Czechoslovak Mathematical Journal

Ivan Žembery

Uniqueness of improper operations

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 1, 19–23

Persistent URL: http://dml.cz/dmlcz/128320

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

UNIQUENESS OF IMPROPER OPERATIONS

IVAN ŽEMBERY, Bratislava

(Received December 4, 1989)

The autor has defined in [1] the concept of an improper operation in a class of similar algebras in connection wist almost equational classes of algebras. An improper operation must be neither a basic operation nor a polynomial of basic operations, but it is preserved by all homomorphisms of the basic operations. We shall define the concept of an improper operation more precisely by means of the underlying functor.

Definition 1. *Let K be a class of similar algebras. Let K denote also the category of this class. Let C be the class of such algebras which result from the algebras of the class K by omitting a system of operations $\{f_i : i \in I\}$. Let C be the category of the class C and let $F: K \to C$ be the underlying functor by which just the operations f_i for $i \in I$ are omitted. If for arbitrary algebras $A, B \in K$*

$$\operatorname{Hom}_K(A,B) = \operatorname{Hom}_C\big(F(A), F(B)\big) \tag{1}$$

holds, then each of the operations f_i for $i \in I$ is called an improper operation with respect to the basic operations of the algebras in the class C.

Now we give some examples of improper operations.

Example 1. Let K be the class of all groups regarded as algebras with one binary operation of multiplication, one nullary operation of the unit and one unary operation of the inverse element. Let C be the class of all groups regarded as algebras with only one binary operation of multiplication. Let $F: K \to C$ be the underlying functor, which omits just the operations of the unit and of the inverse element. Since each mapping between two groups which preserves multiplication, preserves also the unit and the inverse element, the condition (1) is satisfied for the unit and also for the inverse element.

Example 2. On the other hand, the operation of multiplication in groups is not an improper operation with respect to the basic operations of the unit and the inverse element. Let us consider the group $\mathbb{Z}$ of all integers with the addition. The mapping $\mathbb{Z} \to \mathbb{Z}$ defined by $0 \mapsto 0$, $x \mapsto 1$ for $x > 0$ and $2 \mapsto -1$ for $x < 0$ preserves the unit and the inverse element, but it does not preserve the multiplication.

Example 3. Let K be the class of all algebras with one n-ary operation h and n unary operations $f_1, \ldots, f_n$ which satisfy the system of identities

$$h\big(f_1(x), \ldots, f_n(x)\big) = x$$
$$f_i\big(h(x_1, \ldots, x_n)\big) = x_i \qquad \text{for } i = 1, \ldots, n.$$

We show that h is an improper operation with respect to the basic operations $f_1, \ldots, f_n$. Forst we show that every mapping between two algebras of the class K which preserves the operations $f_1, \ldots, f_n$, preserves also the operation h. Let $A, B \in K$ be arbitrary algebras and let $\varphi\colon A \to B$ be an arbitary mapping which preserves the operations $f_1, \ldots, f_n$. Then

$$h(x_1, \ldots, x_n)\varphi = h\big(f_1\big(h(x_1, \ldots, x_n)\varphi\big), \ldots, f_n\big(h(x_1, \ldots, x_n)\varphi\big)\big) =$$
$$h\big(f_1\big(h(x_1, \ldots, x_n)\big)\varphi, \ldots, f_n\big(h(x_1, \ldots, x_n)\big)\varphi\big) = h(x_1\varphi, \ldots, x_n\varphi).$$

Let C be the class of exactly the same algebras as in the class K, the operation h being omitted. Let $F\colon K \to C$ be the underlying functor which omits just the operation h. We have shown that h satisfies (1) and therefore h is an improper operation with respect to the basic operations $f_1, \ldots, f_n$.

Example 4. Let K be the same class of algebras as in Example 3. We show that each of the operations f_i for $i = 1, \ldots, n$ is an improper operation with respect to the basic operation h. First we show that each mapping between two arbitrary algebras preserving the operation h, preserves also each of the operations f_i for $i = 1, \ldots, n$. Let $A, B \in K$ and let $\varphi\colon A \to B$ be an arbitrary mapping preserving the operation h. Then for arbitrary i,

$$f_i(x)\varphi = f_i\big(h\big(f_1(x)\varphi, \ldots, f_n(x)\varphi\big)\big) = f_i\big(h\big(f_1(x), \ldots, f_n(x)\big)\varphi\big) = f_i(x\varphi).$$

Let C be the class of exactly the same algebras as in the class K, the operations $f_1, \ldots, f_n$ being omitted. Let $F\colon K \to C$ be the underlying functor which omits just the operations $f_1, \ldots, f_n$. We have shown that each of the operations f_i satisfies (1) and therefore each of f_i for $i = 1, \ldots, n$ is an improper operation with respect to the basic operation h.

Now we explain the concept of the almost equational class of algebras. An almost equational class of algebras differs from a variety just by the improper operations. Some operations, which are basic in algebras in a variety, are improper in algebras in the almost equational class. The exact definition of an almost equational class of algebras will be given later.

It is well known that an algebraical structure can be defined on one set in different ways. For example, a group on a four-element set can be defined even in two non-isomorphical ways. There is a question whether a similar situation can occur also for improper operations. It is a question of the uniqueness of improper operations in the following sense. If $F\colon K \to C$ is the underlying functor which omits an improper operation f, is it possible that $F(A) = F(B)$ for some $A, B \in K$, $A \neq B$?

The aim of this paper is to show the uniqueness of improper operations and the relationship between a variety and an almost equational class of algebras from the view-point of omitting opperations.

The following theorem concerns the uniqueness of improper operations.

Theorem 1. *Let K be a class of similar algebras and let $F\colon K \to C$ be the underlying functor omitting an improper operation f. Then $F(A) \neq B(F)$ for any $A, B \in K$, $A \neq B$.*

Proof. Let the assumptions of the theorem be satisfied and let $A, B \in K$ be arbitrary algebras with the properties $A \neq B$ and $F(A) = F(B)$. It is obvious that the algebras A and B have the same basic sets and differ just by the improper operation h. Let h be an n-ary improper operation, which will be denoted h^A in the algebra A and h^B in the algebra B. If we do not distinquish between the notation for an algebra and its basic set, then there exist such elements $a_1, \ldots, a_n \in A$ that

$$h^A(a_1, \ldots, a_n) \neq h^B(a_1, \ldots, a_n).$$

Since the algebras A and B have the same basic sets, the identical mapping $\iota\colon A \to B$ can be considered. Because of (1) and $F(A) = F(B)$, the mapping ι is a homomorphism not only as the identical mapping between two algebras in the class C, but also between two algebras in the class K. For the homomorphism ι we have $a_i\iota = a_i$ for $i = 1, \ldots, n$, and since h is an improper operation in the class K,

$$h^A(a_1, \ldots, a_n) = \big(h^A(a_1, \ldots, a_n)\big)\iota = h^B(a_1\iota, \ldots, a_n\iota) = h^B(a_1, \ldots, a_n)$$

holds, which contradicts the assumption $h^A(a_1, \ldots, a_n) \neq h^B(a_1, \ldots, a_n)$. □

In the sequel we will use the concepts of the proper and the improper free algebra and the almost equational class of algebras.

By a free algebra over a class K on the set X we understand any algebra $A \in K$ such that for each algebra $B \in K$ and an arbitrary mapping $X \to B$ there exists a unique extension of this mapping to the homomorphism $A \to B$. Let us observe that a free algebra on the set X need not be generated by the set X. As an example of a free algebra we introduce the free algebra on a one-element set in the class of all groups considered only with one binary operation of multiplication. This group is the set of all integers with respect to addition. It is free on the set $\{1\}$ and the whole group is not generated by the element 1 by means of addition.

If the free algebra on a set X is generated by X, it is called a proper free algebra, otherwise it is called an improper free algebra.

Definition 2. *If the class K in Definition 1 is a variety, then the class C is called an almost equational class with respect to the variety K.*

Example 5. The class of all groups regarded as algebras with one binary operation of multiplicatin is an almost equational class of algebras with respect to the varierty of all groups regarded as algebras with one binary operation of multiplication, one nullary operation of the unit and one unary operation of the inverse element.

Example 6. The class C in Example 3 is an almost equational class with respect to the variety K.

Example 7. The class C in the Example 4 is an almost equational class with respect to the variety K.

When a system of improper operations is ommited from algebras of a variety, an almost equational class is obtained. Of course, each variety can be considered as an almost equational class with respect to itself. However, there is a question whether there exists a variety which is an almost equational class with respect to another variety. The answer to this question is given by the following theorem.

Theorem 2. *Let C be an almost equational class of algebras with respect to a variety K. If C is a variety, then all operations omitted by the corresponding underlying functor $F\colon K \to C$ are polynomials of basic operations of algebras in the class C.*

Proof. Let K and C satisfy the assumptions of the theorem. Let us consider the free algebra in the class K on an infinite countable set X. Since the definition of the free algebra is essentially based on the existence and uniqueness of homomorphisms, $F(A)$ is the free algebra on the set X over the class C. It is obvious that all free algebras in a variety are proper. Since C is a variety, $F(A)$ is a proper free algebra. Let f be an arbitrary n-ary improper operation ommited by the underlying functor F and let $x_1, \ldots, x_n \in X$ be arbitrary elements. The element $f(x_1, \ldots, x_n)$ is a polynomial of the basic operations in the class C, because $F(A)$ is a proper free algebra on the set X. Thus

$$f(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$$

where p is a suitable polynomial. Since this equality holds in the free algebra for the elements of the set X, it holds for arbitrary elements in each algebra in the class C. Therefore the improper operation f omitted by the underlying functor F is a polynomial of the basic operations of algebras in the class C. $\square$

References

[1] *I. Žembery*: Almost equational classes of algebras, Algebra Universalis *23* (1986), 293–307.

Author's address: Matematický ústav SAV, 814 73 Bratislava, Štefánikova 49, Czechoslovakia.