EXTENSION OF VECTOR POLYMEASURES, II

IVAN DOBRAKOV

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ABSTRACT. We prove a necessary and sufficient condition for extension of a vector polymeasure from Cartesian product of rings to the Cartesian product of generated \( \sigma \)-rings.

In this addition to [2], we give a necessary and sufficient condition for the existence of a unique separately countably additive extension \( \gamma: \sigma(R_1) \times \cdots \times \sigma(R_d) \to Y \) of a separately countably additive \( \gamma_0: R_1 \times \cdots \times R_d \to Y \). Here \( R_i \) is a ring of subsets of a non empty set \( T_i \), \( \sigma(R_i) \) is the generated \( \sigma \)-ring, for \( i = 1, \ldots, d \), and \( Y \) is a Banach space.

Since for any sequence \( A_n \in \sigma(R), n = 1, 2, \ldots \), there is a countable subring \( R' \subset R \) such that \( A_n \in \sigma(R') \) for each \( n = 1, 2, \ldots \), see [6; §5, Theorems C and D], the uniqueness of the extension of a vector polymeasure, see [1; Corollary of Lemma 4], implies the following:

**Lemma.** A separately countably additive \( \gamma_0: R_1 \times \cdots \times R_d \to Y \) has a unique separately countably additive extension \( \gamma: \sigma(R_1) \times \cdots \times \sigma(R_d) \to Y \) if and only if \( \gamma_0: R'_1 \times \cdots \times R'_d \to Y \) has a separately countably additive extension \( \gamma: \sigma(R'_1) \times \cdots \times \sigma(R'_d) \to Y \) for any countable subrings \( R'_i \subset R_i, i = 1, \ldots, d \).

Note that [2; Corollary of Theorem 5] gives a necessary and sufficient condition for the extension in the case of countable rings \( R_i, 1, \ldots, d \). The theorem below is not limited, but only reducible, to this case. In a sense, the theorem is a counterpart of [4; Theorem 9] (with iterated limits there) and [5; Theorem 2], which give similar double limit characterizations of \( L_1 \)-representability of bounded multilinear operators on \( \times C_0(T_i) \) and on \( \times C_0(T_i, X_i) \) respectively.

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THEOREM. A separately countably additive \( \gamma_0: R_1 \times \cdots \times R_d \to Y \) has a unique separately countably additive extension \( \gamma: \sigma(R_1) \times \cdots \times \sigma(R_d) \to Y \) if and only if the limits below exist in \( Y \) and

\[
\lim_{n_1 \to \infty} \lim_{k_1 \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots, A_{d,n,k}) = \lim_{n \to \infty} \lim_{k \to \infty} \lim_{n_1 \to \infty} \lim_{k_1 \to \infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots, A_{d,n,k}),
\]

\[
\lim_{n_2 \to \infty} \lim_{k_2 \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(A_{1,n,k}, A_{2,n_2,k_2}, A_{3,n,k}, \ldots, A_{d,n,k}) = \lim_{n \to \infty} \lim_{k \to \infty} \lim_{n_2 \to \infty} \lim_{k_2 \to \infty} \gamma_0(A_{1,n,k}, A_{2,n_2,k_2}, A_{3,n,k}, \ldots, A_{d,n,k}),
\]

\[
\lim_{n_d \to \infty} \lim_{k_d \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(A_{1,n,k}, \ldots, A_{d-1,n,k}, A_{d,n,k,d}) = \lim_{n \to \infty} \lim_{k \to \infty} \lim_{n_d \to \infty} \lim_{k_d \to \infty} \gamma_0(A_{1,n,k}, \ldots, A_{d-1,n,k}, A_{d,n,k,d})
\]

whenever

\[
(A_{1,n,k}, \ldots, A_{d,n,k}) \in R_1 \times \cdots \times R_d, \quad n, k = 1, 2, \ldots
\]

and \( \lim_{n \to \infty} \lim_{k \to \infty} \chi_{A_i,n,k}(t_i) \) exists for each \( t_i \in T_i \) and each \( i = 1, \ldots, d \).

Proof. The necessity of the conditions follows immediately from [1; Theorem 1].

Conversely, assume the conditions of the theorem hold. By Lemma, we may and will suppose that each \( R_i, i = 1, \ldots, d, \) is a countable ring. Having this reduction we obtain the existence of the extension \( \gamma \) by induction in the dimension \( d \). For \( d = 1 \) it follows from Kluvanek's extension theorem, see [7]. Suppose the assertion holds for \( d - 1, \ d > 1, \) and let \( R_i \in R_i, i = 1, \ldots, d \). Then, by the inductive hypothesis, there are uniquely determined separately countably additive extensions \( \gamma_1(R_1, \ldots): \sigma(R_2) \times \cdots \times \sigma(R_d) \to Y \) and \( \gamma_2(\cdot, R_2, \ldots, R_d): \sigma(R_1) \to Y \). Since \( R_1, \ldots, R_d \) are countable rings, by [1; Theorem 11], there are countably additive measures \( \lambda_i: \sigma(R_i) \to [0,1], i = 1, \ldots, d \), such that \( M_i \in \sigma(R_i) \) and \( \lambda_i(M_i) = 0, i = 1, \ldots, d \), imply that \( \gamma_1(R'_1, M_2, \ldots, M_d) = 0 \) for each \( R'_1 \in R_1 \), and \( \gamma_2(M_1, R'_2, \ldots, R'_d) = 0 \) for each \( (R'_2, \ldots, R'_d) \in R_2 \times \cdots \times R_d \).

Let \( (E_1, \ldots, E_d) \in \sigma(R_1) \times \cdots \times \sigma(R_d) \). For each \( i = 1, \ldots, d \) take \( A_i \in (R_i)_{\sigma \delta} \) so that \( E_i \subset A_i \) and \( \lambda_i(A_i - E_i) = 0 \), and let \( A_{i,n,k} \in R_i, n, k = 1, 2, \ldots \) be such that \( A_{i,n,k} \neq A_{i,n} \neq A_i \), see [3; Lemma C in the proof of Theorem 18]. Then

\[
\gamma_1(R_1, E_2, \ldots, E_d) = \gamma_1(R_1, A_2, \ldots, A_d) = \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(R_1, A_{2,n,k}, \ldots, A_{d,n,k})
\]
for each \( R_1 \in R_1 \), and

\[
\gamma_2(E_1, R_2, \ldots, R_d) = \gamma_2(A_1, R_2, \ldots, R_d) = \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(A_{1,n,k}, R_2, \ldots, R_d)
\]

for each \((R_2, \ldots, R_d) \in R_2 \times \cdots \times R_d\), by \([1; \text{Theorem 1}]\).

Suppose \( R_{1,n_1} \in R_1 \), \( n_1 = 1, 2, \ldots \) are pairwise disjoint, and put \( A_{1,2k-1} = R_{1,k} \) and \( A_{1,2k} = \emptyset \) for \( k = 1, 2, \ldots \). Then \( \lim_{n_1 \to \infty} \gamma_1(A_{1,n_1}, E_2, \ldots, E_d) = 0 \). Hence, by Klunéek’s extension theorem, see \([7]\), there is a unique countably additive extension \( \gamma(\cdot, E_2, \ldots, E_d) : \sigma(R_1) \in Y \) of \( \gamma_1(\cdot, E_2, \ldots, E_d) : R_1 \to Y \). Further we have the equalities:

\[
\gamma(A_1, E_2, \ldots, E_d) = \lim_{n_1 \to \infty} \lim_{k_1 \to \infty} \gamma_1(A_{1,n_1,k_1}, E_2, \ldots, E_d)
\]

\[
\quad = \lim_{n_1 \to \infty} \lim_{k_1 \to \infty} \lim_{n \to \infty} \lim_{k \to \infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots, A_{d,n,k})
\]

\[
\quad = \lim_{n \to \infty} \lim_{k \to \infty} \lim_{n_1 \to \infty} \lim_{k_1 \to \infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots, A_{d,n,k})
\]

\[
\quad = \lim_{n \to \infty} \lim_{k \to \infty} \gamma_2(E_1, A_{2,n,k}, \ldots, A_{d,n,k})
\]

by the assumption of the theorem. Since analogous equations hold for any \( A'_1 \in (R_1)_\sigma \) such that \( A'_1 \supset E_1 \) and \( \lambda_1(A'_1 - E_1) = 0 \), we may uniquely define \( \gamma(E_1, E_2, \ldots, E_d) = \gamma(A_1, E_2, \ldots, E_d) \). By the assumption, \( \gamma_1(A_{1,n_1,k_1, \ldots, \cdot}) : \sigma(R_2) \times \cdots \times \sigma(R_d) \to Y \) is separately countably additive for each \( n_1, k_1 = 1, 2, \ldots \), hence \( \gamma(E_1, \ldots) : \sigma(R_2) \times \cdots \times \sigma(R_d) \to Y \) being their set wise iterated limit is also separately countably additive by the (VHSN)-theorem for polymeasures, see the beginning of \([1]\). The theorem is proved.

\[\square\]

It will be of interest to solve the following:

**Problem.** Let \( \gamma_0 : R_1 \times \cdots \times R_d \to Y \) be separately countably additive, and suppose that there is a separately countably additive extension \( \gamma_1 : \sigma(R_1) \times R_2 \times \cdots \times R_d \to Y \) of \( \gamma_0 \), for each \( A_1 \in \sigma(R_1) \) there is a separately countably additive extension \( \gamma_2(A_1, \ldots) : \sigma(R_2) \times R_3 \times \cdots \times R_d \to Y \) of \( \gamma_1(A_1, \ldots) : R_2 \times \cdots \times R_d \to Y \), \( \ldots \), for each \( A_1, \ldots, A_{d-1} \in \sigma(R_1) \times \cdots \times \sigma(R_{d-1}) \) there is a countably additive extension \( \gamma_d(A_1, \ldots, A_{d-1}, \cdot) : \sigma(R_d) \to Y \) of \( \gamma_{d-1}(A_1, \ldots, A_{d-1}) : R_d \to Y \). Assume analogous subsequent extensions exist when we start from \( \sigma(R_2), \ldots, \sigma(R_d) \) and end on \( \sigma(R_1), \ldots, \sigma(R_{d-1}) \) respectively. Are then all the \( d \) final set functions mutually equal on \( \sigma(R_1) \times \cdots \times \sigma(R_d) \)?
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