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ON Z-CONTINUOUS TOLERANCES OF Z-DISTRIBUTIVE LATTICES

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As we known, all lattice tolerances of a distributive lattice form an algebraic frame (cf. [2]), and similarly all frame tolerances (i.e. continuous tolerances) of a frame form a frame with respect to set inclusion (cf. [4]). A natural question arises if there is a suitable generalization of the concepts of distributive lattices on the one hand and of frames on the other, and of the other corresponding concepts such that these new concepts behave like the original ones. It turns out that a slight modification of the concept of order-continuous algebras (cf. [1]) could be helpful.

Definition. Let L be a lattice. An *indexed system* in L is a mapping of an index set I to L.

Notation. Indexed systems are denoted by $\{a_i \mid i \in I\}$.

Definition. An indexed system $\{a_i \mid i \in I\}$ is said to be *finite* if the index set I is finite. An indexed system is said to be *empty* if the index set is empty.

Recall that a *frame* is a complete lattice L in which $a \wedge \bigvee\{a_i \mid i \in I\} = \bigvee\{a \wedge a_i \mid i \in I\}$ for every element $a \in L$ and every indexed system $\{a_i \mid i \in I\}$ in L.

Throughout this paper, Z is a property of indexed systems in a lattice. We further assume:

(A1) If $h\colon L \to K$ is an order-preserving mapping of the lattice L to the lattice K, and an indexed system $\{a_i \mid i \in I\}$ in L satisfies Z, then $\{h(a_i) \mid i \in I\}$ also satisfies Z.

(A2) Finite indexed systems satisfy Z.

(A3) If I_j $(j \in J)$ are pairwise disjoint sets and $\{a_i \mid i \in I_j\}$ $(j \in J)$, $\{\bigvee\{a_i \mid i \in I_j\} \mid j \in J\}$ satisfy Z, then also $\{a_i \mid i \in \bigcup\{I_j \mid j \in J\}\}$ satisfies Z.

(A4) If $\varphi\colon J \to I$ is a bijection and $\{a_i \mid i \in I\}$ satisfies Z, then $\{a_{\varphi(j)} \mid j \in J\}$ also satisfies Z.

Definition. A Z-*system* is an indexed system that satisfies Z.

E x a m p l e s. Finite indexed systems, at most finite indexed systems, at most countable indexed systems, all indexed systems are Z-systems for a suitable Z.

Lemma 1. *If* $\{a_i \mid i \in I\}$ *is a Z-system in a lattice* L *and* p *is a unary algebraic function on* L*, then* $\{p(a_i) \mid i \in I\}$ *is a Z-system in* L.

P r o o f. Every algebraic function on a lattice is order-preserving. We apply (A1). □

Recall that a subset D of a lattice L is *dense* in L if we have $x = \bigvee\{d \in D \mid d \leqslant x\}$ for each element $x \in L$. A *lattice tolerance* of a lattice L is a compatible reflexive and symmetric relation on L.

Definitions. A lattice L is Z-*complete* if every Z-system in L has a supremum in L. The supremum of a Z-system $\{a_i \mid i \in I\}$ is denoted by $\bigsqcup\{a_i \mid i \in I\}$.

A lattice L is Z-*distributive* if it is Z-complete and $a \wedge \bigsqcup\{a_i \mid i \in I\} = \bigsqcup\{a \wedge a_i \mid i \in I\}$ holds for every element $a \in L$ and every Z-system $\{a_i \mid i \in I\}$ in L.

An element a of a Z-complete lattice L is Z-*compact* if every nonempty Z-system $\{a_i \mid i \in I\}$ in L such that $a \leqslant \bigsqcup\{a_i \mid i \in I\}$ contains a finite subsystem $\{a_i \mid i \in I_f \subseteq I\}$ such that $a \leqslant \bigsqcup\{a_i \mid i \in I_f\}$.

A subset D of a Z-complete lattice L is Z-*dense* in L if for each element x of L there exists a Z-systems $\{d_i \mid i \in I\}$ in D such that $x = \bigsqcup\{d_i \mid i \in I\}$.

A Z-complete lattice L is Z-*algebraic* if the set of all its Z-compact elements is Z-dense in L.

A lattice tolerance T on a Z-distributive lattice L is Z-*continuous* if whenever $\{a_i \mid i \in I\}$, $\{b_i \mid i \in I\}$ are Z-systems in L such that $[a_i, b_i] \in T (i \in I)$, then $[\bigsqcup\{a_i \mid i \in I\}, \bigsqcup\{b_i \mid i \in I\}] \in T$.

A Z-*continuous congruence* is a transitive Z-continuous tolerance.

In distributive lattices and in frames, the Z-concepts are as follows:

Z-concept	lattice concept	frame concept
Z-systems	finite indexed systems	indexed systems
Z-complete lattices	lattices	complete lattices
Z-distributive lattices	distributive lattices	frames
Z-compact elements	elements	compact elements
Z-algebraic Z-distributive lattices	distributive lattices	algebraic frames
Z-continuous tolerances	lattice tolerances	frame tolerances
Z-continuous congruences	lattice congruences	frame congruences

Lemma 2. (1) *Z-distributive lattices are distributive.*

(2) *Frames are Z-distributive.*

Proof is obvious. □

Recall that a *topped intersection structure* on a set X is a nonempty set of subsets of X such that it contains X and is closed under arbitrary intersections.

Lemma 3. (1) *All Z-continuous tolerances on a Z-distributive lattice form a topped intersection structure.*

(2) *All Z-continuous congruences on a Z-distributive lattice form a topped intersection structure.*

Proof is obvious. □

Since the set of all Z-continuous tolerances (Z-continuous congruences) on a Z-distributive lattice L, let us denote it by $\mathrm{Z\,Tol}(L)$ $(\mathrm{Z\,Con}(L))$, is a topped intersection structure, we are entitled to speak about Z-continuous tolerances (Z-continuous congruences) *generated* by a relation on L, and in particular about *principal* Z-continuous tolerances (Z-continuous congruences).

Recall some properties of lattice tolerances:

(P1) $[a,b] \in T \Leftrightarrow [a \wedge b, a \vee b] \in T$.

(P2) $[a,b] \in T,\ a \leqslant c \leqslant d \leqslant b \Rightarrow [c,d] \in T$.

(P3) $[a,b] \in T,\ [a,c] \in T \Rightarrow [a, a \vee b \vee c] \in T,\ [a, a \wedge b \wedge c] \in T$.

We know that the principal lattice tolerances of distributive lattices coincide with the principal lattice congruences (cf. [3]), and the following conditions are equivalent for the principal lattice congruence $\Theta(a,b)$ on a distributive lattice:

(i) $[x,y] \in \Theta(a,b)$;

(ii) $(a \vee b) \vee (x \vee y) = (a \vee b) \vee (x \wedge y)$ and $(a \wedge b) \wedge (x \vee y) = (a \wedge b) \wedge (x \wedge y)$;

(iii) $a \vee b \vee x = a \vee b \vee y$ and $a \wedge b \wedge x = a \wedge b \wedge y$.

This will be reffered to as the Principal tolerance lemma.

Lemma 4. *Principal lattice congruences of Z-distributive lattices are Z-continuous.*

Proof. Let L be a Z-distributive lattice, let $c \in L$ and $d \in L$. Denote $a := c \wedge d$ and $b := c \vee d$. We have $\Theta(a,b) = \Theta(c,d)$ in virtue of (P1). If $a = b$, then $\Theta(a,b) = \Delta$, which is Z-continuous. Suppose $a < b$. Let $\{x_i \mid i \in I\}$, $\{y_i \mid i \in I\}$ be Z-systems in L such that $[x_i, y_i] \in \Theta(a,b)$ $(i \in I)$. Then $a \wedge x_i = a \wedge y_i$ and $b \vee x_i = b \vee y_i$ $(i \in I)$ by the Principal tolerance lemma. Now the Z-distributivity of L yields $a \wedge \bigsqcup\{x_i \mid i \in I\} = \bigsqcup\{a \wedge x_i \mid i \in I\} = \bigsqcup\{a \wedge y_i \mid i \in I\} = a \wedge \bigsqcup\{y_i \mid i \in I\}$, and we also have $b \vee \bigsqcup\{x_i \mid i \in I\} = \bigsqcup\{b \vee x_i \mid i \in I\} = \bigsqcup\{b \vee y_i \mid i \in I\} = b \vee \bigsqcup\{y_i \mid i \in I\}$ using Lemma 1. By the Principal tolerance lemma, $[\bigsqcup\{x_i \mid i \in I\}, \bigsqcup\{y_i \mid i \in I\}] \in \Theta(a,b)$. Hence $\Theta(a,b)$ is Z-continuous. □

Proposition 1. *Principal Z-continuous tolerances on Z-distributive lattices coincide with principal lattice congruences.*

Proof. Let L be a Z-distributive lattice, $a \in L$ and $b \in L$. It is obvious that $\Theta(a,b) \subseteq \mathsf{ZT}(a,b) \subseteq \mathsf{ZC}(a,b)$, where $\mathsf{ZT}(a,b)$ is the principal Z-continuous tolerance and $\mathsf{ZC}(a,b)$ is the principal Z-continuous congruence generated by $\{[a,b]\}$. By Lemma 4, $\mathsf{ZC}(a,b) \subseteq \Theta(a,b)$. □

Corollary. *Principal Z-continuous congruences on Z-distributive lattices coincide with principal lattice congruences.*

Proof is obvious. □

The Z-continuous tolerance generated by a relation R will be denoted by $\mathsf{ZT}(R)$. The canonical projections of $L \times L$ onto L will be denoted by p_1 and p_2.

Proposition 2. *Let R be a reflexive and symmetric binary relation on a Z-distributive lattice L. Then the following conditions are equivalent:*

(i) $[a,b] \in \mathsf{ZT}(R)$;

(ii) *there exist finite subsets $M_i (i \in I)$ of R such that both $\{p_1(\bigwedge M_i) \mid i \in I\}$ and $\{p_2(\bigwedge M_i) \mid i \in I\}$ are Z-systems in L and $[a,b] = [\bigsqcup\{p_1(\bigwedge M_i) \mid i \in I\}, \bigsqcup\{p_2(\bigwedge M_i) \mid i \in I\}]$.*

Proof. Denote by T the set of all pairs $[a,b]$ such that condition (ii) is satisfied. It is obvious that $R \subseteq T \subseteq \mathsf{ZT}(R)$, and T is reflexive and symmetric. If we show that T is meet-compatible and Z-continuous, we will prove $T = \mathsf{ZT}(R)$, and consequently $[a,b] \in T$ for any fixed pair $[a,b] \in \mathsf{ZT}(R)$.

T is meet-compatible: Let $a_k = \bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_k\}$ and $b_k = \bigsqcup\{p_2(\bigwedge M_i) \mid i \in I_k\}$ $(k = 1,2)$. Then $a_1 \wedge a_2 = \bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_1\} \wedge \bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_2\} = \bigsqcup\{\bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_1\} \wedge p_1(\bigwedge M_j) \mid j \in I_2\} = \bigsqcup\{\bigsqcup\{p_1(\bigwedge M_i) \wedge p_1(\bigwedge M_j) \mid i \in I_1\} \mid j \in I_2\} = \bigsqcup\{\bigsqcup\{p_1(\bigwedge(M_i \cup M_j)) \mid i \in I_1\} \mid j \in I_2\} = \bigsqcup\{\bigsqcup\{p_1(\bigwedge M_n) \mid n \in I_1 \times \{j\}\} \mid j \in I_2\} = \bigsqcup\{p_1(\bigwedge M_n) \mid n \in I_1 \times I_2\}$, where $M_n := M_i \cup M_j$ for $n := [i,j]$. It is obvious that $\bigcup\{I_1 \times \{j\} \mid j \in I_2\} = I_1 \times I_2$, so that we can apply (A3). For $b_1 \wedge b_2$ we can do a similar computation. T is Z-continuous: Suppose that $\{a_k \mid k \in K\}$ and $\{b_k \mid k \in K\}$ are Z-systems such that $[a_k, b_k] \in T$, for instance $a_k = \bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_k\}$ and $b_k = \bigsqcup\{p_2(\bigwedge M_i) \mid i \in I_k\}$ $(k \in K)$. We can make $I_k - s$ disjoint in virtue of (A4) by taking $I'_k := I_k \times \{k\}$ $(k \in K)$. Since $\{p_1(\bigwedge M_i) \mid i \in \bigcup\{I_k \mid k \in K\}\}$ is a Z-system, we have $\bigsqcup\{a_k \mid k \in K\} = \bigsqcup\{\bigsqcup\{p_1(\bigwedge M_i) \mid i \in I_k\} \mid k \in K\} = \bigsqcup\{p_1(\bigwedge M_i) \mid i \in \bigcup\{I_k \mid k \in K\}\}$. Analogously, $\bigsqcup\{b_k \mid k \in K\} = \bigsqcup\{p_2(\bigwedge M_i) \mid i \in \bigcup\{I_k \mid k \in K\}\}$. Hence T is Z-continuous. □

Proposition 3. *Let $T_j (j \in J)$ be Z-continuous tolerances of a Z-distributive lattice L, denote $R := \bigcup\{T_j \mid j \in J\}$. Then $\mathsf{ZT}(R)$ is the supremum of $\{T_j \mid j \in J\}$ in $\mathsf{Z}\,\mathrm{Tol}(L)$ and the following conditions are equivalent:*

(i) $[a,b] \in \mathsf{ZT}(R)$;

(ii) *there exist finite subsets M_i $(i \in I)$ of R such that both $\{p_1(\bigwedge M_i) \mid i \in I\}$ and $\{p_2(\bigwedge M_i) \mid i \in I\}$ are Z-systems in L and $[a,b] = [\bigsqcup\{p_1(\bigwedge M_i) \mid i \in I\}, \bigsqcup\{p_2(\bigwedge M_i) \mid i \in I\}]$;*

(iii) *there exist finite subsets M_i' $(i \in I)$ of $R \cap \{[u,v] \in L \times L \mid a \wedge b \leqslant u \wedge v,\ u \vee v \leqslant a \vee b\}$ such that both $\{p_1(\bigwedge M_i') \mid i \in I\}$ and $\{p_2(\bigwedge M_i') \mid i \in I\}$ are Z-systems in L and $[a,b] = [\bigsqcup\{p_1(\bigwedge M_i') \mid i \in I\}, \bigsqcup\{p_2(\bigwedge M_i') \mid i \in I\}]$.*

Proof. Let L, T_j $(j \in J)$, R satisfy the assumptions. It is obvious that $\bigvee\{T_j \mid j \in J\}$ in $\mathsf{Z}\,\mathrm{Tol}(L)$ equals $\mathsf{ZT}(R)$. By Proposition 2, (i) is equivalent to (ii). Clearly (iii) implies (ii). It remains to prove that (ii) implies (iii). Let (ii) be valid. Put $M_i' := \{[(x \vee (a \wedge b)) \wedge (a \vee b), (y \vee (a \wedge b)) \wedge (a \vee b)] \mid [x,y] \in M_i\}$. Both $\{p_1(\bigwedge M_i') \mid i \in I\}$ and $\{p_2(\bigwedge M_i') \mid i \in I\}$ are Z-systems by Lemma 1, and if $[x,y] \in T_j$, then also $[(x \vee (a \wedge b)) \wedge (a \vee b), (y \vee (a \wedge b)) \wedge (a \vee b)] \in T_j$. Hence $M_i' \subseteq R \cap \{[u,v] \in L \times L \mid a \wedge b \leqslant u \wedge v,\ u \vee v \leqslant a \vee b\}$. Distributivity yields $p_k(\bigwedge M_i') = (p_k(\bigwedge M_i) \vee (a \wedge b)) \wedge (a \vee b)$ $(i \in I,\ k = 1,2)$. Consequently, $\bigsqcup\{p_1(\bigwedge M_i') \mid i \in I\} = \bigsqcup\{(p_1(\bigwedge M_i) \vee (a \wedge b)) \wedge (a \vee b) \mid i \in I\} = \bigsqcup\{p_1(\bigwedge M_i) \vee (a \wedge b) \mid i \in I\} \wedge (a \vee b) = (\bigsqcup\{p_1(\bigwedge M_i) \mid i \in I\} \vee (a \wedge b)) \wedge (a \vee b) = (a \vee (a \wedge b)) \wedge (a \vee b) = a$. By a similar computation we obtain $\bigsqcup\{p_2(\bigwedge M_i') \mid i \in I\} = b$. □

Proposition 4. *Let L be a Z-distributive lattice. Then $\mathsf{Z}\,\mathrm{Tol}(L)$ is a frame.*

Proof. We must show that for an arbitrary system $\{T_j \mid j \in J\}$ of Z-continuous tolerances of L and any Z-continuous tolerance T of L it is true that $T \cap \bigvee\{T_j \mid j \in J\} \subseteq \bigvee\{T \cap T_j \mid j \in J\}$. The converse inclusion is obvious. Let $[c,d] \in T \cap \bigvee\{T_j \mid j \in J\}$, denote $a := c \wedge d$, $b := c \vee d$. If $a = b$, i.e. $c = d$, we certainly have $[c,d] \in \bigvee\{T \cap T_j \mid j \in J\}$. Suppose $a < b$. It is clear that $[a,b] \in T$ and $[a,b] \in \bigvee\{T_j \mid j \in J\} = \mathsf{ZT}(\bigcup\{T_j \mid j \in J\})$ by Proposition 3. By the same proposition, there exist finite subsets M_i' $(i \in I)$ of $\bigcup\{T_j \mid j \in J\} \cap \{[u,v] \in L \times L \mid a \leqslant u \wedge v,\ u \vee v \leqslant b\}$ such that both $\{p_1(\bigwedge M_i') \mid i \in I\}$ and $\{p_2(\bigwedge M_i') \mid i \in I\}$ are Z-systems in L and $[a,b] = [\bigsqcup\{p_1(\bigwedge M_i') \mid i \in I\}, \bigsqcup\{p_2(\bigwedge M_i') \mid i \in I\}]$. Since $a \leqslant u \wedge v$, $u \vee v \leqslant b$ implies $[u,v] \in T$ by (P2) and (P1), we have $M_i' \subseteq T \cap \bigcup\{T_j \mid j \in J\} = \bigcup\{T \cap T_j \mid j \in J\}$. Therefore, in virtue of Proposition 3, $[a,b] \in \bigvee\{T \cap T_j \mid j \in J\}$, and also $[c,d] \in \bigvee\{T \cap T_j \mid j \in J\}$. □

We know that the frame of all lattice tolerances of a distributive lattice is algebraic with finitely generated lattice tolerances as compact elements (cf. [2]) but, is general, the frame of all frame tolerances is not.

Proposition 5. *Let L be a Z-distributive lattice, let c be a Z-compact element in L. Then $\Theta(a, a \vee c)$ is a compact element in $\mathsf{Z}\,\mathrm{Tol}(L)$.*

P r o o f . Suppose $\Theta(a, a\vee c) \subseteq \bigvee\{T_j \mid j\in J\}$. Then $[a, a\vee c] = [\bigsqcup\{p_1(\bigwedge M_i') \mid i\in I\}, \bigsqcup\{p_2(\bigwedge M_i') \mid i\in I\}]$ for some finite subsets M_i' $(i\in I)$ of $\bigcup\{T_j \mid j\in J\}\cap\{[u,v]\in L\times L \mid a\leqslant u\wedge v,\ u\vee v\leqslant a\vee c\}$. It follows that $c\leqslant a\vee c = \bigsqcup\{p_2(\bigwedge M_i') \mid i\in I\}$. Since c is Z-compact, there exists a finite subset $I_f\subseteq I$ such that $c\leqslant \bigsqcup\{p_2(\bigwedge M_i') \mid i\in I_f\}$, and so $a\vee c = \bigsqcup\{p_2(\bigwedge M_i') \mid i\in I_f\}$. Since $p_1(\bigwedge M_i') = a$ for each index $i\in I$, we have $a = \bigsqcup\{p_1(\bigwedge M_i') \mid i\in I_f\}$. We can conclude that there exists a finite subset $J_f\subseteq J$ such that $\bigcup\{M_i' \mid i\in I_f\}\subseteq\bigcup\{T_j \mid j\in J_f\}$, and therefore $[a, a\vee c]\in\bigvee\{T_j \mid j\in J_f\}$. Finally, $\Theta(a, a\vee c)\subseteq\bigvee\{T_j \mid j\in J_f\}$. □

Proposition 6. *Let L be a Z-algebraic Z-distributive lattice. Then* Z Tol(L) *is an algebraic frame.*

P r o o f . Since every Z-continuous tolerance of L is a supremum of principal Z-continuous tolerances, it suffices to show that every principal Z-continuous tolerance of L is a supremum of compact Z-continuous tolerances. Let $\Theta(c,d)$ be a principal Z-continuous tolerance. Denote $a := c\wedge d$, $b := c\vee d$. If $c = d$, then $\Theta(c,d) = \Delta$ is compact in Z Tol(L). Suppose $a < b$. Because L is Z-algebraic, there exist Z-compact elements $c_i (i\in I)$ such that $b = \bigsqcup\{c_i \mid i\in I\}$. Then also $b = \bigsqcup\{a\vee c_i \mid i\in I\}$. We obtain $\Theta(a, a\vee c_i)\subseteq\Theta(a,b)$ and $\Theta(a,b)\subseteq\bigvee\{\Theta(a,c_i) \mid i\in I\}$. Now, $\Theta(a,c_i)$ $(i\in I)$ are compact by Proposition 5. □

Corollary. *The frames of all frame tolerances of algebraic frames are algebraic.*

P r o o f is obvious. □

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