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ON CONVEXITIES OF LATTICES

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At the problem Session of the Conference on General Algebra (Krems, 1988) E. Fried proposed a problem concerning the "number" of convexities of lattices (cf. [7], p. 255). In the present paper a solution of this problem is given.

Convex sublattices of lattices were investigated by M. Kolibiar [6]. Systems of convex subsets of partially ordered sets were study by M. K. Bennet and G. Birkhoff (cf. [1], [2], [3], [5]).

1.

A nonempty class of lattices will be said to be a convexity if it is closed under homomorphic images, convex sublattices and direct products.

This notion was introduced by Fried [7]. He proposed the following question: What is the "number" of convexities?

Next, he expressed the conjecture that there is no such cardinal. The validity of this conjecture will be proved below.

Let us denote by \( \mathcal{C} \) the collection of all convexities. This collection will be considered to be partially ordered by inclusion.

For a subclass \( X \) of the class \( \mathcal{L} \) of all lattices we denote by

\( H.X \) – the class of all homomorphic images of elements of \( X \),

\( C.X \) – the class of all convex sublattices of elements of \( X \),

\( P.X \) – the class of all direct products of elements of \( X \).

**Lemma 1.1.** (Cf. Fried [7].) *Let \( \emptyset \neq X \subseteq \mathcal{L} \). Then \( HCPX \) is the least convexity containing \( X \).*

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In view of 1.1 the convexity $H C P X$ will be said to be generated by $X$. If $X$ is a one-element class, then $H C P X$ will be called principal. A convexity is said to be modular if all lattices belonging to it are modular. We denote by $\mathcal{C}_{pm}$ the collection of all convexities which are principal and modular.

If $\Omega$ is a congruence relation on a lattice $L$ and $x \in L$, then we denote by $L/\Omega$ the corresponding factor lattice and by $x(\Omega)$ the element of $L/\Omega$ which contains $x$.

For a lattice $L$ we denote by $P(L)$ the set of all pairs $(u,v)$ of elements of $L$ having the property that there exist distinct mutually incomparable elements $x_i \in L$ ($i = 1, 2, 3$) such that $u < x_i < v$ for $i = 1, 2, 3$ and $\{u, v, x_1, x_2, x_3\}$ is a sublattice of $L$.

**Lemma 1.2.** Let $L$ be a modular lattice and let $\Omega$ be a congruence relation on $L$ such that $u(\Omega) = v(\Omega)$ whenever $(u,v) \in P(L)$. Then the lattice $L/\Omega$ is distributive.

**Proof.** By way of contradiction, assume that the lattice $L/\Omega$ fails to be distributive. Then there are elements $x_1, x_2$ and $x_3$ in $L$ such that

(i) $x_1(\Omega), x_2(\Omega)$ and $x_3(\Omega)$ are distinct,

(ii) $x_1(\Omega) \land x_2(\Omega) = x_1(\Omega) \land x_3(\Omega) = x_2(\Omega) \land x_3(\Omega),$

(iii) $x_1(\Omega) \lor x_2(\Omega) = x_1(\Omega) \lor x_3(\Omega) = x_2(\Omega) \lor x_3(\Omega)$.

Let $L_1$ be the sublattice of $L$ generated by the elements $x_1, x_2$ and $x_3$. Further, let $\Omega_1$ be the congruence relation on $L_1$ which is induced by $\Omega$. Let $L^{(3)}$ be the free modular lattice generated by the free generators $x_1^0, x_2^0$ and $x_3^0$. There exists a homomorphism $\varphi$ of $L^{(3)}$ onto $L_1$ such that $\varphi(x_i^0) = x_i$ for $i = 1, 2, 3$. Hence there exists a congruence relation $\Omega_2$ on $L^{(3)}$ having the property that there is an isomorphism $\psi$ of $L^{(3)}/\Omega_2$ onto $L_1/\Omega_1$ such that $\psi(x_i^0(\Omega_2)) = x_i(\Omega_1)$ for $i = 1, 2, 3$.

The conditions (i), (ii) and (iii) remain valid if $\Omega$ is replaced by $\Omega_1$. Hence the same relations remain true if $\Omega$ and $x_i$ are replaced by $\Omega_2$ and $x_i^0$ ($i = 1, 2, 3$); these modified conditions will also be denoted by (i), (ii) and (iii), respectively.

For the elements of $L^{(3)}$ we apply the same notation as in [4], Chap. III, §6 (with the distinction that we now have $x_1^0, x_2^0$ and $x_3^0$ instead of $x, y$ and $z$).

In view of (ii) the relation

(1) \[ O(\Omega_2) = o(\Omega_2) \]

is valid. Analogously, (iii) implies that

(2) \[ i(\Omega_2) = I(\Omega_2) \]

holds.
Next, \((o, i) \in P(L^{(3)})\). If \(\varphi(o) \neq \varphi(i)\), then \((\varphi(o), \varphi(i)) \in P(L_1)\) and hence \(\varphi(o)(\Omega_1) = \varphi(i)(\Omega_1)\). Therefore

\[(3)\quad o(\Omega_2) = i(\Omega_2).\]

The relations (1), (2) and (3) give \(O(\Omega_2) = I(\Omega_2)\), whence \(x^o_1(\Omega_2) = x^o_2(\Omega_2) = x^o_3(\Omega_2)\). In view of the isomorphism \(\psi\) we obtain \(x_1(\Omega_1) = x_2(\Omega_1) = x_3(\Omega_1)\), which contradicts the relation (i). \(\square\)

Let \(\alpha\) be a cardinal, \(\alpha \geq \beta\). We denote by \(L_\alpha\) the lattice consisting of elements \(u, v, x_j\ (j \in J)\), where \(\text{card } J = \alpha\), \(u < x_j < v\), and \(x_j(1)\) is incomparable with \(x_j(2)\) whenever \(j(1)\) and \(j(2)\) are distinct elements of \(J\).

**Lemma 1.3.** Let \(\alpha\) and \(\beta\) be cardinals, \(3 \leq \alpha < \beta\). Then \(L_\alpha\) does not belong to \(HCP\{L_\beta\}\).

**Proof.** By way of contradiction, assume that \(L_\alpha\) belongs to \(HCP\{L_\beta\}\). Thus

(i) there exist lattices \(A_i = \{u_i, v_i, x^i_k\}_{k \in K}\ (i \in I)\) where \(\text{card } K = \beta\), \(u_i < x^i_k < v_i\) for each \(k \in K\), and each \(A_i\) is isomorphic to \(L_\beta\);

(ii) there exist a convex sublattice \(B\) of \(\prod_{i \in I} A_i\) and a congruence relation \(\Omega\) on \(B\) such that \(L_\alpha\) is isomorphic to \(B/\Omega\).

Let \(\varphi\) be an isomorphic of \(L_\alpha\) onto \(B/\Omega\). For each \(t \in L_\alpha\) we denote \(\varphi(t) = t^*\).

The elements of \(L_\alpha\) will be denoted as above.

Choose any \(u' \in u^*\). There exists \(v' \in v^*\) such that \(u' < v'\). Let \(L\) be the interval \([u', v']\) of \(B\) and let \(\Omega_1\) be the congruence relation on \(L\) which is induced by \(\Omega\). Then \(L_\alpha\) is isomorphic to \(L/\Omega_1\). Thus without loss of generality we can assume that \(B = [u', v']\).

For \(z \in \prod_{i \in I} A_i\) and \(i \in I\) let \(z(A_i)\) be the component of \(z\) in the direct factor \(A_i\). Similarly, for \(Z \subseteq \prod_{i \in I} A_i\) we denote \(Z(A_i) = \{z(A_i): z \in Z\}\). Since \(B\) is an interval of \(\prod_{i \in I} A_i\), we get

\[B = \prod_{i \in I} B(A_i).\]

If \(P(B) = \emptyset\), then (since \(B\) is modular) we obtain that \(B\) is distributive, hence \(B/\Omega\) is distributive as well, which is a contradiction. Thus \(P(B) \neq \emptyset\). Hence we can choose \((a, b) \in P(B)\).

We have \([a, b] = \prod_{i \in I} [a(A_i), b(A_i)]\). Put \(I(1) = \{i \in I: \text{card}[a(A_i), b(A_i)] \geq 2\}\). If \(I(1) = \emptyset\), then the interval \([a, b]\) is distributive, which is a contradiction. Hence \(I(1) \neq \emptyset\). Next, since \(A_i\) is isomorphic to \(L_\beta\), for each \(i \in I(1)\) the relations \(a(A_i) = \dots\)

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and \( b(A_i) = v_i \) are valid. We construct elements \( \bar{u}, \bar{v}, \tilde{x}_k \) \((k \in K)\) of \( \prod_{i \in I} A_i \) as follows:

(i) for \( i \in I \setminus I(1) \) we put

\[
\bar{u}(A_i) = \bar{v}(A_i) = \tilde{x}_k(A_i) = a(A_i);
\]

(ii) for \( i \in I(1) \) we put

\[
\bar{u}(A_i) = a(A_i), \quad \bar{v}(A_i) = b(A_i), \quad \tilde{x}_k(A_i) = x^i_k.
\]

Then \( \{\bar{u}, \bar{v}, \tilde{x}_k\}_{k \in K} = \mathbb{Z} \) is a sublattice of \( B \) which is isomorphic to \( L_\beta \).

First, suppose that \( a(\Omega) \neq b(\Omega) \). Then (since the lattice \( Z \) has no proper congruence relations) the elements \( \bar{u}(\Omega), \bar{v}(\Omega), \tilde{x}_k(\Omega) \) \((k \in K)\) are pairwise distinct. Hence \( Z' = \{\bar{u}(\Omega), \bar{v}(\Omega), \tilde{x}_k(\Omega)\}_{k \in K} \) is a sublattice of \( B/\Omega \) which is isomorphic to \( L_\beta \); but this is impossible, since \( B/\Omega \) is isomorphic to \( L_\alpha \).

Therefore \( a(\Omega) = b(\Omega) \). Hence according to 1.2 the lattice \( B/\Omega \) is distributive. We conclude that \( B/\Omega \) cannot be isomorphic to \( L_\alpha \), completing the proof. \( \square \)

**Theorem 1.4.** There exists an injective mapping of the class of all cardinals \( \alpha \) with \( \alpha \geq 3 \) into the class \( C_{pm} \).

**Proof.** For each cardinal \( \alpha \) with \( \alpha \geq 3 \) we put \( f(\alpha) = HCP\{L_\alpha\} \). It is obvious that \( f(\alpha) \in C_{pm} \). According to 1.3, the mapping \( f \) is injective. \( \square \)

Hence we have verified the validity of the conjecture expressed by E. Fried in [7].

2.

Now we shall establish some additional result on convexities of lattices; we shall also propose two open questions.

Though the collection \( C \) fails to be a set we can apply to it the usual notions concerning the partial order.

The least element of \( C \) is the class \( X_0 \) consisting of all one-element lattices; the greatest element of \( C \) is the class \( L \). If \( \{X_i\}_{i \in I} \) is a nonempty subcollection of \( C \), then \( \bigcap_{i \in I} X_i \) is the greatest element of \( C \) contained in all \( X_i \); thus \( \bigcap_{i \in I} X_i = \bigwedge_{i \in I} X_i \). In view of existence of the greatest element in \( C \) we conclude

**Proposition 2.1.** \( C \) is a complete lattice.

Next, 1.1 implies

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Lemma 2.2. Let \( \{X_i\}_{i \in I} \) be a nonempty subcollection of \( \mathcal{C} \). Then \( \bigvee_{i \in I} X_i = HCP \bigcup_{i \in I} X_i \).

In the previous formula, the meaning of \( P \bigcup_{i \in I} X_i \) must be, in fact, considered to be the collection of all lattices \( L \) having the property that there is a set of indices \( I(1) \subseteq I \) such that \( L \) is isomorphic to \( \prod_{i \in I(1)} A_i \), where \( A_i \in X_i \) for each \( i \in I(1) \).

Let \( L(2) \) be a two-element lattice. We denote \( X(2) = HCP\{L(2)\} \).

In [7] (loc. cit) Fried suggested to investigate the convexity which is generated by the two-element lattice.

Proposition 2.3. \( X(2) \) is an atom in \( \mathcal{C} \).

Proof. Obviously \( X_0 < X(2) \). Let \( X \in \mathcal{C} \), \( X_0 < X \leq X(2) \). Hence there exists \( L \in X \) with \( \text{card } L \geq 2 \). Thus there are \( a \) and \( b \) in \( L \) such that \( a < b \). Because of \( L \in X(2) \) we infer that \( [a, b] \in X(2) \) and hence \( [a, b] \) is relatively complemented and distributive. Therefore \( [a, b] \) is a Boolean algebra. There exists a congruence relation \( \Omega \) on \( [a, b] \) such that \( [a, b]/\Omega \) is a two-element lattice. Hence \( L(2) \in X \) and so \( X = X(2) \).

It is clear that if \( X \) is a convexity and \( L \) is an element of \( X \) such that either (i) \( L \) is a Boolean algebra, or (ii) \( L \) is a chain with \( \text{card } L \geq 2 \), then \( X(2) \leq X \).

Let us remark that an element of \( X(2) \) need not be a Boolean algebra. In fact, the following stronger result is valid.

Proposition 2.4. No class of Boolean algebras is a convexity.

Proof. Let \( X \) be a nonempty class of Boolean algebras. By way of contradiction, suppose that \( X \) is a convexity. We apply the same consideration as in the proof of 2.3 and so we conclude that the two-element Boolean algebra \( \{0, 1\} \) belongs to \( X \). Let \( I \) be an infinite set and for each \( i \in I \) let \( A_i = \{0, 1\} \); put \( L = \prod_{i \in I} A_i \). Choose \( I(1) \subseteq I \) such that both \( I(1) \) and \( I \setminus I(1) \) are infinite. Next, let us construct an element \( x \in L \) such that \( x(a_i) = 1 \) if \( i \in I(1) \) and \( x(A_i) = 0 \) otherwise. We denote by \( L_1 \) the set of all \( y \in L \) having the property that the set \( \{ i \in I : y(A_i) \neq x(A_i) \} \) is infinite. Then \( L_1 \) is a convex sublattice of \( L \), whence \( L_1 \in X \). But \( L_1 \) has neither the greatest element nor the least element and thus \( L_1 \) fails to be a Boolean algebra.

Let us remark that if \( L_1 \) is the lattice as in the proof of 2.4, then \( \text{card } L_1 = \aleph_0 \). More generally, we have
Proposition 2.5. Let $X$ be a convexity such that $X^{(2)} \subseteq X$. Then there is $L_1 \in X$ such that $\text{card } L_1 = \aleph_0$.

The proof follows the same idea as in 2.4; it will be omitted.

The following questions remain open:

1. Is $X^{(2)}$ the only atom of $\mathcal{C}$?
2. Let $\alpha$ be an infinite cardinal. We denote by $X$ the class of all lattices $L$ such that, whenever $C$ is a convex chain in $L$, then $\text{card } C < \alpha$. Is $X$ a convexity?

References


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