ON OSCILLATORY SOLUTIONS OF DIFFERENTIAL INEQUALITIES

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Let $-\infty < a < b \leqslant \infty$, $n \geqslant 2$ and let $f_i\colon [a,b) \times \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, n$ fulfil the local Carathéodory conditions. When studying oscillatory solutions of the system

$$(1) \qquad y_i' = f_i(t, y_1, \ldots, y_n), \quad i = 1, \ldots, n$$

it is very often supposed that

$$(2) \qquad \begin{aligned} &\alpha_i f_i(t, x_1, \ldots, x_n) x_{i+1} > 0 \quad \text{for } x_{i+1} \neq 0, \\ &f_i(t, x_1, \ldots, x_n) = 0 \text{ for } x_{i+1} = 0, \quad i = 1, \ldots, n \end{aligned}$$

where $\alpha_i \in \{-1, 1\}$, $x_{n+1} = x_1$, see [3, 4].

$y = (y_1, \ldots, y_n)$ is called a *solution* of (1) if $y_i\colon J = (a,b) \to \mathbf{R}$ is locally absolutely continuous and (1) holds for almost all $t \in J$.

The system (1) leads naturally to be the investigation of properties of a system of differential inequalities

$$(3) \qquad \begin{aligned} &\alpha_i y_i'(t) y_{i+1}(t) > 0 \quad \text{for } y_{i+1}(t) \neq 0, \\ &y_i'(t) = 0 \Leftarrow y_{i+1}(t) = 0, \quad i = 1, \ldots, n \end{aligned}$$

where $\alpha_i \in \{-1, 1\}$, $t \in J$, $y_{n+1} \equiv y_1$.

$y = (y_1, \ldots, y_n)$ is called a *solution* of (3) if $y_i\colon J \to \mathbf{R}$ is locally absolute continuous and (3) holds for all $t \in J$ for which $y_i'(t)$ exists. Denote by T the set of all such solutions. It is evident that T is not empty and that (1), (2) is a special case of (3).

Let n_0 be the entire part of $\frac{n}{2}$ and let $y_{j+kn} \equiv y_j$, $\alpha_{j+kn} = \alpha_j$ be valid for $j \in \{1, \ldots, n\}$, $k \in \{\ldots, -1, 0, 1, \ldots\}$.

A continuous function $z\colon J \to \mathbf{R}$ is called *oscillatory* if $\sup\limits_{t\in[\tau,b)} |z(t)| > 0$ for any $\tau \in J$ and there exists a sequence of its zeros tending to b.

Let $y \in T$, $i \in \{1, \ldots, n\}$ hold. A number τ is called a *simple zero* of y_i if $y_i(\tau) = 0$, $y_{i+1}(\tau) \neq 0$.

Suppose that τ is a simple zero of y_i. It follows from (3) that there exists an interval $[\tau_1, \tau_2] \subset J$ such that $\tau_1 < \tau \leqslant \tau_2$, $y_i'(\tau) \neq 0$, $y_i'(t)$ has a constant sign for almost all $t \in [\tau_1, \tau_2]$ and thus $y_i(t)\ y_i(\bar{t}) < 0$ holds for $t \in [\tau_1, \tau)$, $\bar{t} \in (\tau, \tau_2]$.

In the paper conditions are given under which all zeros of oscillatory functions y_i for $y \in T$ are simple in a left neighbourhood of the number b. We generalize to (3) or (1), (2) similar results obtained for the differential equation of the n-th order in [5] (linear case) and [2] (nonlinear case):

$$(4) \qquad \begin{aligned} &y^{(n)} = f\big(t, y, \dots, y^{(n-1)}\big) \quad \text{in } J \times \mathbf{R}^n, \quad n \geqslant 2, \\ &\alpha f(t, x_1, \dots, x_n) x_1 > 0 \end{aligned}$$

where $\alpha \in \{-1, 1\}$, f is continuous. This equation can be transformed into (1), (2) with $\alpha_1 = \ldots = \alpha_{n-1} = 1$, $\alpha_n = \alpha$.

Let $Z\colon J \to \mathbf{R}$ be continuous. A point $c \in [a, b]$ is called an H-point of Z if there exist sequences $\{\tau_k\}_1^\infty$, $\{\bar{\tau}_k\}_1^\infty$ of numbers from J tending to c such that $Z(\tau_k) = 0$, $Z(\bar{\tau}_k) \neq 0$, $(\tau_k - c)(\bar{\tau}_k - c) > 0$.

Lemma 1. *Let $i, j \in \{1, \dots, n\}$ and $y \in T$ hold. Then $c \in [a, b]$ is an H-point of y_i if and only if c is an H-point of y_j.*

P r o o f. Let $\{\tau_k\}_1^\infty$, $\{\bar{\tau}_k\}_1^\infty$ be increasing sequences of zeros of y_i such that $\tau_k \leqslant \bar{\tau}_k < \tau_{k+1}$, $\lim\limits_{k\to\infty} \tau_k = c$, $y(t) \neq 0$ on $(\tau_k, \bar{\tau}_k)$, $k \in N$. Then there exist numbers t_k, $\bar{t}_k$, $k \in N$ such that $\tau_k < t_k < \bar{t}_k < \bar{\tau}_k$, $y_i'(t_k)$, $y_i'(\bar{t}_k)$ exist and $y_i'(t_k) y_i'(\bar{t}_k) < 0$ is valid. According to (3) we have $y_{i+1}(t_k) y_{i+1}(\bar{t}_k) < 0$ and there exists a zero β_k of y_{i+1}, $t_k < \beta_k < \bar{t}_k$. Thus c is an H-point of y_{i+1}, too. By repeating the considerations for $i+1$, $i+2$, …, n, 1, 2, …, $i-1$ we get the statement of the lemma. The lemma is proved. □

Let $y \in T$, $j \in \{1, \dots, n\}$, and let y_j be oscillatory. Since b is an H-point of y_j, it follows from Lemma 1 that y_i, $i = 1, \dots, n$ is oscillatory, too. Thus we can define: A solution $y \in T$ is oscillatory if every component of y is oscillatory. A point $c \in J$ is an H-point of $y \in T$ if it is an H-point of every component of y. Further, let T_0, $T_0 \subset T$ be the set of oscillatory solutions of (3) for which there exists no H-point in the interval J. The set T_0 is nonempty, it contains e.g. oscillatory solutions of (1), (2), see [3,4].

Lemma 2. *Let $y \in T$, $i \in \{1, \dots, n\}$, $y_i(t) = 0$ on $[c_1, c_2] \subset J$, $c_1 < c_2$ be valid. Then $y_j(t) = 0$ on $[c_1, c_2]$, $j = 1, \dots, n$.*

P r o o f. As $y_i'(t) = 0$ on $[c_1, c_2]$, it follows from (3) that $y_{i+1}(t) = 0$ on $[c_1, c_2]$. By repeating this argument for $i+1$, $i+2$, …, n, 1, …, $i-1$ we get the statement. The lemma is proved. □

Notation. Let $y \in T$. Put $V_n(t) = \prod_{i=1}^{n} y_i(t)$, $S = \{t : t \in J, V_n(t) \neq 0\}$. If $r, k \in \{1, \dots, 2n\}$, $r \leqslant k$, then let us define

$$W_{rk}(t) = \operatorname{card}\{i : r < i \leqslant k, \alpha_{i-1} y_{i-1}(t) y_i(t) < 0\} \text{ for } r < k,$$
$$W_{rr}(t) = 0, \ t \in S.$$

Put $W(t) = W_{1,n+1}(t)$. Further, let $\tau \in J$, $W(\tau) = 0$, $\sum_{i=1}^{n} |y_i(\tau)| \neq 0$ be valid. Let us define integer numbers m, j_i, l_i, $i = 1, \dots, m$ and $B(\tau)$ by the following relations:

$$\begin{aligned}
l_0 &= \min\{s : y_s(\tau) \neq 0, \ 1 \leqslant s \leqslant n\},\\
j_m &= \max\{s : y_s(\tau) \neq 0, \ 1 \leqslant s \leqslant n\},\\
j_i &= \max\{s : y_l(\tau) \neq 0, \ l_{i-1} \leqslant l \leqslant s \leqslant j_m\}, \quad i = 1, \dots, m-1\\
l_i &= \min\{s : y_s(\tau) \neq 0, \ j_i < s < j_m\}, \quad i = 1, \dots, m-1\\
l_m &= n + l_0,
\end{aligned}$$

$$B(\tau) = \sum_{i=1}^{m} \Big\{ l_i - j_i - 1 + \frac{1}{2}((-1)^{l_i - j_i} + 1) \prod_{m=j_i}^{l_i - 1} (\alpha_m) \operatorname{sign}(y_{l_i}(\tau) y_{j_i}(\tau)) \Big\}. \tag{5}$$

Lemma 3. *Let $y \in T$, $0 \leqslant t_0 < \tau < t_1 < b$, $\sum_{i=1}^{n} |y_i(\tau)| > 0$, $V_n(\tau) = 0$ and $V_n(t) \neq 0$ for $t \in [t_0, t_1] - \{\tau\}$ be valid. Then*

$$W(t_0) - W(t_1) = B(\tau) \geqslant 0$$

holds.

Proof. It is clear that the function W is constant on the intervals $[t_0, \tau)$ and $(\tau, t_1]$. According to (5) we get

$$W(t) = W_{1,n+1}(t) = W_{l_0, l_m}(t), \ t \in [t_0, \tau) \cup (\tau, t_1],$$
$$W(t_0) - W(t_1) = \sum_{i=1}^{m} (W_{j_i l_i}(t_0) = W_{j_i l_i}(t_1)). \tag{6}$$

Consider the function $W_{j_i l_i}$. It follows from (5) that $l_i \geqslant j_i + 2$,

$$y_{j_i}(\tau) \neq 0, \ y_s(\tau) = 0 \text{ for } j_i < s < l_i, \ y_{l_i}(\tau) \neq 0. \tag{7}$$

This together with (3) and (7) implies that the following relations are valid in a right (left) neighbourhood of τ for almost all t:

$$\begin{aligned}
&y_{j-1}(\tau) = 0, \quad y_j(t) \neq 0 \Rightarrow \alpha_{j-1} y'_{j-1}(t) y_j(t) > 0\\
&\qquad \Rightarrow \alpha_{j-1} y_{j-1}(t) y_j(t) > 0, \ (< 0)\\
&j = l_i, \ l_i - 1, \ \dots, \ j_i + 2.
\end{aligned} \tag{8}$$

Thus

$$\operatorname{sign} y_{j_i+1}(t_1) = \alpha_{l_i-1} \dots \alpha_{j_i+1} \operatorname{sign} y_{l_i}(t_1),$$
$$\operatorname{sign} y_{j_i+1}(t_0) = (-1)^{l_i-j_i-1} \alpha_{l_i-1} \dots \alpha_{j_i+1} \operatorname{sign} y_{l_i}(t_0)$$

and

$$W_{j_i l_i}(t_1) = \frac{1}{2}\Big(1 - \prod_{m=j_i}^{l_i-1} \alpha_m \operatorname{sign}\big(y_{l_i}(\tau) y_{j_i}(\tau)\big)\Big),$$
$$W_{j_i l_i}(t_0) = l_i - j_i - 1 + \frac{1}{2}\Big(1 - (-1)^{l_i-j_i-1} \prod_{m=j_i}^{l_i-1} \alpha_m \operatorname{sign}\big(y_{l_i}(\tau) y_{j_i}(\tau)\big)\Big).$$

Consequently, we have

$$W_{j_i l_i}(t_0) - W_{j_i l_i}(t_1) = l_i - j_i - 1 + \frac{1}{2}\Big((-1)^{l_i-j_i} + 1\Big) \prod_{m=j_i}^{l_i-1} \alpha_m \operatorname{sign}\big(y_{l_i}(\tau) y_{j_i}(\tau)\big) \geqslant 0$$

and the statement of the lemma follows from (6). The lemma is proved. □

Consequence 1. *Let the assumptions of Lemma* 3 *be fulfilled and, moreover, let there exist numbers* i, j, $0 \leqslant i < j < 2n$ *such that* $y_i(\tau) y_j(\tau) \neq 0$, $y_s(\tau) = 0$ *for* $i < s < j$ *and either* $j - i = 2$, $\alpha_i \alpha_{i+1} \operatorname{sign} y_i(\tau) y_j(\tau) > 0$ *or* $j - i \geqslant 3$ *is valid. Then* $W(t_0) - W(t_1) > 0$.

Lemma 4. *Let* $y \in T$, $0 \leqslant t_0 < \tau_1 \leqslant \tau_2 < t_1 < b$, $y_i \equiv 0$ *on* $[\tau_1, \tau_2]$, $i = 1, \dots, n$ *and* $V_n(t) \neq 0$ *for* $t \in [t_0, t_1] - [\tau_1, \tau_2]$ *be valid. Then* $W(t_0) - W(t_1) > 0$.

P r o o f. The relations (8) are valid in a right (left) neighbourhood of the number τ_2 (τ_1) for $j = n+1, n+2, \dots, 2$ and thus $W(t_1) = n$, $W(t_0) = 0$ holds. The lemma is proved. □

Theorem 1. *Let* $y \in T$ *be valid and let the interval* J *have no* H*-point of this solution. Then the function* W *is nonincreasing on the set* S.

P r o o f. Let $t_1, t_2 \in S$, $t_1 < t_2$ be valid. As J has no H-points of y, the interval $[t_1, t_2]$ can be divided into a finite number of subintervals on which the assumptions of Lemma 3 or Lemma 4 are fulfilled. The theorem is proved. □

R e m a r k. The fact that W is nonincreasing was proved for differential equation of the n-th order in [5], [2]. It is also used in [6] for a cyclic feedback system $y_i' = f_i(y_{i-1}, y_i)$, $i \bmod n$ (the assumptions of f are such that this system can be easily transformed into (1), (2)).

Theorem 2. *Let $y \in T_0$. Then there exists a number $\bar{t} \in J$ such that the following statements hold for $I = [\bar{t}, b)$.*

I. *The zeros of y_i, $i = 1, \dots, n$ are simple on I.*

II. *If $i \in \{1, \dots, n\}$, $c \in I$, $y_i(c) = 0$ is valid, then $\alpha_{i-1}\alpha_i y_{i+1}(c) y_{i-1}(c) < 0$.*

III. *The function $m = W(t)$ is constant on the set $S \cap I$, $m \in \{1, \dots, n-1\}$, and the number $m + \frac{1}{2}\left(1 + \prod_{i=1}^{m} \alpha_i\right)$ is odd.*

IV. *Let $i \in \{1, \dots, n\}$. Between two arbitrary consecutive zeros of y_i lying in I there exists a single zero of y_{i+1}.*

V. *Let $i \in \{2, \dots, n+1\}$. Between two arbitrary consecutive zeros of y_i lying in I there exists a single zero of y_{i-1}.*

Proof. It follows from Theorem 1 that W is increasing on S. As $y \in T_0$, we have $S \cap [\tau, b) \neq \emptyset$ for an arbitrary $\tau \in J$. As W acquires the values from the set $\{0, 1, \dots, n\}$, there exist numbers $\bar{t}$ and m such that $\bar{t} \in S$, $W(t) = m$ for $t \in I \cap S$. The statements I and II follow from Consequence 1 and Lemma 4.

Let us prove the rest of III. The inequality $m \neq 0$ follows directly from $y \in T_0$ and the case I. Thus let $m = n$. Let $\tau \in I$ be an arbitrary zero of y_2. Then it follows from the case II that $\alpha_1\alpha_2 \operatorname{sign}(y_1(t)y_3(t)) < 0$ holds in a left neigbourhood of τ. According to (8) we have $\alpha_2 \operatorname{sign}(y_2(t)y_3(t)) < 0$ and thus we get $\alpha_1 \operatorname{sign}(y_1(t)y_2(t)) > 0$, which contradicts $W(t) = n$. Thus $m < n$. Further, let $\tau \in I \cap S$ be valid. Then the number

$$Z = \prod_{i=1}^{n} \alpha_i y_i(\tau) y_{i+1}(\tau) = \prod_{i=1}^{n} \alpha_i \prod_{j=1}^{n} y_j^2(\tau)$$

is equal to $+1$ ($= -1$) if $\prod_{i=1}^{n} \alpha_i = 1$ ($= -1$). On the other hand, by the definitions of m and $W(\tau)$, $Z = 1$ ($Z = -1$) if $m = W(\tau)$ is even (odd). This yields the rest of the statement III.

The case IV: Let $\bar{t} < \tau_1 < \tau_2$ be consecutive zeros of y_i. It follows from the proof of Lemma 1 that y_{i+1} has a zero in the interval (τ_1, τ_2). The statement will be proved by the indirect proof. Thus, let there exist zeros c_1, c_2 of y_{i+1} such that $\tau_1 < c_1 < c_2 < \tau_2$. Without loss of generality we can suppose that c_1, c_2 are consecutive zeros, $y_{i+1}(t) \neq 0$ on (c_1, c_2). Then according to the statement II we have $\alpha_{i+1}\alpha_i y_{i+2}(c_j) y_i(c_j) < 0$, $j = 1, 2$. Thus $y_{i+2}(c_1)$ and $y_{i+2}(c_2)$ have the same sign and by virtue of (3) the function y'_{i+1} has a constant sign in a neighbourhood of c_1, c_2 (for almost all t). But this contradicts the fact that c_1, c_2 are consecutive zeros of y_{i+1}.

The case V can be proved similarly to IV. The theorem is proved. □

As the system (1), (2) is a special case of (3), we get the following consequence of Theorem 2.

Consequence 2. *Let $y \in T_0$ be a solution of (1), (2). Then the statement of Theorem 2 holds.*

In [1] it is proved that for the equation (4) there exist at most two H-points in the interval J if either n is odd or n is even and $(-1)^{n_0}\alpha = -1$. If n is even and $(-1)^{n_0}\alpha = 1$, then infinitely many H-points may exist in J, see an example in [1]. In the sequel this result will be generalized to the inequalities (3).

Lemma 5. *Let $y \in T$, $1 \in \{1,\dots,n\}$ and either n be odd or n be even and $(-1)^{n_0}\prod_{i=1}^{n}\alpha_i = 1$. Let*

$$y'_{l-1}y_{l+i+1} = \alpha_{l+i}\alpha_{l-i}y_{l-i+1}y'_{l+i}, \quad i = 1,2,\dots,s$$

hold where $s = n - n_0 - 1$. Then the function

$$F(t) = \sum_{i=0}^{n_0-1}(-1)^i\Big(\prod_{j=-1}^{i}\alpha_{l+j}\Big)y_{l-i}(t)y_{l+i+1}(t) + \frac{1}{2}(n-2n_0)(-1)^{n_0}\Big(\prod_{j=0}^{n}\alpha_j\Big)y^2_{l+n_0+1}(t)$$

is nondecreasing on J.

Proof. For almost all $t \in J$ we have

$$\begin{aligned} F'(t) = \alpha_l y'_l y_{l+1} = \sum_{i=1}^{n_0-1}\Big[(-1)^i\Big(\prod_{j=-1}^{i}\alpha_{l+j}\Big)(y'_{l-i}y_{l+i+1} - \alpha_{l+i}\alpha_{l+i}y_{l-i+1}y'_{l+i}\Big] \\ + (-1)^{n_0-1}\Big(\prod_{j=-n_0+1}^{n_0-1}\alpha_{j+1}\Big)y_{l-n_0+1}y'_{l+n_0} \\ + (n-2n_0)(-1)^{n_0}\Big(\prod_{j=1}^{n}\alpha_j\Big)y_{l+n_0+1}y'_{l+n_0+1}. \end{aligned}$$

Using the assumptions of the lemma and the fact that $y_{l+n_0+1} \equiv y_{l-n_0}$ holds for n odd we get for almost all t:

$$F'(t) = \alpha_l y'_l(t)y_{l+1}(t) \quad \text{for } n \text{ odd},$$

$$F'(t) = \alpha_l y'_l(t)y_{l+1}(t) + (-1)^{n_0-1}\Big(\prod_{i=1}^{n}\alpha_i\Big)\alpha_{l+n_0}y'_{l+n_0}(t)y_{l+n_0+1}(t) \quad \text{for } n \text{ even}.$$

Thus according to (3) F is nondecreasing on J. The lemma is proved. □

Theorem 3. *Let the assumptions of Lemma 5 be fulfilled. Then there exist at most two H-points of y in J. If c_1, c_2, $0 \leq c_1 < c_2 < b$ are two H-points then $y_i \equiv 0$ on $[c_1, c_2]$, $i = 1, 2, \ldots, n$. Moreover, if y is oscillatory then the statement of Theorem 2 is valid.*

P r o o f. It follows from the definition of H-points that c is an H-point, and the implication $c \in (0, b) \Rightarrow y_i(c) = 0$, $i = 1, \ldots, n$ holds. Thus, $F(c_1) = F(c_2) = 0$ and according to Lemma 5 we have $F(t) = 0$, $t \in [c_1, c_2]$. This together with (9), (3) yields $y_l'(t) y_{l+1}(t) = 0$ for almost all $t \in [c_1, c_2]$ and thus using (3) we have $y_{l+1} \equiv 0$ on $[c_1, c_2]$. We can conclude by virtue of Lemma 2 that $y_i \equiv 0$ holds on $[c_1, c_2]$, $i = 1$, $\ldots$, n. It is clear that three H-points cannot exist. The theorem is proved. □

Consequence 3. *Let y be a solution of (1), (2), $l \in \{1, \ldots, n\}$. Let either n be odd or n be even and $(-1)^{n_0} \prod_{i=1}^{n} \alpha_i = -1$. Let there exist functions $F_j : J \times \mathbf{R}_n \to (0, \infty)$, $j = 1, \ldots, s$, $s = n - n_0 - 1$ such that F_j fulfil the local Carethéodory conditions and*

$$
\begin{aligned}
f_{l-j}(t, x_1, \ldots, x_n) &= \alpha_{l-j} F_j(t, x_1, \ldots, x_n) x_{l-j+1}, \\
f_{l+j}(t, x_1, \ldots, x_n) &= \alpha_{l+j} F_j(t, x_1, \ldots, x_n) x_{l+j+1}, \quad j = 1, \ldots, s.
\end{aligned}
$$

Then the statement of Theorem 3 holds.

R e m a r k. Suppose that there exist $\varepsilon > 0$ and functions $a_i : J \to (0, \infty)$, $i = 1$, $\ldots$, n such that a_i are locally integrable and

$$
|f_i(t, x_1, \ldots, x_n)| \leqslant a_i(t) \sum_{i=1}^{n} |x_i| \text{ on } J \times [-\varepsilon, \varepsilon]^n.
$$

Then it is clear that the Cauchy problem of (1), (2) with zero initial conditions is uniquely solvable. Thus there exists no H-point of an oscillatory solution y, and the statement of Theorem 2 holds for y.

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