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INDEPENDENT AXIOMATIZATION OF VARIETIES OF LATTICE ORDERED GROUPS

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In the works [1, 2, 3] the author constructed an infinite set of varieties of lattice ordered groups (ℓ-varieties) without independent bases of identities. In connection with this result the following natural question about existence of ℓ-varieties without independent bases in classes of formulae of tight predicate calculus more general than identities (for example, classes of quasidentities or universal formulae) arises.

In this work the example of an ℓ-variety without an independent basis of universal formulae is given.

All general facts and definitions about lattice and linearly ordered groups can be found in the book [4].

The U be the ℓ-variety defined by the system of identities Σ:

a) $(|x|\vee|y|)^{-1}(|[x,y]|\wedge|t|)(|x|\vee|y|)\wedge(|[x,y]|\wedge|t|)^n = \big(|[x,y]|\wedge|t|\big)^n$, $(n \in N,\ n \geqslant 3)$,

b) $(x \wedge y^{-1}x^{-1}y) \vee e = e$,

c) $\big[|[x,y]| \wedge |t|,\ |[x_1,y_1]| \wedge |t_1|\big] = e$, where $[x,y] = x^{-1}y^{-1}xy$ and N is the set of natural numbers.

By $\hat{U}$ we denote the ℓ-variety defined by the system of identities $\hat{\Sigma}$:

a) $(|x| \vee |y|)^{-1}\big(|[x,y]| \wedge |t|\big)(|x| \vee |y|) \wedge \big(|[x,y]| \wedge |t|\big)^3 = \big(|[x,y]| \wedge |t|\big)^3$,

b) $(x \wedge y^{-1}x^{-1}y) \vee e = e$,

c) $\big[|[x,y]| \wedge |t|,\ |[x_1,y_1]| \wedge |t_1|\big] = e$.

It is evident that the ℓ-varieties U, $\hat{U}$ are ℓ-metabelian, $U \subseteq \hat{U}$ and $U \neq \hat{U}$ ([1]).

Proposition 1. *In the lattice of universal classes of ℓ-groups T there is no universal class of ℓ-groups W such that*

1) W covers the ℓ-variety U in the lattice T,

2) $W \subseteq \hat{U}$.

Proof. We assume that there exists a universal class W with properties 1), 2). It should be noted that if $G \in \hat{U}$, then G is representable and there exists an abelian ℓ-ideal A such that the factor-group G/A is abelian. So there exists an ℓ-group G such that $G \in W \setminus U$.

First, we suppose that the ℓ-group G is linearly ordered. In this case there are elements $x_0, y_0, t_0 \in G$ and a natural number $n \in N(n > 3)$ such that

$$\left(|[x_0, y_0]| \wedge |t_0|\right)^3 \leqslant \left(|x_0| \vee |y_0|\right)^{-1}\left(|[x_0, y_0]| \wedge |t_0|\right)\left(|x_0| \vee |y_0|\right) < \left(|[x_0, y_0]| \wedge |t_0|\right)^n.$$

Let us set for the sake of brevity $a = |[x_0, y_0]| \wedge |t_0|$, $b = |x_0| \vee |y_0|$. Then $a \in A$, $b \notin A$ and $a^3 \leqslant b^{-1}ab < a^n$. Let A_1 be the normal convex ℓ-subgroup of A, generated by the element a. We choose an element $c \in A_1$ such that

$$c^3 \leqslant c^{m(c)} \leqslant c^b < c^{m(c)+1},$$

and the natural number $m(c)$ is minimal with this property. Let us consider the ℓ-subgroup $G_1 = \ell\text{-gp}(A_1, b)$ of the linearly ordered group G, generated by the convex ℓ-subgroup A_1 and the element b. It is easy to see that $b > v$ for every $v \in A_1$, and the linearly ordered group $G_1 = A_1 \overleftarrow{\lambda} (b)$ is the lexicographic extension of A_1 by the infinite cyclic group (b). In the linearly ordered group G_1 the inequalities $v^3 \leqslant v^{m(c)} \leqslant v^b$ are true for every element $v \in A_1$. Now, in the linearly ordered group G_1 we choose an ℓ-subgroup $G_2 = \ell\text{-gp}(A_1, b^2) = A_1 \overleftarrow{\lambda} (b^2)$. By $u_l(G)$, $u_l(G_1)$, $u_l(G_2)$ we denote the minimal universal classes of ℓ-groups, containing o-groups G, G_1, G_2, respectively. It is easy to see that $u_l(G) \supseteq u_l(G_1) \supseteq u_l(G_2)$ and $u_l(G_1) \neq u_l(G_2)$. Indeed let us consider the identity

$$(1) \quad (|x| \vee |y|)^{-1}\left(|[x, y]| \wedge |t|\right)(|x| \vee |y|) \wedge \left(|[x, y]| \wedge |t|\right)^{2m(c)} = \left(|[x, y]| \wedge |t|\right)^{2m(c)}.$$

We observe that this identity is not true on the o-group G_1 when $t = x = c$, $y = b$. In this case we have

$$\begin{gathered}[x, y] = [c, b] = c^{-1}b^{-1}cb = c^{-1}c^b \geqslant c^{-1}c^3 = c^2 > e, \\ |[x, y]| \wedge |t| = |[c, b]| \wedge c = c > e, \ \ |x| \vee |y| = c \vee b = b,\end{gathered}$$

and

$$(|x| \vee |y|)^{-1}\left(|[x, y]| \wedge |t|\right)(|x| \vee |y|) = b^{-1}cb < c^{m(c)+1}.$$

But $m(c) \geqslant 3$, hence $m(c) + 1 < 2m(c)$ and therefore

$$b^{-1}cb < c^{m(c)+1} < c^{2m(c)}, \ \ b^{-1}cb \wedge c^{2m(c)} = b^{-1}cb \neq c^{2m(c)}.$$

Now let us prove that the identity (1) is fulfilled on the o-group G_2. If $[x, y] = e$, then the identity (1) is true for these values of variables and so we can suppose that $a_0 = [x, y] \neq e$. Then $|x| \vee |y| = b^{2t}a_1$, where $t > 0$, $a_1 \in A_1$ and

$$\begin{aligned}(|x| \vee |y|)^{-1}\left(|[x, y]| \wedge |t|\right)(|x| \vee |y|) &= (b^{2t}a_1)^{-1}a_0b^{2t}a_1 = a_1^{-1}b^{-2t}a_0b^{2t}a_1 = \\ &= b^{-2t}a_0b^{2t} \geqslant b^{-2}a_0b^2 \geqslant b^{-1}a_0^3b \geqslant a_0^{3m(c)} > a_0^{2m(c)}.\end{aligned}$$

Now it is clear that

$$(|x| \vee |y|)^{-1}\big(|[x,y]| \wedge |t|\big)(|x| \vee |y|) \wedge \big(|[x,y]| \wedge |t|\big)^{2m(c)} = \big(|[x,y]| \wedge |t|\big)^{2m(c)}$$

for all values of variables from the o-group G_2, and the identity (1) is fulfilled on the universal class $u_l(G_2) \vee U$, but is not fulfilled on the universal classes $u_l(G_1) \vee U$, $u_l(G) \vee U$ (here the join operation is considered in the lattice of universal classes of ℓ-groups T).

Moreover $u_l(G_2) \vee U \neq U$ because the identity

$$(2) \quad (|x| \vee |y|)^{-1}\big(|[x,y]| \wedge |t|\big)(|x| \vee |y|) \wedge \big(|[x,y]| \wedge |t|\big)^{4m(c)} = \big(|[x,y]| \wedge |t|\big)^{4m(c)}$$

fails to hold on the o-group G_2 and is true on the ℓ-variety U by definition. So we have the following inclusions: $\hat{U} \supseteq W \supseteq u_l(G) \vee U \supseteq u_l(G_1) \vee U \supseteq u_l(G_2) \vee U$. As all these universal classes are different we have a contradiction with our assumption that the universal class of ℓ-groups W covers U in the lattice T.

Let now the ℓ-group G be not linearly ordered. In this case G is an ℓ-subgroup of the cardinal product $\bar{\Pi} G_i$ of linear ordered groups $G_i (i \in I)$, where $G_i = \varphi_i(G)$ are ℓ-homomorphic images of the ℓ-group G. Since $G \notin U$, there exists elements $x_0, y_0, t_0 \in G$ and a natural number n_0 such that

$$(|x_0| \vee |y_0|)^{-1}\big(|[x_0,y_0]| \wedge |t_0|\big)(|x_0| \vee |y_0|) \wedge \big(|[x_0,y_0]| \wedge |t_0|\big)^{n_0} \neq \big(|[x_0,y_0]| \wedge |t_0|\big)^{n_0}.$$

As $G \in \hat{U}$, we have

$$(|x_0| \vee |y_0|)^{-1}\big(|[x_0,y_0]| \wedge |t_0|\big)(|x_0| \vee |y_0|) \wedge \big(|[x_0,y_0]| \wedge |t_0|\big)^{3} = \big(|[x_0,y_0]| \wedge |t_0|\big)^{3}.$$

Consequently there exists a nonempty subset J of indices I such that the inequalities

$$\begin{aligned}
&\big(|[\varphi_i(x_0), \varphi_i(y_0)]| \wedge |\varphi_i(t_0)|\big)^3 \\
&\leqslant (|\varphi_i(x_0)| \vee |\varphi_i(y_0)|)^{-1}\big(|[\varphi_i(x_0), \varphi_i(y_0)]| \wedge |\varphi_i(t_0)|\big)(|\varphi_i(x_0)| \vee |\varphi_i(y_0)|) \\
&< \big(|[\varphi_i(x_0), \varphi_i(y_0)]| \wedge |\varphi_i(t_0)|\big)^{n_0}
\end{aligned}$$

are true for every $i \in J$.

Let us denote $|x_0| \vee |y_0| = b$, $|[x_0, y_0]| \wedge |t_0| = a$, A_1—the ℓ-ideal of G generated by the element a. Then $A_1 \subseteq A$, where A is the abelian ℓ-ideal of G and

$$\varphi_i^3(a) \leqslant \varphi_i(b)^{-1}\varphi_i(a)\varphi_i(b) < \varphi_i(a)^{n_0}$$

for arbitrary $i \in J \subseteq I$. It is not difficult to prove that for every element $w \in A_1$ the inequality $|w|^3 \leqslant b^{-1}|w|b$ is fulfilled. Indeed every positive element $w \in A_1$ (by definition of A_1) satisfies the inequality

$$e \leqslant w \leqslant a^{g_1} \cdots a^{g_t}$$

for some $g_1, g_2, \ldots, g_t \in G$. By a well-known theorem of Riesz ([4], p. 30) $w = w_1 \cdot w_2 \cdots w_t$, where $e \leqslant w_1 \leqslant a^{g_1}, \ldots, e \leqslant w_t \leqslant a^{g_t}$. So

$$w_k = w_k \wedge a^{g_k} = w_k \wedge |[x_0^{g_k}, y_0^{g_k}]| \wedge |t_0^{g_k}| = |[x_0^{g_k}, y_0^{g_k}]| \wedge |t'_0|$$

for $|t'_0| = |t_0^{g_k}| \wedge |w_k|$. But $G \in \hat{U}$, hence $w_k^b > w_k^3$. As all $w_k (1 \leqslant k \leqslant l)$ are mutually permutable we can multiply all these inequalities $w_k^b \geqslant w_k^3$. As a result we have $w^b \geqslant w^3$. In the ℓ-ideal $A_1 \trianglelefteq G$ there exists an element $e \neq c \in A_1$ and an index $i_0 \in I$ such that

$$(3) \qquad \varphi_{i_0}(c)^3 \leqslant \varphi_{i_0}(c)^{m(c,i_0)} \leqslant \varphi_{i_0}(b)^{-1}\varphi_{i_0}(c)\varphi_{i_0}(b) < \varphi_{i_0}(c)^{(m(c,i_0)+1)} \leqslant \varphi_{i_0}^{n_0}(c)$$

and the number $m(c, i_0)$ is the minimal natural number of the numbers $m(w, i)$, where $i \in I$, $e \neq w \in A_1$. By the definition of the number $m(c, i_0)$ for every element $x \in A_1$ the inequality

$$(4) \qquad x^3 \leqslant x^{m(c,i_0)} \leqslant x^b$$

is fulfilled. Let us consider ℓ-subgroups $H = \ell\text{-gp}(A_1, b)$, $H_1 = \ell\text{-gp}(A_1, b^2)$. It is not difficult to prove that $b > a$ for every element $a \in A_1$, and consequently $H = A_1 \overleftarrow{\lambda} (b)$, $H_1 = A_1 \overleftarrow{\lambda} (b^2)$ are lexicographic extensions of A_1 by the infinite cyclic groups (b) and (b^2), respectively. As $G \geqslant H \geqslant H_1$ we have $u_l(G) \supseteq u_l(H) \supseteq u_l(H_1)$. The inequalities (4) imply that the identity

$$(5) \quad (|x| \vee |y|)^{-1}(|[x,y]| \wedge |t|)(|x| \vee |y|) \wedge (|[x,y]| \wedge |t|)^{2m(c,i_0)} = (|[x,y]| \wedge |t|)^{2m(c,i_0)}$$

is violated on the ℓ-group H under $x = b$, $y = c$, $t = b$. The inequalities (4) imply that this identity is fulfilled on the ℓ-group H_1. The identity (5) is true on the ℓ-variety U by definition, and so

$$U \vee u_l(H) \neq U \vee u_l(H_1), \; U \subseteq U \vee u_l(H_1) \subseteq U \vee u_l(H) \subseteq U \vee u_l(G) \subseteq W.$$

Direct calculation shows that the identity

$$(|x| \vee |y|)^{-1}(|[x,y]| \wedge |t|)(|x| \vee |y|) \wedge (|[x,y]| \wedge |t|)^{4m(c,i_0)} = (|[x,y]| \wedge |t|)^{4m(c,i_0)}$$

is violated on the ℓ-group H_1 under $x = b^2$, $y = c$, $t = b^2$. However this fact contradicts our assumption that the universal class W covers U in the lattice of universal classes T. □

Theorem 1. *U is an ℓ-variety without an independent basis of universal formulae.*

Proof. Indeed, $U \subseteq \hat{U}$, where the ℓ-variety $\hat{U}$ is an ℓ-variety with a finite basis of identities by definition. If the ℓ-variety U has an infinite independent basis of universal formulae, then in the lattice of universal classes T the ℓ-varietty U has infinitely many different covers contained in the ℓ-variety $\hat{U}$ ([5]). This is impossible by Proposition 1. □

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