Czechoslovak Mathematical Journal

Chi Keung Lai; Kar-Ping Shum

A note in inverse and dual semigroups

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 1, 7–10

Persistent URL: http://dml.cz/dmlcz/128321

A NOTE IN INVERSE AND DUAL SEMIGROUPS

C.K. LAI AND K.P. SHUM, Shatin

(Received November 10, 1986)

In studying the structure of inverse semigroups, L. Márkl [2] has proved the following conditions on semigroups with 0 are equivalent:

(i) S is an inverse semigroup and the union of a finite number of its 0-minimal left (right) ideals.

(ii) S is the union of a finite number of its quasi-ideals, and all these quasi-ideals form a special complete system.

(iii) S is an inverse semigroup and the union of a finite number of its 0-minimal quasi-ideals.

(iv) S is an inverse semigroup with finitely many idempotents and every non-zero idempotent is primitive.

However, it has been noticed by him that the following condition:

(v) S is a 0-direct union of finitely many two-sided ideals which are completely 0-simple inverse subsemigroups of S.

is weaker than any one of the conditions (i) to (iv).

In this note, we observe that if each summand in (v) satisfies min-r (that is, Minimum condition on right ideals), then the above conditions are in fact all equaivalent. We shall prove that any one of these conditions is a necessary and sufficiency condition for a semigroup to be semisimple and dual. Thus a characterization theorem for semisimple dual semigroup is obtained. Throughout the paper, every semigroup has 0 and contains more than one element. The reader is referred to O. Steinfeld [7] for all terminology and definitions not given here.

Definition 1. Let A be a subset of a semigroup S. Denote

$$r(A) = \{x \in S \mid Ax = 0\}$$

and

$$\ell(A) = \{x \in S \mid xA = 0\}.$$

A semigroup S is called *dual semigroup* if $\ell r(L) = L$ for every left ideal L in S and $r\ell(R) = R$ for every right ideal R in S.

Definition 2. A semigroup S with min-r is said to be *semisimple* if the radical of S is zero. In other words, a semigroup S is semisimple if S satisfies the descending chain condition on its right ideals and contains no non-trivial nilpotent ideals.

Unlike in ring theory, a semisimple semigroup need not be a dual semigroup. The following is an example.

Example 3. Let $S = \{0, a, b, c, d\}$ with Cayley table

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
d	0	a	b	c	d

Then S is a semisimple semigroup. Since $\{0, a, c\}$ is a right ideal of S such that $r\ell\{0, a, c\} = S$, S is not a dual semigroup.

Lemma 4. *Let S be a dual semigroup. Then S satisfies* min-r *(*min-ℓ*;* max-r*;* max-ℓ*) if and only if S satisfies* max-ℓ*(*max-r*;* min-ℓ*;* min-r*).*

Proof ($\Rightarrow$). Consider the ascending chain of left ideals of S: $L_1 \subseteq L_2 \subseteq \ldots$. Then $r(L_1) \supseteq r(L_2) \subseteq \ldots$ is a descending chain of right ideals of S. By min-r, this chain will be terminated, say, at the nth step. Thus, there exists a positive integer n such that $r(L_n) = r(L_m)$ for all $m \geqslant n$. Apply the duality of S, we have $L_n = \ell r(L_n) = \ell r(L_m) = L_m$ for all $m \geqslant n$. This implies that S satisfies max-ℓ.

($\Leftarrow$). Follows dually as the "only if" part. □

Lemma 5. *If a semigroup is a finite* 0*-direct union of* 0*-minimal right (left) ideals of S, then S satisfies both* min-r *and* max-r *(*min-ℓ *and* max-ℓ*).*

Proof. By assumption, $S = \bigcup_{i=1}^{n} R_i$, where each R_i is a 0-minimal right ideal of S. Let R be an arbitrary non-zero right ideal of S. Then $R \subseteq \bigcup_{i=1}^{k} R_{n_i}$, $\{n_i \mid i = 1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, n\}$. Clearly $R \cap R_{n_i} \neq \{0\}$ for all $i = 1, 2, \ldots, k$. Since each R_{n_i} is 0-minimal, so $R_{n_i} = R \cap R_{n_i} \subseteq R$ for all $i = 1, 2, \ldots, k$. Therefore $\bigcup_{i=1}^{k} R_{n_i} \subseteq R$. Consequently $R = \bigcup_{i=1}^{k} R_{n_i}$. This shows that every right ideal of S is also a finite 0-direct union of 0-minimal right ideals of S. This implies that S satisfies both min-r and max-r. □

Lemma 6. *Let S be a dual semigroup with zero radical. Then the following conditions are equivalent:*

(i) $\overline{\overline{E}} < \infty$.
(ii) *S satisfies* max-r
(iii) *S satisfies* min-r
(iv) *S satisfies* max-ℓ
(v) *S satisfies* min-ℓ.

P r o o f. It was proved by St. Schwarz [3] that S is a 0-direct union of eS for all $e^2 = e \in S$, that is, $S = \bigcup_{e \in E} eS$. It can by seen that each eS is a 0-minimal right ideal of S. In virtue of Schwarz's theorem and Lemma 5, we obtain (i) implies (ii) to (v). In view of Lemma 4, it remains to prove (ii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i). For (ii) $\Rightarrow$ (i), we suppose that the set E is not finite. Then there exists a countable set of idempotents $\{e_i \mid i \in N\} \subseteq S$. Write $S_n = \bigcup_{i=1}^{n} e_i S$. If $S_n = S_{n+1}$, then $\bigcup_{i=1}^{n} e_i S = \bigcup_{i=1}^{n+1} e_i S \supseteq e_{n+1} S$. This implies that $e_{n+1} S \cap e_k S \neq \{0\}$ for some k, which contradicts that the union of $e_i S$'s are 0-directed. Thus $S_n \subsetneqq S_{n+1}$. However, this result contradicts max-r. Consequently E must be a finite set. Thus (ii) $\Rightarrow$ (i). Similarly, (iv) $\Rightarrow$ (i). □

Theorem 7. *The following conditions on a semigroups with* 0 *are equivalent:*
(i) S is a semisimple dual semigroup.
(ii) S is a 0*-direct union of finitely many two-sided ideals which are* 0*-simple dual subsemigroups of S with* min-r.
(iii) S is a 0*-direct union of finitely many two-sided ideals which are completely* 0*-simple inverse subsemigroups of S with* min-r.
(iv) S is a regular semigroup with finitely many orthogonal idempotents.
(v) S is an inverse semigroup with finitely many idempotents and every non-zero idempotent of S is primitive.
(vi) S is an inverse semigroup and can be expressed as the union of a finite number of its quasi-ideals, and these quasi-ideals form a special complete system.
(vii) S is an inverse semigroup and can be expressed as the union of a finite number of its 0*-minimal left (right) ideals.*
(viii) S is an inverse semigroup and can be expressed as the union of a finite nunber of its 0*-minimal quasi-ideals.*

P r o o f. In virtue of the characterization theorem for dual semigroups with zero radicals due to O. Steinfeld in [6] and Lemma 6, we have (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). By Corollary 10.13 in [7], conditions (v) to (viii) are all equivalent. It remains to show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By the Corollary stated in [5], we obtain that (ii) $\Leftrightarrow$ (iii). For (i) $\Leftrightarrow$ (ii), we let S be a semisimple dual semigroup. By Schwarz's decomposition theorem in [4], S is a 0-direct union of two-sided ideals $\{I_i \mid i \in \Gamma\}$ which are 0-simple dual subsemigroups of S. Since every 0-simple dual semigroup contains at least one non-zero idempotent [5], so $E = \bigcup_{i \in \Gamma} (E \cap I_i)$. As $\bar{\bar{E}} < \infty$, by Lemma 6, Γ and $E \cap I_i$ are finite sets for all i. Hence all I_i's satisfy min-r by Lemma 6. Thus (i) $\Rightarrow$ (ii). Conversely, let S be a 0-direct union of finitely many two-sided ideals $\{I_1, I_2, \ldots, I_n\}$ which are 0-simple dual subsemigroups of S with min-r. Then by the converse of Schwarz's decomposition theorem proved in [1], S must be a dual semigroup with zero radical. Moreover, for $i = 1, 2, \ldots, n$, $E \cap I_i$ is a finite set since each I_i is a semisimple dual subsemigroup of S. Thus $E = \bigcup_{i=1}^{n} (E \cap I_i)$ is a finite set. By Lemma 6, S is a semisimple dual semigroup. Thus (ii) $\Rightarrow$ (i). Our proof is completed. □

References

[1] *C.K. Lai* and *K.P. Shum*: Characterizations of dual semigroups, Czechoslovak Math. J. *39* (1989), 518–584.

[2] *L. Márkl*: Structure theorems on certain regular and inverse semigroups, Czechoslovak Math. J. *27* (1977), 388–393.

[3] *St. Schwarz*: On dual semigroups, Czechoslovak Math. J. *10* (1960), 201–230.

[4] *St. Schwarz*: On the structure of dual semigroups, Czechoslovak Math. J. *21* (1971), 461–483.

[5] *St. Schwarz*: Any 0-simple dual semigroup is completely 0-simple, Semigroup Forum *2* (1971), 90–92.

[6] *O. Steinfeld*: A characterization of dual semigroup without nilpotent ideals, Semigroup Forum *14* (1977), 1–5.

[7] *O. Steinfeld*: Quasi-ideals in rings and semigroups, Akademiai Kiadó, Budapest, 1978.

Author's address: Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong.