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OSCILLATORY PROPERTIES OF SOLUTIONS TO A
DIFFERENTIAL INCLUSION OF ORDER n

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The aim of this paper is to investigate the oscillatory as well as the nonoscillatory solutions and their asymptotic behaviour, of the differential inclusion

$$(E) \quad L_n x(t) \in F(t, x(\varphi(t))), \quad n > 1$$

where $L_n x(t)$ is the n -th quasiderivative of $x(t)$ with respect to the continuous functions $a_i(t): J = [t_0, \infty) \rightarrow (0, \infty)$, $i = 0, 1, \dots, n$, $L_0 x(t) = a_0(t)x(t)$, $L_i x(t) = a_i(t)(L_{i-1} x(t))'$, $\int_{t_0}^{\infty} a_i^{-1}(t) dt = \infty$; $F(t, x): J \times \mathbf{R} \rightarrow \{\text{nonempty convex compact subsets of } \mathbf{R}\}$, $\mathbf{R} = (-\infty, \infty)$; $\varphi: J \rightarrow \mathbf{R}$ a continuous function such that $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ as $t \rightarrow \infty$.

Under a solution $x(t)$ of (E) we will understand a solution existing on some ray $[T_x, \infty)$ such that

$$\sup\{|x(t)| : t_1 \leq t < \infty\} > 0 \text{ for any } t_1 > T_x.$$

We will assume existence of such solutions.

Notation. $F(t, x)x > 0$ (< 0) means $yx > 0$ (< 0) for each $y \in F(t, x)$; if $h: J \times \mathbf{R} \rightarrow \mathbf{R}$, then $F(t, x) \geq$ (\leq) $h(t, x)$ means: $y \geq$ (\leq) $h(t, x)$ for each $y \in F(t, x)$. If $B \subset \mathbf{R}$ then $\|B\| = \inf\{|x| : x \in B\}$.

The following basic assumptions will be used:

1° $F(t, x)$ is upper semicontinuous on $J \times \mathbf{R}$;

2° $F(t, 0) = 0$ for each $t \in J$;

3° $F(t, x)x < 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$;

or 4° $F(t, x)x > 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$.

The notions of oscillatory and nonoscillatory solutions will be used in the usual sense.

Consider the inclusion (E) and assume that the assumptions 1°–4° are satisfied. Let $x(t)$ be a nonoscillatory solution of (E). Then from the assumption $\lim_{t \rightarrow \infty} \varphi(t) = \infty$

as $t \rightarrow \infty$ it follows the existence of such $t_1 \geq t_0$ that $x(\varphi(t)) \neq 0$ on $[t_1, \infty)$. Taking into consideration the assumptions $1^\circ-4^\circ$ we get that $x(t)L_n x(t) \neq 0$ on $[t_1, \infty)$. Therefore, $x(t)L_n x(t) > 0$ if $1^\circ, 2^\circ, 4^\circ$ are satisfied and $x(t)L_n x(t) < 0$ if $1^\circ, 2^\circ, 3^\circ$ are satisfied on $[t_1, \infty)$. This implies that there exists $t_2 \geq t_1$ such that each $L_i x(t)$, $i = 0, 1, \dots, n$, has a constant sign on $[t_2, \infty)$. Therefore, each $L_i x(t)$, $i = 0, 1, \dots, n-1$, is monotone on $[t_2, \infty)$, and $\lim L_i x(t)$ as $t \rightarrow \infty$, $i = 0, 1, \dots, n-1$, exists in the extended sense, i.e. $\lim |L_i x(t)|$ is finite or ∞ as $t \rightarrow \infty$ and $i = 0, 1, \dots, n-1$. More detailed considerations [1] lead to the following result: For the nonoscillatory solutions of (E) the following two cases are possible:

- a) $\lim_{t \rightarrow \infty} |L_i x(t)| = \infty$ for $i = 0, 1, \dots, n-1$;
 b) there exists $k \in \{0, 1, \dots, n-1\}$ such that $\lim_{t \rightarrow \infty} L_k x(t)$ is finite,
 $\lim_{t \rightarrow \infty} L_i x(t) = \infty \cdot \operatorname{sgn} x(t)$, $i = 0, 1, \dots, k-1$,
 $\lim_{t \rightarrow \infty} L_i x(t) = 0$, $i = k+1, \dots, n-1$.

Remark 1. The case a) can occur only if the assumptions $1^\circ, 2^\circ, 4^\circ$ are satisfied.

In fact, if the assumptions $1^\circ, 2^\circ, 3^\circ$ are satisfied then $x(t)L_n x(t) < 0$. Therefore, if $x(t) > 0$ then $L_{n-1} x(t)$ decreases and must be ultimately positive. If $x(t) < 0$ then $L_{n-1} x(t)$ increases and must be ultimately negative. Thus $|\lim_{t \rightarrow \infty} L_{n-1} x(t)| < \infty$.

These considerations show that the set of all nonoscillatory solutions of (E) can be divided into disjoint classes in the following way.

Definition 1. We will say that a nonoscillatory solution $x(t)$ of (E) belongs to the class V_n if the case a) occurs. We will say that a nonoscillatory solution $x(t)$ of (E) belongs to the class V_k , $k \in \{0, 1, \dots, n-1\}$, if the case b) occurs.

In the sequel we will use the following notation and lemmas:

Let $t_0 \leq c < t < \infty$. Then

$$P_0(t, c) = 1,$$

$$P_i(t, c) = \int_c^t a_1^{-1}(s_1) \int_c^{s_1} a_2^{-1}(s_2) \dots \int_c^{s_{i-1}} a_i^{-1}(s_i) ds_i \dots ds_1,$$

$$i = 1, 2, \dots, n-1;$$

$$Q_n(t, c) = 1,$$

$$Q_j(t, c) = \int_c^t a_{n-1}^{-1}(s_{n-1}) \int_c^{s_{n-1}} a_{n-2}^{-1}(s_{n-2}) \dots \int_c^{s_{j+1}} a_j^{-1}(s_j) ds_j \dots ds_{n-1},$$

$$j = 1, 2, \dots, n-1.$$

It is easy to see that

$$\lim_{t \rightarrow \infty} P_i(t, c) = \infty, \quad \lim_{t \rightarrow \infty} Q_i(t, c) = \infty, \quad \text{for } i = 1, 2, \dots, n-1,$$

and taking into account the properties of $a_i(t)$, by the l'Hospital rule we get

$$\begin{aligned} \lim_{t \rightarrow \infty} P_i(t, c)P_j^{-1}(t, c) &= 0 \text{ for } 0 \leq i < j \leq n-1, \\ \lim_{t \rightarrow \infty} Q_j(t, c)Q_i^{-1}(t, c) &= 0 \text{ for } 0 < i < j \leq n-1. \end{aligned}$$

Lemma 1 ([1], Lemma 4). *Let $z(t)$ be such that $z(t) \neq 0$ on $[t_1, \infty)$ and $L_n z(t)$ exists on $[t_1, \infty)$. Let $z(t)L_n z(t) \leq 0$ on $[t_1, \infty)$, where the equality may eventually hold at isolated points. Let $k \in \{0, 1, \dots, n-1\}$ be such that b) is fulfilled. Then there exists a $T_1 \geq t_1$ such that $\text{sgn } z(t) = \text{sgn } L_k z(t)$ for $t \geq T_1$.*

If $n+k$ is even then $|L_k z(t)|$ increases on $[T_1, \infty)$ and there exist two constants $0 < c_1 < c_2$ such that for $t > T_1$

$$0 < c_1 < |L_k z(t)| < c_2$$

and

$$0 < c_1 < \lim_{t \rightarrow \infty} |L_0 z(t)P_k^{-1}(t, c)| < c_2, \quad \lim_{t \rightarrow \infty} L_0 z(t)P_{k+1}^{-1}(t, c) = 0.$$

If $n+k$ is odd then $|L_k z(t)|$ decreases on $[T_1, \infty)$ and there exists a constant $c > 0$ such that we have

$$\begin{aligned} 0 < |L_k z(t)| < c \text{ for } t > T_1, \\ 0 \leq \lim_{t \rightarrow \infty} |L_0 z(t)P_k^{-1}(t, c)| < c, \quad \lim_{t \rightarrow \infty} L_0 z(t)P_{k+1}^{-1}(t, c) = 0. \end{aligned}$$

Lemma 2 ([1], Lemma 6). *Let $z(t)$ be such that $z(t) \neq 0$ and $L_n z(t)$ exists, both on $[t_1, \infty)$. Let $z(t)L_n z(t) \geq 0$ for $t \geq t_1$, where the equality may hold at isolated points. Let $k \in \{0, 1, \dots, n-1\}$ be such that b) is fulfilled. Then there exists a $T_1 \geq t_1$ such that $\text{sgn } z(t) = \text{sgn } L_k z(t)$ for $t > T_1$.*

If $n+k$ is odd then $|L_k z(t)|$ increases on $[T_1, \infty)$ and there exist two constants $0 < c_1 < c_2$ such that

$$0 < c_1 < |L_k z(t)| < c_2 \text{ for } t > T_1$$

and

$$0 < c_1 < \lim_{t \rightarrow \infty} |L_0 z(t)P_k^{-1}(t, c)| < c_2, \quad \lim_{t \rightarrow \infty} L_0 z(t)P_{k+1}^{-1}(t, c) = 0.$$

If $n+k$ is even then $|L_k z(t)|$ decreases on $[T_1, \infty)$ and there exists a constant $c_3 > 0$ such that

$$\begin{aligned} 0 < |L_k z(t)| < c_3 \text{ for } t > T_1, \\ 0 \leq \lim_{t \rightarrow \infty} |L_0 z(t)P_k^{-1}(t, c)| < c_3, \quad \lim_{t \rightarrow \infty} L_0 z(t)P_{k+1}^{-1}(t, c) = 0. \end{aligned}$$

Lemma 3 ([2], Lemma 3). Let $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$. Then

$$\lim_{t \rightarrow \infty} L_0 x(t) P_k^{-1}(t, c) = \lim_{t \rightarrow \infty} L_k x(t) = c_k.$$

If $c_k \neq 0$ then there exist constants $\alpha_k > 0$, $\beta_k > 0$ and $T'_k > t_0$ such that

$$(1) \quad \alpha_k a_0^{-1}(t) P_k(t, c) \leq |x(t)| \leq \beta_k a_0^{-1}(t) P_k(t, c), \quad t > T'_k.$$

We will consider two problems. The first problem is to find conditions which guarantee that $\lim L_k x(t) = 0$ as $t \rightarrow \infty$ for each $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$. The second problem is to state conditions which guarantee that the class V_k , $k \in \{0, 1, \dots, n-1\}$, is empty. These problems were discussed in [1], [2], [3] if instead of the inclusion (E) we have a differential equation.

Theorem 1. Let the assumptions 1°-4° be satisfied. Let $G(t, u): Jx[0, \infty) \rightarrow [0, \infty)$ be a continuous and nondecreasing function in u for each fixed $t \in J$, such that

$$G(t, |x|) \leq \|F(t, x)\|, \quad x \in \mathbf{R}.$$

Let $k \in \{0, 1, \dots, n-1\}$. Suppose that

$$(2) \quad \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, \alpha a_0^{-1}(\varphi(s)) P_k(\varphi(s), c)) ds = \infty$$

for all $t \geq T_k$ such that $\varphi(s) > c$ for $s > T_k \geq T'_k$, $c \geq t_0$, and for each $\alpha > 0$, or

$$(3) \quad \limsup_{t \rightarrow \infty} \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, \alpha a_0^{-1}(\varphi(s)) P_k(\varphi(s), c)) ds > 0$$

for each $\alpha > 0$.

Then for each $x(t) \in V_k$ we have $\lim L_k x(t) = 0$ as $t \rightarrow \infty$.

Proof. Let $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$ and let $\lim L_k x(t) = c_k \neq 0$ as $t \rightarrow \infty$. Then

$$0 \leq G(t, |x(\varphi(t))|) \leq \|F(t, x(\varphi(t)))\|, \quad t > T_k,$$

and

$$(4) \quad 0 \leq G(t, |x(\varphi(t))|) \leq |L_n x(t)|, \quad t > T_k.$$

Assume that T_k is such that for $t \geq T_k$, $x(t)$ has a constant sign, $\text{sgn } x(t) = \text{sgn } L_k x(t)$ for $t \geq T_k$ and (1) from Lemma 3 holds. Then the successive integrations on $[t, \infty)$,

$t > T_k$, of (4), by virtue of the fact that $\lim L_i x(t) = 0$ as $t \rightarrow \infty$, $i = k + 1, \dots, n - 1$, give

$$0 \leq \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, |x(\varphi(s))|) ds \leq |L_k x(t) - c_k|.$$

From Lemma 3 we have

$$|x(\varphi(t))| \geq \alpha_k a_0^{-1}(\varphi(t)) P_k(\varphi(t), c).$$

Therefore, $G(t, u)$ being nondecreasing, we get

$$0 \leq \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, \alpha_k a_0^{-1}(\varphi(s)) P_k(\varphi(s), c)) ds \leq |L_k x(t) - c_k|.$$

The expression on the right-hand side is bounded. This leads to a contradiction with (2). If (3) is satisfied then we have once more a contradiction, because $\lim |L_k x(t) - c_k| = 0$ as $t \rightarrow \infty$. \square

Theorem 2. *Let all assumptions of Theorem 1 be satisfied. Then, provided the assumptions 1°, 2°, 3° are satisfied, the sets V_k for $n + k$ even are empty. If the assumptions 1°, 2°, 4° are satisfied then the sets V_k for $n + k$ odd are empty.*

Proof follows from Theorem 1 and from Lemma 1 and 2, respectively. Denote

$$\gamma(t) = \sup\{s \geq t_0 : \varphi(s) \leq t\} \text{ for all } t \geq t_0$$

and

$$m(t) = \max\{\gamma(t), t\}, \quad t \geq t_0.$$

We see that $m(t) \geq t$. From the continuity of $\varphi(t)$ we get $\varphi(s) > t$ for $s > \gamma(t)$ and $\varphi(s) \geq t$ for $s \geq m(t)$, $t \geq t_0$. Evidently $\lim m(t) = \infty$ as $t \rightarrow \infty$.

Consider the class V_k , $k \in \{0, 1, \dots, n - 1\}$. From the properties of the set V_k we get that $\lim L_{n-1} x(t)$ as $t \rightarrow \infty$ is finite for each $x(t) \in V_k$. Then by virtue of the assumptions of Theorem 1, (4) yields

$$(5) \quad 0 \leq \int_t^\infty a_n^{-1}(s) G(s, |x(\varphi(s))|) ds \leq |L_{n-1} x(t)| < \infty.$$

Our forthcoming considerations are based on this fact. Successive integration of (5), together with the fact that $\lim L_i x(t) = 0$ as $t \rightarrow \infty$, $i = k, k + 1, \dots, n - 1$, give

$$(6) \quad 0 \leq \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, |x(\varphi(s))|) ds \leq |L_k x(t)|.$$

a) Assume that $x(t) \in V_k$, $x(t) > 0$ for $t > t_3$, $k > 0$, where t_3 is such that $L_i x(t)$, $i = 0, 1, \dots, n-1$, has a constant sign. Then $L_k x(t) > 0$ for $t > t_3$ and the integration of (6) between u and v , $t_3 \leq u < v$, and the application of Fubini's theorem yield

$$(7) \quad \begin{aligned} 0 &\leq \int_u^v a_n^{-1}(s)G(s, |x(\varphi(s))|) \int_u^s a_k^{-1}(t)Q_{k+1}(s, t) dt ds \\ &+ \int_u^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \int_u^v a_k^{-1}(t)Q_{k+1}(s, t) dt ds \\ &\leq L_{k-1}x(v) - L_{k-1}x(u) \leq L_{k-1}x(v) \end{aligned}$$

because $L_{k-1}x(t) > 0$ for $t > t_3$. It follows from the definition of $Q_{k+1}(s, t)$ than for $t \leq v \leq s$

$$Q_{k+1}(s, t) \geq Q_{k+1}(v, t).$$

Therefore, from (7) we get

$$(8) \quad 0 \leq \int_u^v a_k^{-1}(t)Q_{k+1}(v, t) dt \int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) ds \leq L_{k-1}x(v).$$

Repeating this procedure $(k-1)$ -times, we get

$$(9) \quad \begin{aligned} 0 &\leq \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \dots \int_u^{t_{k-1}} a_k^{-1}(t_k) Q_{k+1}(t_{k-1}, t_k) dw_k \\ &\cdot \int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) ds \leq L_0x(v) \end{aligned}$$

for $t_3 \leq u < v$, where $dw_k = dt_k dt_{k-1} \dots dt_1$. Denote

$$(10) \quad R_k(v, u) = \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \dots \int_u^{t_{k-1}} a_k^{-1}(t_k) Q_{k+1}(t_{k-1}, t_k) dw_k.$$

Then we have

$$(11) \quad 0 \leq R_k(v, u) \int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) ds \leq L_0x(v), \quad t_3 \leq u < v.$$

The monotonicity of G and the properties of $m(t)$ yield

$$\begin{aligned}
 (12) \quad |L_0 x(v)| &\geq R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, |x(\varphi(s))|) ds \\
 &= R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 x(\varphi(s))|) ds.
 \end{aligned}$$

But $|L_0 x(t)|$ is nondecreasing, $\varphi(s) \geq v$ for $s \geq m(v)$ and $G(t, z)$ is nondecreasing in z . Therefore, (12) implies

$$(13) \quad |L_0 x(v)| \geq R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 x(v)|) ds.$$

for $t_3 \leq u < v$. Once more by virtue of the monotonicity of $G(t, z)$ we get

$$\begin{aligned}
 &\int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 x(v)|) ds \\
 &\geq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(\tau) G(\tau, a_0^{-1}(\varphi(\tau)) |L_0 x(v)|) d\tau ds.
 \end{aligned}$$

Denote

$$(14) \quad p(v) = \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 x(v)|) ds$$

Then we have

$$(15) \quad p(v) \geq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) R_k(v, u) p(v)) ds.$$

From (5) and (14) we obtain

$$\begin{aligned}
 |L_{n-1} x(m(v))| &\geq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, |x(\varphi(s))|) ds \\
 &\geq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 x(v)|) ds = p(v) \geq 0
 \end{aligned}$$

and

$$0 \leq \lim_{v \rightarrow \infty} p(v) \leq \lim_{v \rightarrow \infty} L_{n-1}x(m(v)) = 0.$$

Thus

$$(16) \quad \lim_{v \rightarrow \infty} p(v) = 0.$$

b) Let $x(t) \in V_k$, $x(t) < 0$ for $T \geq t_3$, $k > 0$. Then $\text{sgn } L_k x(t) = \text{sgn } x(t) = -1$ and from (6) we get

$$0 \leq \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) G(s, |(\varphi(s))|) ds \leq -L_k x(t), \quad t \geq t_3.$$

Similar considerations as in the case a) lead to the inequalities (13), (15) and equality (16). \square

Now we are able to prove the following theorems:

Theorem 3. *Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed $t \geq t_0$*

$$(17) \quad z^{-1}G(t, z) \text{ is nondecreasing for } z > 0,$$

and for $k \in \{1, 2, \dots, n-1\}$,

$$(18) \quad \limsup_{v \rightarrow \infty} R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s) c^{-1} G(s, a_0^{-1}(\varphi(s))c) ds > 1$$

for some $c > 0$. Then the set V_k is empty.

Proof. Let $x(t) \in V_k$, $k \in \{1, 2, \dots, n-1\}$. Then $\lim_{t \rightarrow \infty} |L_0 x(t)| = \infty$. Therefore, for $c > 0$ there exists $v_1 > u \geq t_3$ such that $|L_0 x(v)| > c$ for all $v > v_1$. Then from (13) and (17) we obtain

$$1 \geq R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s) a_0^{-1}(\varphi(s)) \frac{G(s, a_0^{-1}(\varphi(s))c)}{a_0^{-1}(\varphi(s))c} ds,$$

which contradicts (18). \square

Theorem 4. Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed $t \geq t_0$

$$(19) \quad z^{-1}G(t, z) \text{ is nonincreasing for } z > 0,$$

and for $k \in \{1, 2, \dots, n-1\}$

$$(20) \quad \limsup_{v \rightarrow \infty} \int_{m(v)}^{\infty} a_n^{-1}(s) c^{-1} G(s, R_k(v, u) a_0^{-1}(\varphi(s)) c) ds > 1$$

for some $c > 0$. Then the set V_k is empty.

Proof. Let $K \in \{1, 2, \dots, n-1\}$ and $x(t) \in V_k$. Because $\lim p(v) = 0$ as $v \rightarrow \infty$ and $p(v) > 0$ for $v > u$, for $c > 0$ there exists $v_2 \geq u \geq t_3$ such that $c > p(v)$ for all $v > v_2$. Taking into account (15) and (19) we obtain

$$1 \geq \int_{m(v)}^{\infty} a_n^{-1}(s) a_0^{-1}(\varphi(s)) R_k(v, u) \frac{G(s, a_0^{-1}(\varphi(s)) R_k(v, u) c)}{a_0^{-1}(\varphi(s)) R_k(v, u) c} ds$$

for all $v > v_2$. This leads to a contradiction with (20). \square

Definition 2. We will say that the inclusion (E) has property A if, provided n is even, all solutions of (E) are oscillatory and, provided n is odd, each solution $x(t)$ of (E) is either oscillatory or $\lim L_i x(t) = 0$ as $t \rightarrow \infty$ for $i = 0, 1, \dots, n-1$.

Definition 3. We will say that the inclusion (E) has property B if for n even each solution $x(t)$ of (E) is either oscillatory or $\lim L_i x(t) = 0$ as $t \rightarrow \infty$ for $i = 0, 1, \dots, n-1$ or it belongs to the class V_n , i.e. $\lim |L_i x(t)| = \infty$ as $t \rightarrow \infty$ for $i = 0, 1, \dots, n-1$, and for n odd each solution $x(t)$ of (E) either is oscillatory or belongs to the class V_n .

Now, from the Theorems 1–4 we obtain the final theorem:

Theorem 5. Let all assumptions of Theorem 1 be satisfied.

a) If the assumptions $1^\circ, 2^\circ, 3^\circ$ are satisfied and if (17) and (18) (or (19) and (20)) hold for $k = 1, 2, \dots, n-1$, then the inclusion (E) has property A.

b) If the assumptions $1^\circ, 2^\circ, 4^\circ$ are satisfied and if (17) and (18) (or (19) and (20)) hold for $k = 1, 2, \dots, n-1$, then the inclusion (E) has property B.

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