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THREE-POINT BOUNDARY VALUE PROBLEM FOR NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATION WITH PARAMETER

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1. INTRODUCTION

Consider the second-order differential equation

$$y'' - q(t)y = f(t, y, y', \mu) \tag{1}$$

in which $q \in C^0(J)$, $f \in C^0(J \times \mathbb{R}^2 \times I)$, $q(t) > 0$ for $t \in J$, where $J = \langle t_1, t_3 \rangle$, $I = \langle a, b \rangle$, $-\infty < t_1 < t_3 < \infty$, $-\infty < a < b < \infty$, containing a parameter μ. Let $t_2 \in \mathbb{R}$, $t_1 < t_2 < t_3$ be an arbitrary fixed number. The problem considered is to determine sufficient conditions on q and f quaranteeing that it is possible to choose the parameter μ so that there exists a solution y of (1) satisfying either the boundary conditions

$$y(t_1) = y(t_2) = y(t_3) = 0 \tag{2}$$

or the boundary conditions

$$y(t_1) = y'(t_1) = y(t_3) = 0. \tag{3}$$

The uniqueness of solutions of the boundary value problems (BVP for short) (1), (2) and (1), (3) is also discussed.

Sufficient conditions for a two-parameter differential equation $y'' + \big(q(t, \lambda, \mu) + r(t)\big)y = 0$ having a nontrivial solution y satisfying (2) are given in [2].

Let u, v be solutions of the equation

$$y'' = q(t)y \quad \big(q \in C^0(J),\ q(t) > 0 \text{ for } t \in J\big), \tag{q}$$

$u(t_1) = 0$, $u'(t_1) = 1$, $v(t_1) = 1$, $v'(t_1) = 0$ and

$$r(t,s) := u(t)v(s) - u(s)v(t) \quad \text{for} \quad (t,s) \in J^2. \tag{4}$$

Then the following lemma can be proved.

Lemma 1. *Let r be defined by* (4). *Then*

$$r(t,s) > 0 \quad \text{for} \quad t_1 \leqslant s < t \leqslant t_3, \quad r(t,s) < 0 \quad \text{for} \quad t_1 \leqslant t < s \leqslant t_3 \tag{5}$$

and

$$\big(r_1'(t,s) :=\big)\ \frac{\partial r}{\partial t}(t,s) > 1 \quad \text{for} \quad (t,s) \in J^2, \quad t \neq s.$$

Proof. Let $s \in J$ be an arbitrary fixed number. Setting $z(t) := r(t,s)$ for $t \in J$, then z is a solution of (q), $z(s) = 0$, $z'(s) = 1$ and $z'(t) = r_1'(t,s)$. Since $q(t) > 0$ on J, it is easy to verify that $z(t) < 0$ for $t_1 \leqslant t < s$ (provided $s > t_1$), $z(t) > 0$ for $s < t \leqslant t_3$ (provided $s < t_3$) and $z'(t) > 1$ for $t \in J$, $t \neq s$. This proves Lemma 1. □

Lemma 2. *Let $h \in C^0(J)$ and let y be a solution of the equation*

$$y'' - q(t)y = h(t) \tag{6}$$

satisfying the boundary conditions

$$y(t_1) = y(t_2) = 0. \tag{7}$$

Then

$$y(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)h(s)\mathrm{d}s + \int_{t_2}^{t} r(t,s)h(s)\mathrm{d}s, \quad t \in J. \tag{8}$$

Proof. Setting $y_0(t) := \int_{t_2}^{t} r(t,s)h(s)\mathrm{d}s$ for $t \in J$, then y_0 is a solution of (6), $y_0(t_2) = y_0'(t_2) = 0$, and the function

$$y(t) := \frac{r(t_2,t)}{r(t_1,t_2)} y_0(t_1) + y_0(t) \quad \text{for} \quad t \in J,$$

is a solution of (6) (which is then unique) satisfying (7). □

Lemma 3. *Let $h \in C^0(J \times I)$, let $h(t, .)$ be an increasing function on I for every fixed $t \in J$ and*

$$h(t,a)h(t,b) \leqslant 0 \quad \text{for} \quad t \in J. \tag{9}$$

Then there exists a unique μ_0, $\mu_0 \in J$ such that the equation

$$y'' - q(t)y = h(t,\mu) \tag{10}$$

with $\mu = \mu_0$ has a solution y (which is then unique) satisfying (2).

Proof. Let $y(t,\mu)$ be the solution of (10), $y(t_1,\mu) = y(t_2,\mu) = 0$. By Lemma 2

$$y(t_3,\mu) = \frac{r(t_2,t_3)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)h(s,\mu)\mathrm{d}s + \int_{t_2}^{t_3} r(t_3,s)h(s,\mu)\mathrm{d}s, \quad \mu \in I.$$

Since $r(t_2,t_1) > 0$, $r(t_2,t_3) < 0$, $r(t_1,s) < 0$ for $s \in (t_1,t_3\rangle$ and $r(t_3,s) > 0$ for $s \in \langle t_1,t_3)$ by Lemma 1, the function $y(t_3,.)$ is an increasing continuous function on I and by virtue of (9) we have $y(t_3,a) \leqslant 0$, $y(t_3,b) \geqslant 0$. Consequently, there exists a unique μ_0, $\mu_0 \in I$ such that $y(t_3,\mu_0) = 0$. BVP (q), (2) has only the trivial solution and thus BVP (10), (2) with $\mu = \mu_0$ has a unique solution. □

Let r_1, r_2 be positive constants, $r_1 > 0$, $r_2 > 0$. In what follows we shall assume that q and f satisfy some of the following assumptions:

$$|f(t,y_1,y_2,\mu)| \leqslant q(t)r_1 \quad \text{for} \quad (t,y_1,y_2,\mu) \in D \times I, \quad \text{where} \quad D := J \times \langle -r_1,r_1\rangle \times \langle -r_2,r_2\rangle\,; \tag{11}$$

$$f(t,y_1,y_2,.) \quad \text{is an increasing function on} \quad I \quad \text{for every fixed} \quad (t,y_1,y_2) \in D; \tag{12}$$

$$f(t,y_1,y_2,a)f(t,y_1,y_2,b) \leqslant 0 \quad \text{for} \quad (t,y_1,y_2) \in D; \tag{13}$$

$$\min\Big\{ \big(A + r_1 \max_{t\in J} q(t)\big)(t_3 - t_1),\ 2\sqrt{r_1}\sqrt{A + r_1 \max_{t\in J} q(t)} \Big\} \leqslant r_2, \quad \text{where} \quad A := \max_{(t,y_1,y_2,\mu)\in D\times I} |f(t,y_1,y_2,\mu)|. \tag{14}$$

Remark 1. If the function f may be written in the form $f(t,y_1,y_2,\mu) = g(t,y_1,y_2) + \mu \cdot \varphi(t)$ with $g \in C^0(J \times \mathbb{R}^2)$, $\varphi \in C^0(J)$, $\varphi(t) > 0$ on J, then f satisfies assumption (12) for arbitrary positive constants r_1, r_2. If, in addition, g is bounded on $J \times \mathbb{R}^2$, then assumption (11) is satisfied for an arbitrary positive constant r_2 and a sufficient large positive constant r_1.

Remark 2. A function $\alpha \in C^2(J)$ ($\beta \in C^2(J)$) is called an upper (lower) solution of BVP (1), (2) if $\alpha''(t) - q(t)\alpha(t) \leqslant f(t, \alpha(t), \alpha'(t), \mu)$, $\alpha(t_1) \geqslant 0$, $\alpha(t_2) \geqslant 0$, $\alpha(t_3) \geqslant 0$ ($\beta''(t) - q(t)\beta(t) \geqslant f(t, \beta(t), \beta'(t), \mu)$, $\beta(t_1) \leqslant 0$, $\beta(t_2) \leqslant 0$, $\beta(t_3) \leqslant 0$) for $(t, \mu) \in J \times I$. It follows from assumptions (11) that $\alpha(t) \equiv r_1$ ($\beta(t) \equiv -r_1$) is an upper (lower) solution of BVP (1), (2).

Lemma 4. *Assume that assumptions* (11)–(14) *are satisfied for positive constant* r_1, r_2. *Then for every* φ, $\varphi \in C^1(J)$, $|\varphi^{(i)}(t)| \leqslant r_{i+1}$ *for* $t \in J$, $i = 0, 1$, *there exists a unique* μ_0, $\mu_0 \in I$ *such that the equation*

$$y'' - q(t)y = f(t, \varphi(t), \varphi'(t), \mu) \tag{15}$$

with $\mu = \mu_0$ *has a solution* y *(which is then unique) satisfying* (2). *For this solution* y *the inequalities*

$$|y^{(i)}(t)| \leqslant r_{i+1}, \quad t \in J, \quad i = 0, 1, \tag{16}$$

hold.

Proof. Let $\varphi \in C^1(J)$, $|\varphi^{(i)}(t)| \leqslant r_{i+1}$ for $t \in J$, $i = 0, 1$, and $h(t, \mu) := f(t, \varphi(t), \varphi'(t), \mu)$ for $(t, \mu) \in J \times I$. Then $|h| \leqslant A$, from (13) we get

$$h(t, a) \leqslant 0, \quad h(t, b) \geqslant 0 \quad \text{for} \quad t \in J,$$

and $h(t, .)$ is an increasing continuous function on J for all fixed $t \in J$ by assumption (12). In this situation we may apply Lemma 3 and thus there exists a unique μ_0, $\mu_0 \in I$ such that equation (10) with $\mu = \mu_0$ has a unique solution y satisfying (2).

We now prove inequalities (16). Let $|y(t) \leqslant |y(\xi)| > r_1$ be satisfied for some $\xi \in (t_1, t_3)$ and $t \in J$. If $y(\xi) > r_1$ ($y(\xi) < -r_1$) then $y''(\xi) > 0$ ($y''(\xi) < 0$) by assumption (11), and therefore y does not have a local maximum (minimum) at the point $t = \xi$, which is a contradiction. Thus $|y(t)| \leqslant r_1$ on J.

Let $r_2 \geqslant \left(A + r_1 \max_{t \in J} q(t)\right)(t_3 - t_1)$. Since $y'(\xi) = 0$, we obtain from the equality $y'(t) = \int_\xi^t \left(q(s)y(s) + h(s, \mu_0)\right) \mathrm{d}s$ that

$$|y'(t)| \leqslant \left(A + r_1 \max_{t \in J} q(t)\right)|t - \xi| \leqslant \left(A + r_1 \max_{t \in J} q(t)\right)(t_3 - t_1) \leqslant r_2$$

for $t \in J$.

Let $r_2 \geqslant 2\sqrt{r_1}\sqrt{A + r_1 \max_{t \in J} q(t)}$ and $y'(\xi_1) = 0$ for some $\xi_1 \in J$. Multiplying both sides of the equality

$$y''(t) = q(t)y(t) + h(t, \mu_0), \quad t \in J,$$

by $2y'(t)$ and integrating from ξ_1 to t $(\in J)$, we obtain

$$y'^2(t) = 2\int_{\xi_1}^t \big(q(s)y(s)y'(s) + h(s,\mu_0)y'(s)\big)\,\mathrm{d}s.$$

Then we get

$$y'^2(t) \leqslant 2\big(A + r_1 \max_{t\in J} q(t)\big)\Big|\int_{\xi_1}^t y'(s)\mathrm{d}s\Big| \leqslant 4r_1\big(A + r_1 \max_{t\in J} q(t)\big)$$

on every interval J_1, $J_1 \subset J$, $\xi_1 \in J_1$, $y'(t) \geqslant 0$ $(\leqslant 0)$ for $t \in J_1$, and this yields

$$|y'(t)| \leqslant 2\sqrt{r_1}\sqrt{A + r_1 \max_{t\in J} q(t)} \leqslant r_2 \quad \text{for} \quad t \in J.$$

□

If the function $f(t, y_1, y_2, \mu)$ does not depend explicitly on y_2, then we may write equation (1) in the form

$$y'' - q(t)y = f_1(t, y, \mu) \tag{17}$$

where $f_1 \in C^0(J \times \mathbb{R} \times I)$. From Lemma 2 and its proof it immediately follows:

Lemma 5. *Let $r > 0$ be a positive constant and*

(18) $|f_1(t,y,\mu)| \leqslant rq(t)$ *for* $(t,y,\mu) \in H \times I$, *where* $H := J \times \langle -r, r\rangle$;

(19) $f_1(t,y,.)$ *is an increasing function on I for every fixed $(t,y) \in H$;*

(20) $f(t,y,a)f(t,y,b) \leqslant 0$ *for* $(t,y) \in H$.

Then for every φ, $\varphi \in C^0(J)$, $|\varphi(t)| \leqslant r$ for $t \in J$ there exists a unique μ_0, $\mu_0 \in I$ such that the equation

$$y'' - q(t)y = f_1(t, \varphi(t), \mu) \tag{21}$$

with $\mu = \mu_0$ has a solution y (which is then unique) satisfying (2). For this solution y the inequality

$$|y(t)| \leqslant r \quad \text{for} \quad t \in J$$

holds.

Remark 3. Let the assumptions of Lemma 5 be satisfied,

$$A_1 := \max_{(t,y,\mu)\in H\times I} |f_1(t,y,\mu)|,$$

$\varphi \in C^0(J)$, $|\varphi(t)| \leqslant r$ for $t \in J$ and let y be the solution of BVP (21), (2) with $\mu = \mu_0$. Then there exists ξ, $\xi \in I$: $y'(\xi) = 0$ and from the equality $y'(t) = \int_\xi^t \big(q(s)y(s) + f_1(s,\varphi(s),\mu_0)\big)\mathrm{d}s$ we get $|y'(t)| \leqslant \big(A_1 + r\max_{t\in J} q(t)\big)(t_3 - t_1)$ for $t \in J$.

Theorem 1. *Suppose that assumptions (11)–(14) are satisfied for positive constants r_1, r_2. Then there exists μ_0, $\mu_0 \in I$ such that BVP (1), (2) with $\mu = \mu_0$ has a solution y satisfying (16).*

Proof. Let $X = C^1(J)$ be a Banach space with the norm $\|y\| = \max_{t \in J}(|y(t)| + |y'(t)|)$ for $y \in X$, $\mathcal{K} := \{y;\ y \in X, |y^{(i)}(t)| \leqslant r_{i+1}$ for $t \in J,\ i = 0, 1\}$ and $B := A + r_1 \max_{t \in J} q(t)$. $\mathcal{K}$ is a closed bounded convex subset of X, $\mathcal{K} \subset X$. For every φ, $\varphi \in \mathcal{K}$ there exists (by Lemma 4) a unique μ_0, $\mu_0 \in I$ such that equation (15) with $\mu = \mu_0$ has a unique solution y satisfying (2) and (16). Setting $T(\varphi) = y$ we obtain an operator T, $T\colon \mathcal{K} \to \mathcal{K}$. We shall prove that T is a completely continuous operator.

Let $\{y_n\}$, $y_n \in \mathcal{K}$ be a convergent sequence, $\lim_{n\to\infty} y_n = y$, and $z_n = T(y_n)$, $z = T(y)$. Then (by Lemma 4) there exist a sequence $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$z_n(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s) f\big(s, y_n(s), y_n'(s), \mu_n\big)\,\mathrm{d}s + \int_{t_2}^{t} r(t,s) f\big(s, y_n(s), y_n'(s), \mu_n\big)\,\mathrm{d}s, \quad t \in J, \tag{22}$$

and

$$z(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s) f\big(s, y(s), y'(s), \mu_0\big)\,\mathrm{d}s + \int_{t_2}^{t} r(t,s) f\big(s, y(s), y'(s), \mu_0\big)\,\mathrm{d}s, \quad t \in J.$$

If $\{\mu_n\}$ is not a convergent sequence, then there exist convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$. Putting $n = k_n$ and $n = r_n$ in (22) and taking limits on both sides of (22) we obtain

$$\lim_{n\to\infty} z_{k_n}(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s) f\big(s, y(s), y'(s), \lambda_1\big)\,\mathrm{d}s + \int_{t_2}^{t} r(t,s) f\big(s, y(s), y'(s), \lambda_1\big)\,\mathrm{d}s \tag{23}$$

and

$$\lim_{n\to\infty} z_{r_n}(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s) f\big(s, y(s), y'(s), \lambda_2\big)\,\mathrm{d}s + \int_{t_2}^{t} r(t,s) f\big(s, y(s), y'(s), \lambda_2\big)\,\mathrm{d}s \tag{24}$$

uniformly on J, respectively. Since $f(t,y(t),y'(t),\lambda_1) < f(t,y(t),y'(t),\lambda_2)$ for $t\in J$ by assumption (12) then it follows from (23), (24) and Lemma 1 that $\lim_{n\to\infty} z_{k_n}(t_3) < \lim_{n\to\infty} z_{r_n}(t_3)$, which contradicts $z_n(t_3) = 0$ for all $n\in\mathbb{N}$. Consequently, $\{\mu_n\}$ is a convergent sequence; let $\lim_{n\to\infty}\mu_n = \mu^*$. Then $\lim_{n\to\infty} f(t,y_n(t),y_n'(t),\mu_n) = f(t,y(t),y'(t),\mu^*)$ uniformly on J and taking limits on both sides of (22) we get

$$(z^*(t) :=) \lim_{n\to\infty} z_n(t) = \frac{r(t_2,t)}{r(t_2,t_1)}\int_{t_1}^{t_2} r(t_1,s)f(s,y(s),y'(s),\mu^*)\,\mathrm{d}s + \int_{t_2}^{t} r(t,s)f(s,y(s),y'(s),\mu^*)\,\mathrm{d}s$$

uniformly on J. The function z^* is a solution of the equation

$$z'' - q(t)z = f(t,y(t),y'(t),\mu^*)$$

and $z^*(t_1) = z^*(t_2) = z^*(t_3) = 0$, consequently by Lemma 4 we get $\mu^* = \mu_0$, $z^* = z$. Next, uniformly on J we have

$$\begin{aligned}\lim_{n\to\infty} z_n'(t) &= \lim_{n\to\infty}\Big[-\frac{r_1'(t,t_2)}{r(t_2,t_1)}\int_{t_1}^{t_2} r(t_1,s)f(s,y_n(s),y_n'(s),\mu_n)\,\mathrm{d}s \\ &\quad + \int_{t_2}^{t} r_1'(t,s)f(s,y_n(s),y'(s),\mu_n)\,\mathrm{d}s\Big] \\ &= -\frac{r_1'(t,t_2)}{r(t_2,t_1)}\int_{t_1}^{t_2} r(t_1,s)f(s,y(s),y'(s),\mu_0)\,\mathrm{d}s \\ &\quad + \int_{t_2}^{t} r_1'(t,s)f(s,y(s),y'(s),\mu_0)\,\mathrm{d}s = z'(t).\end{aligned}$$

Hence $\lim_{n\to\infty} T(y_n) = T(y)$ and T is a continuous operator on $\mathcal{K}$.

Let $y\in\mathcal{K}$ and $z = T(y)$. Then $z\in\mathcal{K}$, $z(t_1) = z(t_2) = z(t_3) = 0$ and

$$z''(t) = q(t)z(t) + f(t,y(t),y'(t),\mu_0) \quad \text{for} \quad t\in J \tag{25}$$

with some $\mu_0\in I$. From (25) it follows that $|z''(t)| \leqslant B$ for $t\in J$, thus $T(\mathcal{K})\subset\mathcal{L} := \{y, y\in C^2(J), |y(t)|\leqslant r_1, |y'(t)|\leqslant r_2, |y''(t)|\leqslant B \text{ for } t\in J\}\subset\mathcal{K}$, and since $\mathcal{L}$ is a compact subset of X, $T(\mathcal{K})$ is a relative compact subset of X.

By Schauder's fixed point theorem there exists $y\in\mathcal{K}$: $T(y) = y$, i.e. there exists $\mu_0\in I$ such that

$$y''(t) - q(t)y(t) = f(t,y(t),y'(t),\mu_0) \quad \text{for} \quad t\in J$$

and y satisfies (2) and (16). □

Corollary 1. *Assume that assumptions* (12), (13) *are satisfied for positive constants* r_1, r_2 *and* $r_1(t_3 - t_1)\max_{t\in J} q(t) \leqslant r_2$. *Then there is* $\delta > 0$ *such that for every* ε, $0 < \varepsilon \leqslant \delta$ *there exists* μ_ε, $\mu_\varepsilon \in I$ *such that BVP*

$$y'' - q(t)y = \varepsilon f(t, y, y', \mu),$$
$$y(t_1) = y(t_2) = y(t_3) = 0$$

with $\mu = \mu_\varepsilon$ *has a solution* y *satisfying* (16).

P r o o f. Let $A := \max_{(t,y_1,y_2,\mu)\in D\times I} |f(t, y_1, y_2, \mu)|$ and

$$\delta := \min\Big\{\frac{r_1}{A}\min_{t\in J} q(t), \frac{1}{A}\Big(\frac{r_2}{t_3 - t_1} - r_1 \max_{t\in J} q(t)\Big)\Big\}.$$

Then εf for $0 < \varepsilon \leqslant \delta$ satisfies the same assumptions as f in Theorem 1 and thus Corollary 1 follows immediately from Theorem 1. □

E x a m p l e 1. Let $t_2 \in \big(0, \frac{1}{3}\big)$. Consider BVP

$$y'' - q(t)y = t^3 \cos y + ty'^n + \mu p(t), \tag{26}$$
$$y(0) = y(t_2) = y\Big(\frac{1}{3}\Big) = 0,$$

where p, $q \in C^0(J_1)$, $1 \leqslant p(t) \leqslant 2$, $\frac{8}{3\pi} \leqslant q(t) \leqslant \frac{10}{3\pi}$ for $t \in \langle 0, \frac{1}{3}\rangle =: J_1$, n is a positive integer and $\mu \in \langle -\frac{4}{9}, \frac{4}{9}\rangle$. One can easily check that the assumptions of Theorem 1 are satisfied with $r_1 = \frac{\pi}{2}$, $r_2 = 1$ and thus there exists μ_0, $\mu_0 \in \langle -\frac{4}{9}, \frac{4}{9}\rangle$ such that BVP (26) with $\mu = \mu_0$ has a solution y satisfying $|y(t)| \leqslant \frac{\pi}{2}$, $|y'(t)| \leqslant 1$ for $t \in J_1$.

Corollary 2. *Assume that assumptions* (11)–(14) *are satisfied for positive constants* r_1, r_2. *Then there exists* μ_0, $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (3) *and* (16).

P r o o f. Let $\{x_n\}$, $x_n \in (t_1, t_3)$ be a convergent sequence, $\lim_{n\to\infty} x_n = t_1$. By Theorem 1 there exists a sequence $\{\mu_n\}$, $\mu_n \in I$ such that equation (1) with $\mu = \mu_n$ has a solution y_n satisfying

$$y_n(t_1) = y_n(x_n) = y_n(t_3) = 0$$

and

$$\big|y_n^{(i)}(t)\big| \leqslant r_{i+1} \quad \text{for} \quad t \in J; \quad i = 0, 1; \quad n \in \mathbb{N}.$$

Since $\{\mu_n\}$ is a bounded sequence we may assume, without loss of generality, that $\{\mu_n\}$ is convergent, $\lim_{n\to\infty} \mu_n = \mu_0$. From the equalities $y_n''(t) = q(t)y_n(t) + f\big(t, y_n(t), y_n'(t), \mu_n\big)$ we obtain

$$|y_n''(t)| \leqslant r_1 \max_{t\in J} q(t) + A \quad \text{for} \quad t \in J \quad \text{and} \quad n \in \mathbb{N}.$$

Let $\xi_n \in (t_1, x_n)$ be such numbers that $y_n'(\xi_n) = 0$, $n \in \mathbb{N}$. Using Ascoli's theorem we may choose a subsequence $\{y_{k_n}(t)\}$ of $\{y_n(t)\}$ such that $\{y_{k_n}^{(j)}(t)\}$ are uniformly convergent on J for $j = 0, 1, 2$. Then $y(t) := \lim_{n\to\infty} y_{k_n}(t)$, $t \in J$, is a solution of (1) with $\mu = \mu_0$, $y(t_1) = y(t_3) = 0$, $|y^{(i)}(t)| \leqslant r_{i+1}$ for $t \in J$ and $i = 0, 1$. Since $y'(t) = \lim_{n\to\infty} y_{k_n}'(t)$ uniformly on J, $y_{k_n}'(\xi_{k_n}) = 0$ and $\lim_{n\to\infty} \xi_{k_n} = t_1$, we have $y'(t_1) = 0$. □

R e m a r k 4. If assumptions of Corollary 2 are satisfied, then it is obvious from the proof of Corollary 2 that there exists μ_0, $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfying (16) and

$$y(t_1) = y(t_3) = y'(t_3) = 0.$$

For BVP (17), (2) we have the following results:

Theorem 2. *Let assumptions (18)–(20) be satisfied with a positive constant r. Then there exists μ_0, $\mu_0 \in I$ such that BVP (17), (2) with $\mu = \mu_0$ has a solution y and*

$$|y(t)| \leqslant r, \quad |y'(t)| \leqslant \big(A_1 + r \max_{t\in J} q(t)\big)(t_3 - t_1) \quad \textit{for} \quad t \in J,$$

where $A_1 := \max_{(t,y,\mu)\in H\times I} |f_1(t, y, \mu)|$.

P r o o f. Let $B_1 =: A_1 + r \max_{t\in J} q(t)$, let X be the Banach space defined in the same way as in the proof of Theorem 1 and $\mathcal{K}_1 := \{y;\ y \in X, |y(t)| \leqslant r, |y'(t)| \leqslant B_1(t_3 - t_1)$ for $t \in J\}$. $\mathcal{K}_1$ is a closed bounded convex subset of X, $\mathcal{K}_1 \subset X$. For every φ, $\varphi \in \mathcal{K}_1$ there exists (by Lemma 5) a unique μ_0, $\mu_0 \in I$ such that equation (21) with $\mu = \mu_0$ has a unique solution y satisfying (2) and $y \in \mathcal{K}_1$ (by Remark 3). Setting $T(\varphi) = y$ we obtain an operator T, $T\colon \mathcal{K}_1 \to \mathcal{K}_1$. The next part of the proof is very similar to that of Theorem 1 and therefore is omitted. □

E x a m p l e 2. Let n be a positive integer, let ν, c be constants, $\nu \geqslant 0$, $c > 2$ and $t_2 \in (0, 1)$. Consider BVP

$$\begin{aligned} &y'' - q(t)y = t^\nu y^n + \varphi(t) + \mu, \\ &y(0) = y(t_2) = y(1) = 0, \end{aligned} \tag{27}$$

where q, $\varphi \in C^0(J_2)$, $q(t) \geqslant c$, $0 < \varphi(t) \leqslant c - 2$ for $t \in \langle 0, 1\rangle =: J_2$ and $\mu \in \langle 1 - c, 1\rangle$. The assumptions of Theorem 2 are satisfied with $r = 1$. By Theorem 2 there exists μ_0, $\mu_0 \in \langle 1 - c, 1\rangle$ such that BVP (27) with $\mu = \mu_0$ has a solution y and $|y(t)| \leqslant 1$, $|y'(t)| \leqslant 2c - 2 + \max_{t\in J_2} q(t)$ for $t \in J_2$.

Corollary 3. *Assume that assumptions* (19) *and* (20) *are satisfied with a positive constant* r. *Then there is* $\delta > 0$ *such that for every* ε, $0 < \varepsilon \leqslant \delta$ *there exists* μ_ε, $\mu_\varepsilon \in I$ *such that BVP*

$$y'' - q(t)y = \varepsilon f_1(t, y, \mu),$$
$$y(t_1) = y(t_2) = y(t_3) = 0,$$

with $\mu = \mu_\varepsilon$ *has a solution* y *satisfying*

$$|y(t)| \leqslant r, \quad |y'(t)| \leqslant \big(A_1 + r \max_{t\in J} q(t)\big)(t_3 - t_1) \quad \text{for} \quad t \in J,$$

where A_1 *is defined in Theorem* 2.

Proof. Let $\delta := \frac{r}{A_1} \min_{t\in J} q(t)$. If $0 < \varepsilon \leqslant \delta$ then εf satisfies the same assumptions as f in Theorem 2. Therefore Corollary 3 follows immediately from Theorem 2. □

Corollary 4. *Suppose that assumptions* (18)–(20) *are satisfied for a positive constant* r. *Then there exists* μ_0, $\mu_0 \in I$ *such that equation* (17) *with* $\mu = \mu_0$ *has a solution* y *satisfying* (3) *and*

$$|y(t)| \leqslant r, \quad |y'(t)| \leqslant \big(A_1 + r \max_{t\in J} q(t)\big)(t_3 - t_1) \quad \text{for} \quad t \in J,$$

where A_1 *is defined in Theorem* 2.

Proof. Let $\{x_n\}$, $x_n \in (t_1, t_3)$ be a convergent sequence, $\lim_{n\to\infty} x_n = t_1$. Then (by Theorem 2) there exists a sequence $\{\mu_n\}$, $\mu_n \in I$, such that equation (17) with $\mu = \mu_n$ has a solution y_n satisfying

$$y_n(t_1) = y_n(x_n) = y(t_3) = 0,$$
$$|y_n(t)| \leqslant r, \quad |y_n'(t)| \leqslant \big(A_1 + r \max_{t\in J} q(t)\big)(t_3 - t_1)$$

and

$$|y_n''(t)| = \big|q(t)y_n(t) + f_1\big(t, y_n(t), \mu_n\big)\big| \leqslant A_1 + r \max_{t\in J} q(t) \quad \text{for} \quad t \in J, \quad n \in \mathbb{N}.$$

Since $\{\mu_n\}$ is a bounded sequence we may assume, without loss of generality, that $\{\mu_n\}$ is convergent and $\lim_{n\to\infty} \mu_n = \mu_0$. Let $\xi_n \in (t_1, x_n)$ be such numbers for which $y_n'(\xi_n) = 0$. Then by Ascoli's theorem we may choose a subsequence $\{y_{k_n}(t)\}$ of $\{y_n(t)\}$ such that $\{y_{k_n}^{(j)}(t)\}$ are uniformly convergent on J for $j = 0, 1, 2$. The function $y(t) := \lim_{n\to\infty} y_{k_n}(t)$, $t \in J$, is a solution of (17) with $\mu = \mu_0$, $y(t_1) = y(t_3) = 0$, $|y(t)| \leqslant r$, $|y'(t)| \leqslant \big(A_1 + r \max_{t\in J} q(t)\big)(t_3 - t_1)$ for $t \in J$. Since $y'(t) = \lim_{n\to\infty} y_{k_n}'(t)$ uniformly on J, $y_{k_n}'(\xi_{k_n}) = 0$ and $\lim_{n\to\infty} \xi_n = t_1$, we have $y'(t_1) = 0$. □

R e m a r k 5. If the assumptions of Corollary 4 are satisfied then we can prove the existence of μ_0, $\mu_0 \in I$ such that equation (17) with $\mu = \mu_0$ has a solution y satisfying

$$y(t_1) = y(t_3) = y'(t_3) = 0$$

and

$$|y(t)| \leqslant r, \quad |y'(t)| \leqslant \big(A_1 + r \max_{t \in J} q(t)\big)(t_3 - t_1) \quad \text{for} \quad t \in J.$$

4. Uniqueness theorem

Lemma 6. *Let r_1, r_2 be positive constants, $S = \{y;\ y \in C^1(J), \big|y^{(i)}(t)\big| \leqslant r_{i+1}$ for $t \in J, i = 0, 1\}$. Assume*

$$\begin{aligned}(28)\qquad &|f(t, y_1, y_2, \mu) - f(t, z_1, z_2, \mu)| \leqslant h_1(t)|y_1 - z_1| + h_2(t)|y_2 - z_2| \\ &\textit{for} \quad (t, y_1, y_2, \mu), (t, z_1, z_2, \mu) \in J \times \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I,\end{aligned}$$

where $h_1, h_2 \in C^0(J)$ and at least one of the following four assumptions holds:

$$(29)\qquad \int_{t_1}^{t_2} \Big[\Big(\exp \int_{t_1}^{s} h_2(\tau)\mathrm{d}\tau\Big) \cdot \int_{t_1}^{s} \big(q(\tau) + h_1(\tau)\big)\mathrm{d}\tau\Big]\mathrm{d}s \leqslant 1,$$

$$(30)\qquad \int_{t_1}^{t_2} \big[\big(q(s) + h_1(s)\big)(s - t_1) + h_2(s)\big]\mathrm{d}s \leqslant 1,$$

$$(31)\qquad \int_{t_2}^{t_3} \Big[\Big(\exp \int_{t_2}^{s} h_2(\tau)\mathrm{d}\tau\Big) \cdot \int_{t_2}^{s} \big(q(\tau) + h_1(\tau)\big)\mathrm{d}\tau\Big]\mathrm{d}s \leqslant 1,$$

$$(32)\qquad \int_{t_2}^{t_3} \big[\big(q(s) + h_1(s)\big)(s - t_2) + h_2(s)\big]\mathrm{d}s \leqslant 1.$$

If BVP (1), (2) with $\mu = \mu_0$, $\mu_0 \in I$ has a solution y, $y \in S$, then this solution is unique in S.

P r o o f. Let $y_1, y_2 \in S$ be solutions of BVP (1), (2) with $\mu = \mu_0$, $\mu_0 \in I$ and define $w := y_1 - y_2$. Since $w(t_1) = w(t_2) = 0$ there exists a $\xi \in (t_1, t_2)$: $|w(t)| \leqslant |w(\xi)|$ for $t \in \langle t_1, t_2\rangle$.

Let assumptions (29) be satisfied. Using Gronwall's lemma for the inequality

$$(33)\qquad |w'(t)| \leqslant \Big|\int_{\xi}^{t} \big[\big(q(s) + h_1(s)\big)|w(s)| + h_2(s)|w'(s)|\big]\mathrm{d}s\Big|, \quad t \in \langle t_1, t_2\rangle$$

we get

$$|w'(t)| \leqslant \Big(\exp \int_{\xi}^{t} h_2(s)\mathrm{d}s\Big) \cdot \int_{\xi}^{t} \big(q(s) + h_1(s)\big)|w(s)|\mathrm{d}s, \quad t \in \langle \xi, t_2\rangle\,.$$

For all $t \in \langle \xi, t_2 \rangle$ we have

$$|w(t)-w(\xi)| \leqslant \int_\xi^t |w'(s)|\mathrm{d}s \leqslant \int_\xi^t \Big[\Big(\exp\int_\xi^s h_2(\tau)\mathrm{d}\tau\Big)\cdot\int_\xi^s (q(\tau)+h_1(\tau))|w(\tau)|\mathrm{d}\tau\Big]\mathrm{d}s$$

and thus, if $w(\xi) \neq 0$, we obtain

$$\begin{aligned}|w(\xi)| = |w(t_2)-w(\xi)| &\leqslant \int_\xi^{t_2} \Big[\Big(\exp\int_\xi^s h_2(\tau)\mathrm{d}\tau\Big)\cdot\int_\xi^s (q(\tau)+h_1(\tau))|w(\tau)|\mathrm{d}\tau\Big]\mathrm{d}s \\ &< |w(\xi)|\int_{t_1}^{t_2} \Big[\Big(\exp\int_{t_1}^s h_2(\tau)\mathrm{d}\tau\Big)\cdot\int_{t_1}^s (q(\tau)+h_1(\tau))\mathrm{d}\tau\Big]\mathrm{d}s.\end{aligned}$$

Then

$$1 < \int_{t_1}^{t_2} \Big[\Big(\exp\int_{t_1}^s h_2(\tau)\mathrm{d}\tau\Big)\cdot\int_{t_1}^s (q(\tau)+h_1(\tau))\mathrm{d}\tau\Big]\mathrm{d}s,$$

which contradicts (29). Therefore $w(\xi)=0$ and $y_1(t)=y_2(t)$ for $t \in \langle t_1, t_2 \rangle$.

Now, let assumptions (30) be satisfied. From (33) and $|w(t)| \leqslant \int_{t_1}^t |w'(s)|\mathrm{d}s$ for $t \in J$ we get

$$\begin{aligned}|w'(t)| &\leqslant \Big|\int_\xi^t \Big[(g(s)+h_1(s))\cdot\int_{t_1}^s |w'(\tau)|\mathrm{d}\tau + h_2(s)|w'(s)|\Big]\mathrm{d}s\Big| \\ &\leqslant \int_{t_1}^{t_2} \Big[(q(s)+h_1(s))\cdot\int_{t_1}^s |w'(\tau)|\mathrm{d}\tau + h_2(s)|w'(s)|\Big]\mathrm{d}s, \quad t \in \langle t_1, t_2 \rangle.\end{aligned}$$

Putting $X(t) := \max_{t_1 \leqslant s \leqslant t} |w'(s)|$ for $t \in \langle t_1, t_2 \rangle$ then, if $X(t_2) \neq 0$, we obtain

$$|w'(t)| < X(t_2)\int_{t_1}^{t_2} \big[(q(s)+h_1(s))(s-t_1)+h_2(s)\big]\mathrm{d}s, \quad t \in \langle t_1, t_2 \rangle$$

and thus

$$X(t_2)\Big(1-\int_{t_1}^{t_2} \big[(q(s)+h_1(s))(s-t_1)+h_2(s)\big]\mathrm{d}s\Big) < 0,$$

which contradicts (30). This shows that $X(t_2)=0$, consequently $w'(t)=0$ for $t \in \langle t_1, t_2 \rangle$ and since $w(t_1)=0$ we get $w(t)=0$ on $\langle t_1, t_2 \rangle$, that is $y_1(t)=y_2(t)$ for $t \in \langle t_1, t_2 \rangle$.

By the existence and uniqueness theorem for equation (1) we get $y_1(t)=y_2(t)$ for $t \in J$.

If assumptions (31) (or (32)) is satisfied, then the proof is very similar and therefore is omitted. □

R e m a r k 6. It is evident from the proof of Lemma 6 that assumptions (29)–(32) may be replaced by the assumptions

$$\int_{t_1}^{t_2}\Big[\Big(\exp\int_s^{t_2} h_2(\tau)\,\mathrm{d}\tau\Big)\cdot\int_s^{t_2}\big(q(\tau)+h_1(\tau)\big)\,\mathrm{d}\tau\Big]\,\mathrm{d}s\leqslant 1,$$
$$\int_{t_1}^{t_2}\big[\big(q(s)+h_1(s)\big)(t_2-s)+h_1(s)\big]\,\mathrm{d}s\leqslant 1,$$
$$\int_{t_2}^{t_3}\Big[\Big(\exp\int_s^{t_3} h_2(\tau)\,\mathrm{d}\tau\Big)\cdot\int_s^{t_3}\big(q(\tau)+h_1(\tau)\big)\,\mathrm{d}\tau\Big]\,\mathrm{d}s\leqslant 1,$$
$$\int_{t_2}^{t_3}\big[\big(q(s)+h_1(s)\big)(t_3-s)+h_2(s)\big]\,\mathrm{d}s\leqslant 1.$$

E x a m p l e 3. Consider BVP (26) as in Example 1, where $n=3$. Assumption (28) is satisfied for $h_1(t)=t^3$ and $h_2(t)=3t$ with an arbitrary positive constant r_1 and $r_2=1$. If BVP (26) with $\mu=\mu_0$ $\big(\in\langle-\frac{4}{9},\frac{4}{9}\rangle\big)$ has a solution y satisfying $|y'(t)|\leqslant 1$ on $\langle 0,\frac{1}{3}\rangle$ (by Example 1 such μ_0 and y exist if $r_1\geqslant\frac{\pi}{2}$), then this solution y is unique in the set $\big\{y;\ y\in C^2(\langle 0,\frac{1}{3}\rangle),\,|y'(t)|\leqslant 1 \text{ for } t\in\langle 0,\frac{1}{3}\rangle\big\}$ since

$$\int_{t_1}^{t_3}\big[\big(q(s)+h_1(s)\big)(s-t_1)+h_2(s)\big]\,\mathrm{d}s=\int_0^{\frac{1}{3}}\big[\big(q(s)+s^3\big)s+3s\big]\,\mathrm{d}s$$
$$\leqslant\frac{5}{3\pi}\Big(\frac{1}{3}\Big)^2+\frac{1}{5}\Big(\frac{1}{3}\Big)^5+\frac{3}{2}\Big(\frac{1}{3}\Big)^2<1.$$

Lemma 6 and its proof immediately yield

Corollary 5. *Let $r>0$ be a positive constant, $S_1=\{y;\ y\in C^0(J),\,|y(t)|\leqslant r \text{ for } t\in J\}$. Assume*

$$(34)\quad |f_1(t,y,\mu)-f_1(t,z,\mu)|\leqslant h(t)|y-z|\quad\text{for}\quad (t,y,\mu),(t,z,\mu)\in J\times\langle -r,r\rangle\times I,$$

where $h\in C^0(J)$ and at least one from the following four assumptions holds:

$$(35)\qquad \int_{t_1}^{t_2}\int_{t_1}^{s}\big(q(\tau)+h(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s\leqslant 1,\quad \int_{t_1}^{t_2}\big(q(s)+h(s)\big)(s-t_1)\,\mathrm{d}s\leqslant 1,$$
$$\int_{t_2}^{t_3}\int_{t_2}^{s}\big(q(\tau)+h(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s\leqslant 1,\quad \int_{t_2}^{t_3}\big(q(s)+h(s)\big)(s-t_2)\,\mathrm{d}s\leqslant 1.$$

If BVP (17), (2) with $\mu=\mu_0$ ($\in I$) has a solution y, $y\in S_1$, then this solution y is unique in S_1.

R e m a r k 7. Assumptions (34) in Corollary 5 may be replaced by the assumptions

$$\int_{t_1}^{t_2}\int_{s}^{t_2}(q(\tau)+h(\tau))\mathrm{d}\tau\mathrm{d}s\leqslant 1,\qquad \int_{t_1}^{t_2}(q(s)+h(s))(t_2-s)\mathrm{d}s\leqslant 1,$$

$$\int_{t_2}^{t_3}\int_{s}^{t_3}(q(\tau)+h(\tau))\mathrm{d}\tau\mathrm{d}s\leqslant 1,\qquad \int_{t_2}^{t_3}(q(s)+h(s))(t_3-s)\mathrm{d}s\leqslant 1.$$

E x a m p l e 4. Consider BVP (27) as in Example 2. Assumption (34) holds for $h(t)=n$ and $r=1$. If BVP (27) has for $\mu=\mu_0$ ($\in\langle 1-c,1\rangle$) a solution y, $y\in S_2:=\{y;\ y\in C^2(\langle 0,1\rangle),\ |y(t)|\leqslant 1 \text{ for } t\in\langle 0,1\rangle\}$ (by Example 2 such μ_0 and y exist) and $t_2\in(0,1)$ satisfies at least one from the conditions

$$\int_0^{t_2}\int_0^{s}q(\tau)\mathrm{d}\tau+\frac{n}{2}t_2^2\leqslant 1,\qquad \int_0^{t_2}sq(s)\mathrm{d}s+\frac{n}{2}t_2^2\leqslant 1,$$

$$\int_{t_2}^{1}\int_{t_2}^{s}q(\tau)\mathrm{d}\tau\mathrm{d}s+\frac{n}{2}(1-t_2)^2\leqslant 1,\qquad \int_{t_2}^{1}q(s)(s-t_2)\mathrm{d}s+\frac{n}{2}(1-t_2)^2\leqslant 1,$$

$$\int_0^{t_2}\int_s^{t_2}q(\tau)\mathrm{d}\tau\mathrm{d}s+\frac{n}{2}t_2^2\leqslant 1,\qquad \int_0^{t_2}q(s)(t_2-s)\mathrm{d}s+\frac{n}{2}t_2^2\leqslant 1,$$

$$\int_{t_2}^{1}\int_s^{1}q(\tau)\mathrm{d}\tau\mathrm{d}s+\frac{n}{2}(1-t_2)^2\leqslant 1,\qquad \int_{t_2}^{1}q(s)(1-s)\mathrm{d}s+\frac{n}{2}(1-t_2)^2\leqslant 1,$$

then this solution y is unique in S_2.

Lemma 7. *Let assumption* (12) *be satisfied for positive constants* r_1, r_2. *Let* $\frac{\partial f}{\partial y_1}(t,y_1,y_2,\mu)$, $\frac{\partial f}{\partial y_2}(t,y_1,y_2,\mu)\in C^0(D_2)$ *and*

$$q(t)+\frac{\partial f}{\partial y_1}(t,y_1,y_2,\mu)\geqslant 0\quad \text{for}\quad (t,y_1,y_2,\mu)\in D_2,\tag{36}$$

where $D_2=D\times I$. *Define* $S:=\{y;\ y\in C^1(J),\ |y^{(i)}(t)|\leqslant r_{i+1}$ *for* $t\in J$ *and* $i=0,1\}$.

If BVP (1), (2) *with* $\mu=\mu_0$, $\mu_0\in I$ *has a solution* y, $y\in S$, *then* μ_0 *and* y *are unique.*

P r o o f. Let y_1 and y_2 be solutions of BVP (1), (2) with $\mu=\mu_1$ and $\mu=\mu_2$, respectively, $y_1,y_2\in S$, $\mu_1,\mu_2\in I$, $\mu_1\leqslant\mu_2$. Using Taylor's formula we get

$$\begin{aligned}
f(t,y_1(t),y_1'(t),\mu_1)-f(t,y_2(t),y_2'(t),\mu_2)&=\\
&=\big(f(t,y_1(t),y_1'(t),\mu_1)-f(t,y_1(t),y_1'(t),\mu_2)\big)\\
&\quad+\big(f(t,y_1(t),y_1'(t),\mu_2)-f(t,y_2(t),y_1'(t),\mu_2)\big)\\
&\quad+\big(f(t,y_2(t),y_1'(t),\mu_2)-f(t,y_2(t),y_2'(t),\mu_2)\big)\\
&=\big(f(t,y_1(t),y_1'(t),\mu_1)-f(t,y_1(t),y_1'(t),\mu_2)\big)\\
&\quad+g(t)(y_1(t)-y_2(t))+h(t)(y_1(t)-y_2(t))'
\end{aligned}$$

with $g, h \in C^0(J)$ and $q(t) + g(t) \geqslant 0$ on J by (36). Setting $w := y_1 - y_2$ then if $\mu_1 < \mu_2$, we have

$$w''(t) < \big(q(t) + g(t)\big)w(t) + h(t)w'(t) \quad \text{for} \quad t \in J \tag{37}$$

by (12) and if $\mu_1 = \mu_2$, we have

$$w''(t) = \big(q(t) + g(t)\big)w(t) + h(t)w'(t) \quad \text{for} \quad t \in J \tag{38}$$

Let $\mu_1 < \mu_2$. If $w'(t_1) \leqslant 0$ then using (37) and Tschaplygin's lemma (see e.g. [1], p. 195) we get $w(t) < 0$ on $(t_1, t_3\rangle$, which contradicts $w(t_2) = w(t_3) = 0$. If $w'(t_1) > 0$ then there exists η, $\eta \in (t_1, t_2\rangle$ such that $w(t) > 0$ for $t \in (t_1, \eta)$, $w(\eta) = 0$ and $w'(\eta) \leqslant 0$. Therefore $w(t) < 0$ on $(\eta, t_3\rangle$, which is a contradiction with $w(t_3) = 0$.

Let $\mu_1 = \mu_2$. Since $q(t)+g(t) \geqslant 0$ for $t \in J$, the equation $y'' = \big(q(t)+g(t)\big)y+h(t)y'$ is disconjugate on J, consequently in virtue of $w(t_1) = w(t_2) = w(t_3) = 0$ we have $w = 0$ and $y_1 = y_2$. This completes the proof. □

Lemma 8. *Let assumptions* (19) *be satisfied for a positive constant* r. *Let* $\frac{\partial f_1}{\partial y}(t, y, \mu) \in C^0(H_1)$ *and*

$$q(t) + \frac{\partial f_1}{\partial y}(t, y, \mu) \geqslant 0 \quad \textit{for} \quad (t, y, \mu) \in H_1, \tag{39}$$

where $H_1 = H \times I$. *Define* $S_1 = \{y;\ y \in C^0(J),\ |y(t)| \leqslant r \text{ for } t \in J\}$.

If BVP (17), (2) *with* $\mu = \mu_0$, $\mu_0 \in I$ *has a solution* y, $y \in S_1$, *then* μ_0 *and* y *are unique.*

The proof is entirely analogous to the proof of Lemma 7.

Theorem 3. *Suppose that assumptions* (11)–(14) *are satisfied for positive constants* r_1, r_2. *Let* $\frac{\partial f}{\partial y_1}(t, y_1, y_2, \mu)$, $\frac{\partial f}{\partial y_2}(t, y_1, y_2, \mu)$ *be continuous on* $D \times I$ *and let* S *be defined as in Lemma* 7.

If (36) *holds then BVP* (1), (2) *has a solution* y, $y \in S$ *for a single value of the parameter* μ $(\in I)$. *Moreover, this solution* y *is unique in the set* S.

The proof follows immediately from Theorem 1 and Lemma 7.

Example 5. Consider BVP (26) as in Example 1. Since $q(t)+\frac{\partial f}{\partial y_1}(t, y_1, y_2, \mu) = q(t)-t^3 \sin y_1 \geqslant \frac{8}{3\pi} - \left(\frac{1}{3}\right)^3 > 0$ for $(t, y_1, y_2, \mu) \in \langle 0, \frac{1}{3}\rangle \times \mathbf{R}^2 \times \langle -\frac{4}{9}, \frac{4}{9}\rangle$ then Example 1 and Theorem 3 imply, that BVP (26) has a solution y for a single value of the parameter μ $(\in \langle -\frac{4}{9}, \frac{4}{9}\rangle)$. This solution is unique in the set $\{y;\ y \in C^2(\langle 0, \frac{1}{3}\rangle),\ |y(t)| \leqslant \frac{\pi}{2},\ |y'(t)| \leqslant 1 \text{ for } t \in \langle 0, \frac{1}{3}\rangle\}$.

Theorem 4. *Suppose that assumptions* (18)–(20) *are satisfied for a positive constant r. Let* $\frac{\partial f_1}{\partial y} \in C^0(H \times I)$ *and let* S_1 *be defined as in Lemma* 8.

If (39) *holds then BVP* (17), (2) *has a solution* y, $y \in S_1$ *for a single value of the parameter* μ $(\in I)$. *Moreover, this solution* y *is unique in the set* S_1.

The proof follows immediately from Theorem 2 and Lemma 8.

Example 6. Consider BVP (27) as in Example 2 with the additional assumption that n is an odd integer. Then $q(t) + \frac{\partial f_1}{\partial y}(t, y, \mu) = q(t) + nt^\nu y^{n-1} > 0$ for $(t, y, \mu) \in \langle 0, 1\rangle \times \langle -1, 1\rangle \times \langle 1 - c, 1\rangle$. Example 2 and Theorem 4 imply that BVP (27) has a solution y for a single value of parameter μ $(\in \langle 1 - c, 1\rangle)$. This solution y is unique in the set $\{y;\ y \in C^2(\langle 0, 1\rangle),\ |y(t)| \leqslant 1 \text{ for } t \in \langle 0, 1\rangle\}$.

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