Marko Švec
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OSCILLATORY PROPERTIES OF SOLUTIONS TO A
DIFFERENTIAL INCLUSION OF ORDER n

MARKO ŠVEC, Bratislava

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The aim of this paper is to investigate the oscillatory as well as the nonoscillatory solutions and their asymptotic behaviour, of the differential inclusion

(E) \[ L_n x(t) \in F(t, x(\varphi(t))), \quad n > 1 \]

where \( L_n x(t) \) is the n-th quasiderivative of \( x(t) \) with respect to the continuous functions \( a_i(t) : J = [t_0, \infty) \to (0, \infty), \quad i = 0, 1, \ldots, n \), \( L_0 x(t) = a_0(t)x(t) \), \( L_i x(t) = a_i(t)(L_{i-1} x(t))' \), \( \int a_i^{-1}(t) \, dt = \infty \); \( F(t, x) : J \times \mathbb{R} \to \{ \text{nonempty convex compact subsets of } \mathbb{R} \} \), \( \mathbb{R} = (-\infty, \infty) \); \( \varphi : J \to \mathbb{R} \) a continuous function such that \( \lim \varphi(t) = \infty \) as \( t \to \infty \).

Under a solution \( x(t) \) of (E) we will understand a solution existing on some ray \([T_x, \infty)\) such that

\[ \sup \{ |x(t)| : t_1 \leq t < \infty \} > 0 \text{ for any } t_1 > T_x. \]

We will assume existence of such solutions.

Notation. \( F(t, x)x > 0 \) \((< 0)\) means \( yx > 0 \) \((< 0)\) for each \( y \in F(t, x) \); if \( h : J \times \mathbb{R} \to \mathbb{R} \), then \( F(t, x) \geq (\leq) h(t, x) \) means: \( y \geq (\leq) h(t, x) \) for each \( y \in F(t, x) \). If \( B \subset \mathbb{R} \) then \( ||B|| = \inf \{|x| : x \in B\} \).

The following basic assumptions will be used:

1° \( F(t, x) \) is upper semicontinuous on \( J \times \mathbb{R} \);
2° \( F(t, 0) = 0 \) for each \( t \in J \);
3° \( F(t, x)x < 0 \) for each \( (t, x) \in J \times \mathbb{R}, \ x \neq 0 \);
4° \( F(t, x)x > 0 \) for each \( (t, x) \in J \times \mathbb{R}, \ x \neq 0 \).

The notions of oscillatory and nonoscillatory solutions will be used in the usual sense.

Consider the inclusion (E) and assume that the assumptions 1°–4° are satisfied. Let \( x(t) \) be a nonoscillatory solution of (E). Then from the assumption \( \lim \varphi(t) = \infty \)
as \( t \to \infty \) it follows the existence of such \( t_1 \geq t_0 \) that \( x(\varphi(t)) \neq 0 \) on \([t_1, \infty)\). Taking into consideration the assumptions 1°-4° we get that \( x(t)L_n x(t) \neq 0 \) on \([t_1, \infty)\). Therefore, \( x(t)L_n x(t) > 0 \) if 1°, 2°, 4° are satisfied and \( x(t)L_n x(t) < 0 \) if 1°, 2°, 3° are satisfied on \([t_1, \infty)\). This implies that there exists \( t_2 \geq t_1 \) such that each \( L_i x(t), \) \( i = 0, 1, \ldots, n, \) has a constant sign on \([t_2, \infty)\). Therefore, each \( L_i x(t), \) \( i = 0, 1, \ldots, n-1, \) is monotone on \([t_2, \infty)\), and \( \lim L_i x(t) \) as \( t \to \infty, i = 0, 1, \ldots, n-1, \) exists in the extended sense, i.e. \( \lim|L_i x(t)| \) is finite or \( \infty \) as \( t \to \infty \) and \( i = 0, 1, \ldots, n-1. \)

More detailed considerations [1] lead to the following result: For the nonoscillatory solutions of (E) the following two cases are possible:

a) \( \lim_{t \to \infty} |L_i x(t)| = \infty \) for \( i = 0, 1, \ldots, n-1; \)

b) there exists \( k \in \{0, 1, \ldots, n-1\} \) such that \( \lim_{t \to \infty} L_k x(t) \) is finite,

\[
\begin{align*}
\lim_{t \to \infty} L_i x(t) &= \infty \cdot \text{sgn} \ x(t), \ i = 0, 1, \ldots, k-1, \\
\lim_{t \to \infty} L_i x(t) &= 0, \ i = k+1, \ldots, n-1.
\end{align*}
\]

Remark 1. The case a) can occur only if the assumptions 1°, 2°, 4° are satisfied.

In fact, if the assumptions 1°, 2°, 3° are satisfied then \( x(t)L_n x(t) < 0. \) Therefore, if \( x(t) > 0 \) then \( L_{n-1} x(t) \) decreases and must be ultimately positive. If \( x(t) < 0 \) then \( L_{n-1} x(t) \) increases and must be ultimately negative. Thus \( |\lim_{t \to \infty} L_{n-1} x(t)| < \infty. \)

These considerations show that the set of all nonoscillatory solutions of (E) can be divided into disjoint classes in the following way.

Definition 1. We will say that a nonoscillatory solution \( x(t) \) of (E) belongs to the class \( V_n \) if the case a) occurs. We will say that a nonoscillatory solution \( x(t) \) of (E) belongs to the class \( V_k, \ k \in \{0, 1, \ldots, n-1\}, \) if the case b) occurs.

In the sequel we will use the following notation and lemmas:

Let \( t_0 \leq c < t < \infty \). Then

\[
\begin{align*}
P_0(t,c) &= 1, \\
P_i(t,c) &= \int_c^t a_i^{-1}(s_1) \int_c^s a_2^{-1}(s_2) \cdots \int_c^{s_{i-1}} a_i^{-1}(s_i) \, ds_i \cdots ds_1, \\
&= 1, \\
Q_n(t,c) &= 1, \\
Q_j(t,c) &= \int_c^t a_{n-1}^{-1}(s_{n-1}) \int_c^s a_{n-2}^{-1}(s_{n-2}) \cdots \int_c^{s_{j+1}} a_j^{-1}(s_j) \, ds_j \cdots ds_{n-1}, \\
&= 1, \\
&= 0.
\end{align*}
\]
It is easy to see that
\[ \lim_{t \to \infty} P_i(t, c) = \infty, \quad \lim_{t \to \infty} Q_i(t, c) = \infty, \quad \text{for } i = 1, 2, \ldots, n - 1, \]
and taking into account the properties of \( a_i(t) \), by the l'Hospital rule we get
\[ \lim_{t \to \infty} P_i(t, c) P_j^{-1}(t, c) = 0 \quad \text{for } 0 \leq i < j \leq n - 1, \]
\[ \lim_{t \to \infty} Q_j(t, c) Q_i^{-1}(t, c) = 0 \quad \text{for } 0 < i < j \leq n - 1. \]

**Lemma 1** ([1], Lemma 4). Let \( z(t) \) be such that \( z(t) \neq 0 \) on \([t_1, \infty)\) and \( L_n z(t) \) exists on \([t_1, \infty)\). Let \( z(t) L_n z(t) \leq 0 \) on \([t_1, \infty)\), where the equality may eventually hold at isolated points. Let \( k \in \{0, 1, \ldots, n - 1\} \) be such that b) is fulfilled. Then there exists a \( T_1 \geq t_1 \) such that \( \text{sgn } z(t) = \text{sgn } L_k z(t) \) for \( t \geq T_1 \).

If \( n + k \) is even then \( |L_k z(t)| \) increases on \([T_1, \infty)\) and there exist two constants \( 0 < c_1 < c_2 \) such that for \( t > T_1 \)
\[ 0 < c_1 < |L_k z(t)| < c_2 \]
and
\[ 0 < c_1 < \lim_{t \to \infty} |L_0 z(t) P_k^{-1}(t, c)| < c_2, \quad \lim_{t \to \infty} L_0 z(t) P_{k+1}^{-1}(t, c) = 0. \]

If \( n + k \) is odd then \( |L_k z(t)| \) decreases on \([T_1, \infty)\) and there exists a constant \( c > 0 \) such that we have
\[ 0 < |L_k z(t)| < c \text{ for } t > T_1, \]
\[ 0 \leq \lim_{t \to \infty} |L_0 z(t) P_k^{-1}(t, c)| < c, \quad \lim_{t \to \infty} L_0 z(t) P_{k+1}^{-1}(t, c) = 0. \]

**Lemma 2** ([1], Lemma 6). Let \( z(t) \) be such that \( z(t) \neq 0 \) and \( L_n z(t) \) exists, both on \([t_1, \infty)\). Let \( z(t) L_n z(t) \geq 0 \) for \( t \geq t_1 \), where the equality may hold at isolated points. Let \( k \in \{0, 1, \ldots, n - 1\} \) be such that b) is fulfilled. Then there exists a \( T_1 \geq t_1 \) such that \( \text{sgn } z(t) = \text{sgn } L_k z(t) \) for \( t > T_1 \).

If \( n + k \) is odd then \( |L_k z(t)| \) increases on \([T_1, \infty)\) and there exist two constants \( 0 < c_1 < c_2 \) such that
\[ 0 < c_1 < |L_k z(t)| < c_2 \text{ for } t > T_1 \]
and
\[ 0 < c_1 < \lim_{t \to \infty} |L_0 z(t) P_k^{-1}(t, c)| < c_2, \quad \lim_{t \to \infty} L_0 z(t) P_{k+1}^{-1}(t, c) = 0. \]

If \( n + k \) is even then \( |L_k z(t)| \) decreases on \([T_1, \infty)\) and there exists a constant \( c_3 > 0 \) such that
\[ 0 < |L_k z(t)| < c_3 \text{ for } t > T_1, \]
\[ 0 \leq \lim_{t \to \infty} |L_0 z(t) P_k^{-1}(t, c)| < c_3, \quad \lim_{t \to \infty} L_0 z(t) P_{k+1}^{-1}(t, c) = 0. \]
Lemma 3 ([2], Lemma 3). Let \( x(t) \in V_k, k \in \{0, 1, \ldots, n - 1\} \). Then
\[
\lim_{t \to \infty} L_0 x(t) P_k^{-1}(t, c) = \lim_{t \to \infty} L_k x(t) = c_k.
\]
If \( c_k \neq 0 \) then there exist constants \( \alpha_k > 0, \beta_k > 0 \) and \( T'_k > t_0 \) such that
\[
(1) \alpha_k a^{-1}_0(t) P_k(t, c) \leq |x(t)| \leq \beta_k a^{-1}_0(t) P_k(t, c), \quad t > T'_k.
\]

We will consider two problems. The first problem is to find conditions which guarantee that \( \lim L_k x(t) = 0 \) as \( t \to \infty \) for each \( x(t) \in V_k, k \in \{0, 1, \ldots, n - 1\} \). The second problem is to state conditions which guarantee that the class \( V_k, k \in \{0, 1, \ldots, n - 1\} \), is empty. These problems were discussed in [1], [2], [3] if instead of the inclusion (E) we have a differential equation.

**Theorem 1.** Let the assumptions 1°-4° be satisfied. Let \( G(t, u) : Jx[0, \infty) \to [0, \infty) \) be a continuous and nondecreasing function in \( u \) for each fixed \( t \in J, s \lt t \), such that
\[
G(t, |x|) \leq \| F(t, x) \|, \quad x \in R.
\]
Let \( k \in \{0, 1, \ldots, n - 1\} \). Suppose that
\[
(2) \int_t^\infty a^{-1}_n(s) Q_{k+1}(s, t) G(s, \alpha a^{-1}_0(\varphi(s)) P_k(\varphi(s), c)) \, ds = \infty
\]
for all \( t \geq T_k \) such that \( \varphi(s) > c \) for \( s > T_k \geq T'_k, c \geq t_0 \), and for each \( \alpha > 0 \), or
\[
(3) \limsup_{t \to \infty} \int_t^\infty a^{-1}_n(s) Q_{k+1}(s, t) G(s, \alpha a^{-1}_0(\varphi(s)) P_k(\varphi(s), c)) \, ds > 0
\]
for each \( \alpha > 0 \).

Then for each \( x(t) \in V_k \) we have \( \lim L_k x(t) = 0 \) as \( t \to \infty \).

**Proof.** Let \( x(t) \in V_k, k \in \{0, 1, \ldots, n - 1\} \) and let \( \lim L_k x(t) = c_k \neq 0 \) as \( t \to \infty \). Then
\[
0 \leq G(t, |x(\varphi(t))|) \leq \| F(t, x(\varphi(t))) \|, \quad t > T_k,
\]
and
\[
(4) 0 \leq G(t, |x(\varphi(t))|) \leq |L_n x(t)|, \quad t > T_k.
\]
Assume that \( T_k \) is such that for \( t \geq T_k, x(t) \) has a constant sign, \( \text{sgn} x(t) = \text{sgn} L_k x(t) \) for \( t \geq T_k \) and (1) from Lemma 3 holds. Then the successive integrations on \([t, \infty)\),
$t > T_k$, of (4), by virtue of the fact that \( \lim_{t \to \infty} L_i x(t) = 0 \) as \( t \to \infty \), \( i = k + 1, \ldots, n - 1 \), give

\[
0 \leq \int_{s_0}^{\infty} a_n^{-1}(s) Q_{k+1}(s, t) G(s, x(\varphi(s))) \, ds \leq |L_k x(t) - c_k|.
\]

From Lemma 3 we have

\[
|x(\varphi(t))| \geq \alpha_k a_0^{-1}(\varphi(t)) P_k(\varphi(t), c).
\]

Therefore, \( G(t, u) \) being nondecreasing, we get

\[
0 \leq \int_{s_0}^{\infty} a_n^{-1}(s) Q_{k+1}(s, t) G(s, \alpha_0^{-1}(\varphi(s)) P_k(\varphi(s), c)) \, ds \leq |L_k x(t) - c_k|.
\]

The expression on the right-hand side is bounded. This leads to a contradiction with (2). If (3) is satisfied then we have once more a contradiction, because \( \lim_{t \to \infty} |L_k x(t) - c_k| = 0 \) as \( t \to \infty \).

\[\Box\]

**Theorem 2.** Let all assumptions of Theorem 1 be satisfied. Then, provided the assumptions 1°, 2°, 3° are satisfied, the sets \( V_k \) for \( n + k \) even are empty. If the assumptions 1°, 2°, 4° are satisfied then the sets \( V_k \) for \( n + k \) odd are empty.

**Proof** follows from Theorem 1 and from Lemma 1 and 2, respectively. Denote

\[
\gamma(t) = \sup \{ s \geq t_0 : \varphi(s) \leq t \} \quad \text{for all } t \geq t_0
\]

and

\[
m(t) = \max \{ \gamma(t), t \}, \quad t \geq t_0.
\]

We see that \( m(t) \geq t \). From the continuity of \( \varphi(t) \) we get \( \varphi(s) > t \) for \( s > \gamma(t) \) and \( \varphi(s) \geq t \) for \( s \geq m(t), \, t \geq t_0 \). Evidently \( \lim_{t \to \infty} m(t) = \infty \) as \( t \to \infty \).

Consider the class \( V_k, \, k \in \{0, 1, \ldots, n - 1\} \). Form the properties of the set \( V_k \) we get that \( \lim_{t \to \infty} L_{n-1} x(t) \) as \( t \to \infty \) is finite for each \( x(t) \in V_k \). Then by virtue of the assumptions of Theorem 1, (4) yields

\[
0 \leq \int_{s_0}^{\infty} a_n^{-1}(s) G(s, x(\varphi(s))) \, ds \leq |L_{n-1} x(t)| < \infty.
\]

Our forthcoming considerations are based on this fact. Successive integration of (5), together with the fact that \( \lim_{t \to \infty} L_i x(t) = 0 \) as \( t \to \infty \), \( i = k, k + 1, \ldots, n - 1 \), give

\[
0 \leq \int_{s_0}^{\infty} a_n^{-1}(s) Q_{k+1}(s, t) G(s, x(\varphi(s))) \, ds \leq |L_k x(t)|.
\]
a) Assume that \( x(t) \in V, \ x(t) > 0 \) for \( t > t_3, \ k > 0 \), where \( t_3 \) is such that 
\( L_i x(t), \ i = 0, 1, \ldots, n - 1, \) has a constant sign. Then \( L_k x(t) > 0 \) for \( t > t_3 \) and the integration of (6) between \( u \) and \( v \), \( t_3 \leq u < v \), and the application of Fubini's theorem yield

\[
0 \leq \int_u^v a_n^{-1}(s)G(s, |x(\varphi(s))|) \int_u^s a_k^{-1}(t)Q_{k+1}(s, t) \, dt \, ds
\]

(7)
\[
+ \int_u^v a_n^{-1}(s)G(s, |x(\varphi(s))|) \int_u^s a_k^{-1}(t)Q_{k+1}(s, t) \, dt \, ds
\]

\[
\leq L_{k-1} x(v) - L_{k-1} x(u) \leq L_{k-1} x(v)
\]

because \( L_{k-1} x(t) > 0 \) for \( t > t_3 \). It follows from the definition of \( Q_{k+1}(s, t) \) than for \( t \leq v \leq s \)

\[
Q_{k+1}(s, t) \geq Q_{k+1}(v, t).
\]

Therefore, from (7) we get

(8) \[
0 \leq \int_u^v a_k^{-1}(t)Q_{k+1}(v, t) \, dt \int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \, ds \leq L_{k-1} x(v).
\]

Repeating this procedure \((k - 1)\)-times, we get

\[
0 \leq \int_u^v a_k^{-1}(t_1) \int_{t_1}^{t_1} a_k^{-1}(t_2) \ldots \int_{t_1}^{t_k-1} a_k^{-1}(t_k)Q_{k+1}(t_{k-1}, t_k) \, dw_k
\]

(9)
\[
\int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \, ds \leq L_0 x(v)
\]

for \( t_3 \leq u < v \), where \( dw_k = dt_k \, dt_{k-1} \ldots dt_1 \). Denote

(10) \[
R_k(v, u) = \int_u^v a_k^{-1}(t_1) \int_{t_1}^{t_1} a_k^{-1}(t_2) \ldots \int_{t_1}^{t_k-1} a_k^{-1}(t_k)Q_{k+1}(t_{k-1}, t_k) \, dw_k.
\]

Then we have

(11) \[
0 \leq R_k(v, u) \int_v^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \, ds \leq L_0 x(v), \ t_3 \leq u < v.
\]
The monotonicity of \( G \) and the properties of \( m(t) \) yield

\[
|L_0 x(v)| \geq R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \, ds
\]

(12)

\[
= R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 x(\varphi(s))|) \, ds.
\]

But \( |L_0 x(t)| \) is nondecreasing, \( \varphi(s) \geq v \) for \( s \geq m(v) \) and \( G(t, z) \) is nondecreasing in \( z \). Therefore, (12) implies

\[
|L_0 x(v)| \geq R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 x(v)|) \, ds.
\]

(13)

for \( t_3 \leq u < v \). Once more, by virtue of the monotonicity of \( G(t, z) \) we get

\[
\int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 x(v)|) \, ds
\]

\[
\geq \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(\tau)G(\tau, a_0^{-1}(\varphi(\tau))|L_0 x(v)|) \, d\tau \, ds.
\]

Denote

\[
p(v) = \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 x(v)|) \, ds
\]

(14)

Then we have

\[
p(v) \geq \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))R_k(v, u)p(v)) \, ds.
\]

(15)

From (5) and (14) we obtain

\[
|L_{n-1} x(m(v))| \geq \int_{m(v)}^\infty a_n^{-1}(s)G(s, |x(\varphi(s))|) \, ds
\]

\[
\geq \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 x(v)|) \, ds = p(v) \geq 0
\]
and

$$0 \leq \lim_{v \to \infty} p(v) \leq \lim_{v \to \infty} L_{n-1}x(m(v)) = 0.$$ 

Thus

(16) \hspace{1cm} \lim_{v \to \infty} p(v) = 0. 

b) Let $x(t) \in V_k$, $x(t) < 0$ for $T \geq t_3$, $k > 0$. Then $\text{sgn} \ L_kx(t) = \text{sgn} \ x(t) = -1$ and from (6) we get

$$0 \leq \int_{t}^{\infty} a_n^{-1}(s)Q_{k+1}(s,t)G(s,|\varphi(s)|) \ ds \leq -L_kx(t), \ t \geq t_3.$$ 

Similar considerations as in the case a) lead to the inequalities (13), (15) and equality (16). □

Now we are able to prove the following theorems:

**Theorem 3.** Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed $t \geq t_0$

(17) \hspace{1cm} z^{-1}G(t,z) \text{ is nondecreasing for } z > 0,

and for $k \in \{1, 2, \ldots, n-1\}$,

(18) \hspace{1cm} \lim_{v \to \infty} \sup \ R_k(v,u) \int_{m(v)}^{\infty} a_n^{-1}(s)c^{-1}G(s, a_0^{-1}(\varphi(s))c) \ ds > 1

for some $c > 0$. Then the set $V_k$ is empty.

**Proof.** Let $x(t) \in V_k$, $k \in \{1, 2, \ldots, n-1\}$. Then $\lim_{t \to \infty} |L_0x(t)| = \infty$. Therefore, for $c > 0$ there exists $v_1 > u \geq t_3$ such that $|L_0x(v)| > c$ for all $v > v_1$. Then from (13) and (17) we obtain

$$1 \geq R_k(v,u) \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}(\varphi(s))G(s, a_0^{-1}(\varphi(s))c) \ ds,$$

which contradicts (18). □
Theorem 4. Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed \( t \geq t_0 \)
(19) \( z^{-1}G(t, z) \) is nonincreasing for \( z > 0 \),
and for \( k \in \{1, 2, \ldots, n-1\} \)
(20) \[ \lim_{v \to \infty} \sup_{t \in [a, b]} \int_{m(v)}^{\infty} a_n^{-1}(s)c^{-1}G(s, R_k(v, u))a_0^{-1}(\varphi(s))c \ ds > 1 \]
for some \( c > 0 \). Then the set \( V_k \) is empty.

Proof. Let \( K \in \{1, 2, \ldots, n-1\} \) and \( x(t) \in V_k \). Because \( \lim p(v) = 0 \) as \( v \to \infty \) and \( p(v) > 0 \) for \( v > u \), for \( c > 0 \) there exists \( v_2 \geq u \geq t_3 \) such that \( c > p(v) \) for all \( v > v_2 \). Taking into account (15) and (19) we obtain

\[ 1 \geq \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}(\varphi(s))R_k(v, u)G(s, a_0^{-1}(\varphi(s))R_k(v, u)c) ds \]
for all \( v > v_2 \). This leads to a contradiction with (20). \( \square \)

Definition 2. We will say that the inclusion \( (E) \) has property A if, provided \( n \) is even, all solutions of \( (E) \) are oscillatory and, provided \( n \) is odd, each solution \( x(t) \) of \( (E) \) is either oscillatory or \( \lim L_i x(t) = 0 \) as \( t \to \infty \) for \( i = 0, 1, \ldots, n-1 \).

Definition 3. We will say that the inclusion \( (E) \) has property B if for \( n \) even each solution \( x(t) \) of \( (E) \) is either oscillatory or \( \lim L_i x(t) = 0 \) as \( t \to \infty \) for \( i = 0, 1, \ldots, n-1 \) or it belongs to the class \( V_n \), i.e. \( \lim|L_i x(t)| = \infty \) as \( t \to \infty \) for \( i = 0, 1, \ldots, n-1 \), and for \( n \) odd each solution \( x(t) \) of \( (E) \) either is oscillatory or belongs to the class \( V_n \).

Now, from the Theorems 1–4 we obtain the final theorem:

Theorem 5. Let all assumptions of Theorem 1 be satisfied.

a) If the assumptions 1°, 2°, 3° are satisfied and if (17) and (18) (or (19) and (20)) hold for \( k = 1, 2, \ldots, n-1 \), then the inclusion \( (E) \) has property A.

b) If the assumptions 1°, 2°, 4° are satisfied and if (17) and (18) (or (19) and (20)) hold for \( k = 1, 2, \ldots, n-1 \), then the inclusion \( (E) \) has property B.

References


Author's address: Matematicko-fyzikálna fakulta, Katedra matematickej analýzy, Mlynská dolina, 842 15 Bratislava, Czechoslovakia.