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BIFURCATION OF PERIODIC SOLUTIONS TO DIFFERENTIAL INEQUALITIES IN $\mathbb{R}^3$

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1. INTRODUCTION

Consider the inequality

$$U(t) \in K \text{ for } t \in [0, T),$$
$$\langle U(t) - A_\lambda U(t) - G(\lambda, U(t)), v - U(t) \rangle \geq 0$$
for all $v \in K$, a.a. $t \in [0, T)$,

where $K$ is a closed convex cone with its vertex at the origin in $\mathbb{R}^3$, $A_\lambda$ is a real $3 \times 3$ matrix depending continuously on a real parameter $\lambda$, $G: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a continuous mapping locally lipschitzian in the variable $u$ and satisfying the usual condition

$$\lim_{u \to 0} \frac{G(\lambda, u)}{|u|} = 0 \text{ uniformly on compact } \lambda\text{-intervals.}$$

Under certain assumptions concerning the eigenvalues of $A_\lambda$ and a relation of the cone $K$ to the eigenvectors of $A_\lambda$, we prove the existence of a bifurcation point $\lambda_f$ at which periodic solutions to the inequality (1) bifurcate from the branch of trivial solutions.

Main results of the paper are contained in Theorems 1, 2. While Theorem 1 contains the basic idea of our approach, Theorem 2 is in fact its consequence and can serve as a tool for verifying periodic bifurcation in examples (see Section 5). Both theorems are proved by elementary means. We investigate the solutions of (1) and those of the linearized inequality

$$U(t) \in K \text{ for } t \in [0, +\infty),$$
$$\langle U(t) - A_\lambda U(t), v - U(t) \rangle \geq 0 \text{ for all } v \in K, \text{ a.a. } t \in [0, +\infty).$$
Note that a different approach to the investigation of bifurcations of periodic solutions to inequalities in $\mathbb{R}^n$ based on degree theory is described in [3], [4]. Further, recall that a bifurcation of stationary solutions to variational inequalities has been studied by several authors during the last 15 years (see e.g. [2], [5], [6], [8] and the references therein).

2. Main Results

Our assumptions concerning the matrix $A_\lambda$ and the convex cone $K$ will be the following: $A_\lambda$ has eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$, $-\nu(\lambda)$ which depend continuously on $\lambda \in \mathbb{R}$ and eigenvectors $\vec{u} \pm i\vec{v}, \vec{w}$ independent of $\lambda$. Let $f_i: \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, N$ be convex functions continuously differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$ and satisfying $f_i(rx_1, rx_2) = rf_i(x_1, x_2)$, $i = 1, \ldots, N$ for all $r > 0$. We shall assume that the cone $K$ is of the form

$$K = \{u \in \mathbb{R}^3; x_3 \geq f_i(x_1, x_2), i = 1, 2, \ldots, N\},$$

where $x = [x_1, x_2, x_3]$ is the vector of the coordinates of $u$ with respect to the basis $\{\vec{u}, \vec{v}, \vec{w}\}$, i.e. $u = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$. Moreover, we assume that

$$K \neq \{u \in \mathbb{R}^3; x_3 \geq 0\},$$

i.e. not all the functions $f_i$ are zero, and also that near any point $v \in K, v \neq 0$ the cone $K$ can be locally described in terms of at most two of the functions $f_1, \ldots, f_N$. More precisely, we impose the following condition on $K$:

for any $v \in K, v \neq 0$ there exist a pair of indices $1 \leq i, j \leq N$

and an open neighbourhood $W$ of the point $v$ such that

$$W \cap K = \{u \in W; x_3 \geq f_i(x_1, x_2), x_3 \geq f_j(x_1, x_2)\}.$$

Remark 1. By a solution of inequality (1) on $[0, T)$ we mean an absolutely continuous function satisfying (1). The following assertions are obtained by standard considerations from the existence results for general differential inclusions [1]. For any $u \in K, \lambda \in \mathbb{R}$ the solution of (1) satisfying $U(0) = u$ exists and is unique at least on some interval $[0, T)$, $T > 0$. This solution will be denoted by $U_\lambda(t, u)$. If $T_0 > 0$ and $U_\lambda(t, u)$ is bounded on any subinterval $[0, T)$ of $[0, T_0)$ on which it exists then $U_\lambda(t, u)$ exists on $[0, T_0)$. This together with simple a priori estimates (see Lemma 2.1 in [4]) imply that for any $T > 0$, $\lambda > 0$ there is $R > 0$ such that $U_\lambda(t, u)$ exists on $[0, T)$ for any $u \in K$, $|u| \leq R$, $|\lambda| \leq \Lambda$. Particularly, for any $u \in K, \lambda \in \mathbb{R}$ there exists a unique solution of (3) satisfying $U(0) = u$ on the whole interval $[0, +\infty)$. It will be denoted by $U_{\lambda, 0}(t, u)$. 340
The symbol $(\cdot, \cdot)$ will stand for the usual inner product in $\mathbb{R}^3$ with the corresponding norm denoted by $| \cdot |$. We denote by $\langle \cdot, \cdot \rangle$ the inner product $\langle u, v \rangle = (x, y)$, where $x, y$ are the vectors of the coordinates of $u, v$ with respect to the basis $\{\bar{u}, \bar{v}, \bar{w}\}$.

We set

$$S = \{r\bar{w}; r \in \mathbb{R}\}.$$  

Any continuous function $U : [0, T] \to \mathbb{R}^3 \setminus S$ can be uniquely written as

$$U(t) = g(t)[\cos(\varphi_0 - \varphi(t))\bar{u} + \sin(\varphi_0 - \varphi(t))\bar{v}] + X_3(t)\bar{w},$$

where $\varphi_0 \in [0, 2\pi)$, $g(t) > 0$, $\varphi(t)$, $X_3(t)$ are continuous functions defined on $[0, T]$ and $\varphi$ satisfies $\varphi(0) = 0$. Hence, for any $u \in K \setminus S$, $\lambda \in \mathbb{R}$ we can define $\varphi_\lambda(t, u)$ as the function $\varphi(t)$ corresponding to $U(t) = U_\lambda(t, u)$ on an interval $[0, T)$ on which $U_\lambda(t, u) \notin S$. Similarly, we define $\varphi_{\lambda, 0}(t, u)$ as the function $\varphi(t)$ corresponding to $U_{\lambda, 0}(t, u)$ on $[0, +\infty)$ (see also Lemma 2,(1)).

Remark 2. Let $U(t) = U_\lambda(t, u) \notin S$ for all $t \in [0, T]$ and let $X(t)$ be the vector of the coordinates of $U(t)$ with respect to the basis $\{\bar{u}, \bar{v}, \bar{w}\}$, i.e. $U(t) = X_1(t)\bar{u} + X_2(t)\bar{v} + X_3(t)\bar{w}$. It follows easily from the definition of $\varphi_\lambda(t, u)$ that

$$\dot{\varphi}_\lambda(t, u) = \frac{\langle \dot{U}(t), X_2(t)\bar{u} - X_1(t)\bar{v} \rangle}{X_1^2(t) + X_2^2(t)}, \ t \in [0, T).$$

For $u \in K \setminus S$, $\lambda \in \mathbb{R}$ we define

$$T(\lambda, u) = \inf\{t > 0; \varphi_\lambda(t, u) = 2\pi\}$$

and use the symbol $T_0(\lambda, u)$ in the linearized case (3). We note that $T(\lambda, u) = +\infty$ if one of the following three cases occurs:

- $\varphi_\lambda(t, u) < 2\pi$ for all $t > 0$;
- there exists $T > 0$ such that $\varphi_\lambda(t, u) < 2\pi$ for all $t \in [0, T)$ and $U_\lambda(T, u) \in S$;
- $U_\lambda(t, u)$ is defined only on $[0, T)$ and $\varphi_\lambda(t, u) < 2\pi$ for all $t \in [0, T)$.

Consider the inequality

$$u \in K,$$

$$(\mu u - A_\lambda u, v - u) \geq 0 \text{ for all } v \in K.$$  

A real number $\mu$ is called an eigenvalue of the inequality (7) (for a given $\lambda \in \mathbb{R}$) if there exists a nontrivial $u$ satisfying (7). Any such $u$ is called an eigenvector of (7) corresponding to $\mu$.  

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We define
\[ g(u) = \frac{x_3}{\sqrt{x_1^2 + x_2^2}} \text{ for } u \notin S, u = x_1\bar{u} + x_2\bar{v} + x_3\bar{w}, \]
\[ \tau = \max\{g(u); 0 \neq u \in \partial K\}. \]

**Remark 3.** In any cone \( K \) of the form (4) there exists at least one vector \( v \) satisfying

(8) \[ 0 \neq v \in \partial K, \ g(v) = \tau. \]

(This \( v \) represents the ray which is the closest one to \( S \) with respect to \( \langle \cdot, \cdot \rangle \) among those lying on \( \partial K \).)

We denote by \( T_K(u) \) the contingent cone to \( K \) at a point \( u \in K \), i.e.
\[ T_K(u) = \text{cl}\left( \bigcup_{h>0} \bigcup_{v \in K} h(v-u) \right). \]

**Theorem 1.** Let \([\lambda_1, \lambda_2] \subset \mathbb{R}\) be an interval and \( v \) an arbitrary fixed element satisfying (8). Assume

(9) \[ T_0(\lambda, v) < +\infty \quad \text{for } \lambda_1 \leq \lambda \leq \lambda_2, \]
(10) \[ \alpha(\lambda) + \nu(\lambda) > 0 \quad \text{for } \lambda_1 \leq \lambda \leq \lambda_2, \]
(11) \[ \beta(\lambda) > 0 \quad \text{for } \lambda_1 \leq \lambda \leq \lambda_2, \]
(12) \[ |U_{\lambda,0}(T_0(\lambda, v), v)| < |v| \quad \text{for } \lambda = \lambda_1, \]
(13) \[ |U_{\lambda,0}(T_0(\lambda, v), v)| > |v| \quad \text{for } \lambda = \lambda_2. \]

Then for any sufficiently small \( r > 0 \) there exists \( \lambda \in (\lambda_1, \lambda_2) \) such that \( U_\lambda(\cdot, rv) \) is a periodic solution of the inequality (1). There is at least one bifurcation point \( \lambda_1 \in (\lambda_1, \lambda_2) \) at which periodic solutions of (1) bifurcate from the branch of trivial solutions.

**Idea of the proof** of Theorem 1 (see Section 4 for details). The conditions (9), (10), (11) and Lemmas 2, 3 enable us to prove that the solution of the linearized inequality (3) starting from the particular initial condition \( v \) satisfies \( \varphi_{\lambda,0}(T_0(\lambda, v), v) > 0 \) when \( \lambda \in [\lambda_1, \lambda_2] \). As a result, Lemma 1,(vi) implies \( T(\lambda, rv) < +\infty \) for all \( \lambda \in [\lambda_1, \lambda_2] \) and \( r > 0 \) small. Combining Lemma 3 and Remark 5 we conclude that \( U_\lambda(T(\lambda, rv), rv) = k(\lambda, r)rv \) where \( k(\lambda, r) \) is a positive function defined on \([\lambda_1, \lambda_2] \times (0, R)\). The conditions (12), (13) ensure \( k(\lambda_1, r) < r < k(\lambda_2, r) \). Since \( k \) is continuous in the variable \( \lambda \) we obtain for any sufficiently small \( r > 0 \) a value \( \lambda \in [\lambda_1, \lambda_2] \) such that \( k(\lambda, r) = r \). Thus we get \( U_\lambda(T, rv) = rv \) where \( T = T(\lambda, rv) \) and \( rv \) is the initial condition of a periodic solution. □
**Theorem 2.** Let \([\Lambda_1, \Lambda_2] \subset \mathbb{R}\) be an arbitrary interval. Assume

\[
(14) \quad \alpha(\lambda) + \nu(\lambda) = 0, \alpha(\lambda) < 0 \quad \text{for } \lambda = \Lambda_1, \\
(15) \quad \alpha(\lambda) + \nu(\lambda) > 0 \quad \text{for } \Lambda_1 < \lambda \leq \Lambda_2, \\
(16) \quad \beta(\lambda) > 0 \quad \text{for } \Lambda_1 \leq \lambda \leq \Lambda_2, \\
(17) \quad 0 \neq u \in \partial K \implies A_\lambda u \notin T_K(u) \quad \text{for } \lambda = \Lambda_2, \\
(18) \quad 0 \neq u \in \partial K \implies (A_\lambda u, u) > 0 \quad \text{for } \lambda = \Lambda_2.
\]

In addition, assume \(\mu > 0\) whenever \(\mu\) is an eigenvalue of (7) corresponding to an eigenvector \(u \in \partial K\) for some \(\lambda \in [\Lambda_1, \Lambda_2]\).

Then to any sufficiently small \(r > 0\) there exist \(\lambda \in (\Lambda_1, \Lambda_2)\) and \(u \in K, \ |u| = r\) such that \(U_\lambda(\cdot, u)\) is a periodic solution of (1).

**Idea of the proof of Theorem 2** (see Section 4 for details). We shall find an interval \([\Lambda_1, \Lambda_2] \subset [\Lambda_1, \Lambda_2]\) for which the assumptions (9)-(13) are fulfilled. As in Theorem 1 the solutions of the inequality (3) starting at \(v\) are investigated. First we prove by using (14) that the solution \(U_{\lambda,0}(t, v)\) of the inequality (3) with \(\lambda = \Lambda_1\) is simultaneously a solution of the linear differential equation \(\dot{U}(t) = A_\lambda U(t)\). Making use of the explicit form of this solution (see Remark 4) and of Lemma 1 we find \(T_0(\lambda, v) < +\infty\) and \(|U_{\lambda,0}(T_0(\lambda, v), v)| < |v|\) for all \(\lambda\) close to \(\Lambda_1\). Hence \(\lambda_1\) satisfying (12) is obtained. To find \(\lambda_2\) we consider two cases: either \(T_0(\lambda, v) < +\infty\) for all \(\lambda \in [\Lambda_1, \Lambda_2]\) or there is a \(\delta \in (\Lambda_1, \Lambda_2]\) such that \(T_0(\delta, v) = +\infty\) and \(T_0(\lambda, v) < +\infty\) for all \(\lambda \in [\Lambda_1, \delta)\). In the first case we use the assumptions (17), (18) and Lemma 4 to get the inequality \(|U_{\lambda,0}(T_0(\lambda, v), v)| > |v|\) for \(\lambda = \Lambda_2\) and we can put \(\lambda_2 = \Lambda_2\). In the case of \(T_0(\delta, v) = +\infty\) we use Lemma 2 to prove

\[
\frac{U_{\delta,0}(t, v)}{|U_{\delta,0}(t, v)|} \to u \quad \text{for } t \to +\infty
\]

where \(u \in \partial K\) is an eigenvector of (7). By our assumption, the corresponding eigenvalue \(\mu\) is positive, which permits us to show \(|U_{\delta,0}(t, v)| \to +\infty\) as \(t \to +\infty\). This in turn leads to the inequality (13) with some \(\lambda_2 < \delta, \lambda_2\) close to \(\delta\).

\[\square\]

**3. Some General Remarks**

Let \(C \subset \mathbb{R}^3\) be a nonempty closed convex set and \(w \in \mathbb{R}^3\) an arbitrary vector. The nearest point (with respect to the norm \(|\cdot|\)) to \(w\) in the set \(C\) will be hereafter referred to as the projection of \(w\) onto \(C\).

We introduce some additional notation:
$K_i = \{ u \in \mathbb{R}^3; x_3 \geq f_i(x_1, x_2) \}, \ 1 \leq i \leq N,$

$T_i(u)$ for $u \in K_i$ is the contingent cone to $K_i$ at a point $u$,

$n_i(u)$ is the unit inner normal to $\partial K_i$ at a point $u \in \partial K_i$,

$P_u w$ for $u \in K, \ w \in \mathbb{R}^3$ is the projection of $w$ onto $T_K(u)$,

$P_u^i w$ for $u \in K_i, \ w \in \mathbb{R}^3$ is the projection of $w$ onto $T_i(u),$

$L$ is the $3 \times 3$ matrix with columns $\bar{u}, \bar{v}, \bar{w}$ and $B_\lambda = L^{-1} A_\lambda L$ is the canonical form of $A_\lambda,$ i.e.

$$B_\lambda = \begin{pmatrix}
\alpha(\lambda) & \beta(\lambda) & 0 \\
-\beta(\lambda) & \alpha(\lambda) & 0 \\
0 & 0 & -\nu(\lambda)
\end{pmatrix}.$$ 

While points in $\mathbb{R}^3$ are usually denoted by $u = [u_1, u_2, u_3],$ vector functions with values in $\mathbb{R}^3$ are denoted for instance by $U(t) = [U_1(t), U_2(t), U_3(t)].$ Throughout the paper the symbols $\dot{U}(t), \ U_\lambda(t, u), \ \varphi_{\lambda, 0}(t, u)$ etc. denote the right derivatives of the corresponding functions.

**Remark 4.** Let $U(t) = X_1(t)\bar{u} + X_2(t)\bar{v} + X_3(t)\bar{w}, \ X(t) = [X_1(t), X_2(t), X_3(t)]$ be the solution of the equation $\dot{U}(t) = A_\lambda U(t)$ with the initial condition $U(0) = v.$

Then $\dot{X}(t) = B_\lambda X(t), \ t \geq 0$ and

$$X_1(t) = e^{\alpha(\lambda)t}(X_1(0) \cos \beta(\lambda)t + X_2(0) \sin \beta(\lambda)t),$$

$$X_2(t) = e^{\alpha(\lambda)t}(X_2(0) \cos \beta(\lambda)t - X_1(0) \sin \beta(\lambda)t),$$

$$X_3(t) = e^{-\nu(\lambda)t}X_3(0).$$

**Remark 5.** Let $v \in K$ satisfy (8) and let $T(\lambda, v) < +\infty$ for some $\lambda \in \mathbb{R}.$ Then

(i) $g(U_\lambda(T(\lambda, v), v)) \leq \tau$ implies $U_\lambda(T(\lambda, v), v) = kv$ with some $k > 0.$

For any $u \in \mathbb{R}^3 \setminus S$

(ii) $g(u) \geq \tau$ implies $u \in K$ and $g(u) > \tau$ implies $u \in \text{int } K.$

The proof of these assertions follows directly from the definitions of the function $g$ and of the number $\tau.$

**Remark 6.** Let $u \in K, \ w \in T_K(u), \ z \in \mathbb{R}^3.$ Then it is easy to see that

$$w = P_u z \iff (w - z, x - w) \geq 0 \ \text{for all } x \in T_K(u).$$

Thus it follows from the definition of the cone $T_K(u)$ that $P_u z$ is the unique point in $T_K(u)$ with the property

$$\langle P_u z - z, P_u z \rangle = 0,$$

$$\langle P_u z - z, v - u \rangle \geq 0 \ \text{for all } v \in K.$$
Remark 7. An absolutely continuous function \( U : [0, T) \rightarrow K \) is a solution of the inequality (1) if and only if

\[
\dot{U}(t) = P_{U(t)}(A\lambda U(t) + G(\lambda, U(t))) \quad \text{for a. a. } t \in [0, T)
\]

(see [1]).

Remark 8. Any solution \( U : [0, T) \rightarrow K \) of (1) is right differentiable, its right derivative is right continuous in the interval \([0, T)\) and the equation (22) holds for all \( t \in [0, T) \). For the proof see [7].

Remark 9. Any eigenvalue \( \mu \) of the inequality (7) with the corresponding eigenvector \( u \) satisfies

\[
\mu|u|^2 = (A\lambda u, u).
\]

Further, it follows from Remark 6 that for any \( u \in K \) and \( \mu \in \mathbb{R} \) the inequality (7) is equivalent to

\[
\mu u = P_u A\lambda u.
\]

Remark 10. Suppose that at a point \( t = t_0 \) the solution \( U(t) = U_{\lambda,0}(t, u) \) of the inequality (3) satisfies the equation \( \dot{U}(t) = A\lambda U(t) \). Then \( \varphi_{\lambda,0}(t_0, u) = \beta(\lambda) \) (see Remark 4). This occurs for instance when \( U_{\lambda,0}(t_0, u) \in \text{int } K \). More generally, it follows from Remark 8 that if \( U(t) \) is a solution of (1) such that \( U(t) \in \text{int } K \) for all \( t \in [t_1, t_2] \) then the equation \( \dot{U}(t) = A\lambda U(t) + G(\lambda, U(t)) \) holds on this interval.

Remark 11. For any solution \( U : [0, T) \rightarrow K \) of the inequality (3) we have

\[
(\dot{U}(t) - A\lambda U(t), U(t)) = 0, \quad t \in [0, T).
\]

Lemma 1. To any \( T > 0, \Lambda > 0 \) there exists \( R > 0 \) such that for any sequences \( \lambda_n \rightarrow \lambda, |\lambda| < \Lambda, u_n \in K, u_n \rightarrow u, |u| < R \) we have

(i) \( U_{\lambda_n}(\cdot, u_n) \rightarrow U_{\lambda}(\cdot, u) \) uniformly on \([0, T]\),

(ii) if \( U_\lambda(t, u) \not\in S \) for \( t \in [0, T] \) then \( \varphi_{\lambda_n}(\cdot, u_n) \rightarrow \varphi_{\lambda}(\cdot, u) \) uniformly on \([0, T]\),

(iii) if \( T(\lambda, u) < T, \varphi_{\lambda}(T(\lambda, u), u) > 0 \) then \( T(\lambda_n, u_n) \rightarrow T(\lambda, u) \).

Let \( \lambda_n \rightarrow \lambda \in \mathbb{R}, 0 \neq u_n \in K, u_n \rightarrow 0, \frac{u_n}{|u_n|} \rightarrow w \in \mathbb{R}^3, \) let \( T > 0 \) be arbitrary. Then

(iv) \( \frac{U_{\lambda_n}(\cdot, u_n)}{|u_n|} \rightarrow U_{\lambda,0}(\cdot, w) \) uniformly on \([0, T]\),

(v) if \( w \not\in S \) then \( \varphi_{\lambda_n}(\cdot, u_n) \rightarrow \varphi_{\lambda,0}(\cdot, w) \) uniformly on \([0, T]\),

(vi) if \( w \not\in S, T_0(\lambda, w) < +\infty \) and \( \varphi_{\lambda,0}(T_0(\lambda, w), w) > 0 \) then \( T(\lambda_n, u_n) \rightarrow T_0(\lambda, w) \).

For the proof see Theorems 2.1, 2.2 and Consequence 2.2 in [4].
Observation 1. Let \( u = x_1 \dot{u} + x_2 \dot{v} + x_3 \dot{w} \in \partial K \setminus \{0\} \) and \( w \in \partial T_i(u) \) for some \( i, 1 \leq i \leq N \). Then \( \langle w, x_2 \dot{v} - x_1 \dot{v} \rangle = 0 \) implies \( w = \mu u, \mu \in \mathbb{R} \).

Observation 2. Let \( u_n \in K, u_n \to u \). Then for any vector \( v \in T_K(u) \) there exists a sequence \( v_n \to v \) satisfying \( v_n \in T_K(u_n), n = 1, 2, \ldots \).

The proof of this observation follows from results proved in [1].

Observation 3. Let \( u_n \in K, z_n \in \mathbb{R}^3, u_n \to u, z_n \to z \). Then the following implications hold:

(i) If \( P_{u_n}z_n \to w, P_{u_n}z_n \in \partial T_K(u_n), n = 1, 2, \ldots \) then \( w \in \partial T_i(u) \) for some \( i, 1 \leq i \leq N \).

(ii) If \( P_{u_n}z_n \to w, w \in T_K(u) \) then \( w = P_u z \).

(iii) If there exists \( j, 0 \leq j \leq N \) such that \( u, u_n \in \partial K_1 \cap \partial K_2 \cap \ldots \cap \partial K_j \cap \text{int } K_{j+1} \cap \ldots \cap \text{int } K_N, n = 1, 2, \ldots \) then \( P_{u_n}z_n \to P_u z \).

Proof. (i) Since \( P_{u_n}z_n \in \partial T_K(u_n) \) we have \( (P_{u_n}z_n, n_i(u_n)) = 0 \) with some \( 1 \leq i_n \leq N \). We may suppose that the sequence \( i_n \) is constant and therefore

\[
(P_{u_n}z_n, n_i(u_n)) = 0, \quad n = 1, 2, \ldots.
\]

From the continuity of the normal \( n_i(\cdot) \) we conclude \( (w, n_i(u)) = 0 \) and therefore \( w \in \partial T_i(u) \).

(ii) Take an arbitrary \( v \in T_K(u) \). Observation 2 implies \( v_n \to v \) for some sequence \( v_n \in T_K(u_n), n = 1, 2, \ldots \). We have

\[
|v_n - z_n| \geq |P_{u_n}z_n - z_n|
\]

and consequently \( |v - z| \geq |w - z| \). This inequality, holding for all \( v \in T_K(u) \), together with \( w \in T_K(u) \) implies \( w = P_u z \).

(iii) The case \( j = 0 \) is trivial. Let \( j \geq 1 \). As \( |P_{u_n}z_n| \leq |z_n| \) and \( z_n \) is convergent, the sequence \( P_{u_n}z_n \) is bounded. Therefore it is sufficient to prove the implication

\[
P_{u_n}z_n \to w \implies w = P_u z.
\]

However, for \( n = 1, 2, \ldots \) we have

\[
(P_{u_n}z_n, n_i(u_n)) \geq 0, \quad i = 1, 2, \ldots, j.
\]

Consequently, \( w, n_i(u) \geq 0, i = 1, 2, \ldots, j \), and \( w \) belongs to \( T_K(u) = T_1(u) \cap \ldots \cap T_j(u) \). Now we use (ii) to prove \( w = P_u z \). \( \Box \)
Observation 4. Let \( u \in K, w \in \mathbb{R}^3 \) be arbitrary vectors.

If \( P_u w \in \text{int}T_{j+1}(u) \cap \ldots \cap \text{int}T_N(u) \) where \( 1 \leq j \leq N - 1 \) then \( P_u w \) coincides with the projection of \( w \) onto \( T_1(u) \cap T_2(u) \cap \ldots \cap T_j(u) \).

Further, \( P_u w = w \) whenever \( P_u w \in \text{int}T_K(u) \).

Proof. Denote by \( \Pi \) the set \( T_1(u) \cap \ldots \cap T_j(u) \). We have \( P_u w \in \Pi \) and therefore it is sufficient to prove \( (P_u w - w, x - P_u w) \geq 0 \) for all \( x \in \Pi \) (see Remark 6). Choose \( x \in \Pi \). Then \( (1 - t)P_u w + tx \in \Pi \), \( 0 \leq t \leq 1 \). Moreover,

\[
(1 - t)P_u w + tx \in T_{j+1}(u) \cap \ldots \cap T_N(u) \quad \text{for } t > 0, \ t \text{ small.}
\]

Hence \( P_u w + t(x - P_u w) \in T_K(u) \) for some \( t > 0 \). Since \( P_u w \) is the projection of \( w \) onto \( T_K(u) \) we have

\[
(P_u w - w, x - P_u w) = \frac{1}{t}(P_u w - w, P_u w + t(x - P_u w) - P_u w) \geq 0.
\]

\( \Box \)

4. Proof of Main Results

Lemma 2. Let \( \lambda \in \mathbb{R}, \beta(\lambda) > 0 \), and let \( v \in K \setminus S \). Then

(I) \( U_{\lambda,0}(t, v) \notin S \) for all \( t > 0 \),

(II) if \( \varphi_{\lambda,0}(t_0, v) = 0 \) then \( U_{\lambda,0}(t_0, v) \) is an eigenvector of (7) and \( \varphi_{\lambda,0}(t, v) = 0 \) for all \( t > t_0 \),

(III) if

\[
\lim_{t \to +\infty} \varphi_{\lambda,0}(t, v) = \varphi
\]

then

\[
\lim_{t \to +\infty} \frac{U_{\lambda,0}(t, v)}{|U_{\lambda,0}(t, v)|} = u \in \partial K
\]

where \( u \) is an eigenvector of (7),

(IV) if \( \varphi_{\lambda,0}(t_0, v) \leq 0 \) for some \( t_0 \geq 0 \) then \( \varphi_{\lambda,0}(t, v) \leq 0 \) for all \( t \geq t_0 \),

(V) if \( T_0(\lambda, v) < +\infty \) then \( \varphi_{\lambda,0}(t, v) > 0 \) for all \( t \in [0, T_0(\lambda, v)] \).

Proof. Throughout the proof we shall write \( U(t) = U_{\lambda,0}(t, v), \varphi(t) = \varphi_{\lambda,0}(t, v) \).

(I) If the statement were false there would exist \( t_0 > 0 \) such that \( U(t_0) \in S, U(t) \notin S \) for all \( t \in [0, t_0) \). Remark 11 implies

\[
\frac{d}{dt}(|U(t)|^2) = 2(\dot{U}(t), U(t)) = 2(A_\lambda U(t), U(t)) \geq -C|U(t)|^2
\]
with some $C > 0$. Thus $|U(t)|^2 \geq e^{-Ct}|v|^2$ and therefore $U(t) \neq 0$ for all $t > 0$. Now it follows from the assumption (4) that $U(t_0) \in \text{int } K$. Therefore $U(t)$ is also a solution of the equation $\dot{U}(t) = A_\lambda U(t)$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$, $\varepsilon > 0$ small. However, one can see from Remark 4 that no solution to this equation starting from a point $u \notin S$ can reach $S$ in a finite time.

(II) Let $u = U(t_0)$, $w = \dot{U}(t_0)$ and let $u = x_1 \bar{u} + x_2 \bar{v} + x_3 \bar{w}$. It follows from (I) that $u \notin S$, and Remark 2 yields
\[
\langle w, x_2 \bar{v} - x_1 \bar{v} \rangle = 0.
\]

We have $w \in T_K(u)$ by Remarks 7, 8. We shall prove $w \in \partial T_K(u)$. Indeed, if $w \in \text{int } T_K(u)$ we would obtain from Remark 8
\[
P_u A_\lambda u = \dot{U}(t_0) \in \text{int } T_K(u),
\]
and Observation 4 would imply $P_u A_\lambda u = A_\lambda u$. Hence $\dot{U}(t_0) = A_\lambda U(t_0)$ and Remark 10 would yield $\varphi(t_0) = A(u) > 0$.

Now $w \in \partial T_K(u)$ implies $w \in \partial T_i^1(u)$ for some $i$, $1 \leq i \leq N$ and thus Observation 1 together with (25) yields $P_u A_\lambda u = w = \mu u$ with some $\mu \in \mathbb{R}$. By Remark 9 we conclude that $u$ is an eigenvector of (7).

Let us set $V(t) = e^{\mu t}u$ and prove that $V(t) = U_{\lambda,0}(t, u)$. Indeed, using (7) we get
\[
(\dot{V}(t) - A_\lambda V(t), z - V(t)) = (\mu e^{\mu t}u - e^{\mu t}A_\lambda u, z - e^{\mu t}u)
\]
\[
= e^{2\mu t}(\mu u - A_\lambda u, e^{-\mu t}z - u) \geq 0
\]
for all $z \in K$, $t \geq 0$.

Consequently, since $V(0) = U(t_0)$, we have $\dot{U}(t) = \dot{V}(t-t_0) = \mu e^{\mu(t-t_0)}u = e^{\mu(t-t_0)}w$ for $t \geq t_0$ and so the statement follows from (25) by Remark 2.

(III) To prove that the limit in (24) exists we shall verify that there is exactly one $u \in \mathbb{R}^3$ that satisfies
\[
\frac{U(t_n)}{|U(t_n)|} \rightarrow u \text{ for some } t_n \rightarrow +\infty.
\]

Let us prove that (26) implies $u \in \partial K$. Suppose there is $u \in \text{int } K$ satisfying (26). Then $U_{\lambda,0}(t, u) \in \text{int } K$ for all $t$ in a small interval $[0, T]$ and Lemma 1,(1) yields
\[
U_{\lambda,0} \left(t, \frac{U(t_n)}{|U(t_n)|}\right) \in \text{int } K, \ t \in [0, T]
\]
for \( n \) sufficiently large. Hence \( U(t + t_n) = U_{\lambda,0}(t, U(t_n)) \in \text{int} K, \ t \in [0, T] \) and therefore \( \dot{U}(t) = A_{\lambda} U(t), \ t \in [t_n, t_n + T) \). By Remark 10

\[
\varphi(t_n + T) - \varphi(t_n) = \int_0^T \dot{\varphi}(t_n + t)\,dt = \int_0^T \beta(\lambda)\,ds = T\beta(\lambda) > 0,
\]

which is a contradiction as (23) yields \( \varphi(t_n + T) - \varphi(t_n) \rightarrow 0 \) for \( n \rightarrow +\infty \). We have proved that (26) implies \( u \in \partial K \). Finally, it follows from (4) that there is exactly one vector \( u \in \partial K \) with a given argument (determined by (23)) and a given norm \( |u| = 1 \).

To show that \( u \) is an eigenvector of (7) we shall prove \( \dot{\varphi}_{\lambda,0}(0, u) = 0 \) and then use (II). Suppose for a moment that \( \dot{\varphi}_{\lambda,0}(0, u) > 0 \). Then \( \varphi_{\lambda,0}(T, u) > 0 \) for some \( T > 0 \) and Lemma 1 together with (24) yields

\[
0 < \varepsilon < \varphi_{\lambda,0} \left( \frac{T}{U(t)} \right) = \varphi_{\lambda,0}(T, U(t))
\]

for \( t \) large and some \( \varepsilon > 0 \). Since \( \varphi_{\lambda,0}(0, w) = 0 \) for all \( w \in K \setminus S \) we have \( \varphi_{\lambda,0}(T, U(t)) = \varphi(t + T) - \varphi(t) \) and so the last inequality contradicts (23). By excluding in a similar way the inequality \( \dot{\varphi}_{\lambda,0}(0, u) < 0 \) we complete the proof of (III).

(IV) It follows from (I) (and Remark 8) that \( \varphi(t), \dot{\varphi}(t) \) are defined for all \( t \geq 0 \). We set \( t_1 = \inf\{t > t_0: \dot{\varphi}(t) > 0\} \) and suppose \( t_0 < t_1 < +\infty \). It follows from Remark 8 that \( \lim_{t \to t_1-} \dot{\varphi}(t) = \dot{\varphi}(t_1) \) and so \( \dot{\varphi}(t_1) \geq 0 \). On the other hand, if \( \dot{\varphi}(\bar{t}) = 0 \) for some \( \bar{t} \in [t_0, t_1] \) we would obtain from (II) that \( \dot{\varphi}(t) = 0 \) for all \( t \geq \bar{t} \) which would contradict the assumption \( t_1 < +\infty \).

Thus we are left with the situation

\[
\begin{align*}
(27) & \quad \dot{\varphi}(t_1) > 0, \\
(28) & \quad \dot{\varphi}(t) < 0, \quad t \in [t_0, t_1).
\end{align*}
\]

To show that (27) and (28) contradict each other we shall prove

\[
\lim_{t \to t_1-} \dot{U}(t) = \dot{U}(t_1)
\]

and therefore

\[
\lim_{t \to t_1-} \dot{\varphi}(t) = \dot{\varphi}(t_1).
\]

First, note that because of (6) we may suppose

\[
K = \{ u \in \mathbb{R}^3; x_3 \geq f_i(x_1, x_2), x_3 \geq f_j(x_1, x_2) \}
\]
where \( i, j \) are not necessarily distinct indices. Indeed, all our considerations will be confined to a suitable neighborhood of the point \( u = U(t_1) \). Now Remark 10 and (28) imply \( U(t) \in \partial K \) for all \( t \in [t_0, t_1) \) and therefore \( U(t_1) \in \partial K \). Moreover, we may suppose \( x_3 = f_i(x_1, x_2) = f_j(x_1, x_2) \). Indeed, if \( f_j(x_1, x_2) < x_3 \) then \( u \in \partial K \) would imply \( f_i(x_1, x_2) = x_3 \) and we could take \( i = j \) in (31). Thus the normals \( n_i(u), n_j(u) \) are defined and we first consider the case where \( n_i(u) = n_j(u) \). We have \( T_K(u) = T_i(u) = T_j(u) \) and therefore \( P_u A_{i\lambda} u = P_{u_1}^i A_{i\lambda} u = P_{u_1}^j A_{i\lambda} u \). To prove (29) it is sufficient to show \( P_{u_1}^i A_{i\lambda} u_n \to P_u A_{i\lambda} u \) whenever \( u_n \to u, u_n \in \partial K, n = 1, 2, \ldots \) (see Remark 8). The continuity of the normals \( n_i, n_j \) implies

\[
  u_n \in \partial K_i \cap \text{int } K_j \implies P_{u_1}^i A_{i\lambda} u_n = P_{u_1}^i A_{i\lambda} u_n \\
  u_n \in \text{int } K_i \cap \partial K_j \implies P_{u_1}^i A_{i\lambda} u_n = P_{u_1}^i A_{i\lambda} u_n \\
  u_n \to P_u A_{i\lambda} u.
\]

Recalling Observation 3, (iii) we find

\[
  u_n \in \partial K_i \cap \partial K_j \implies P_{u_1}^i A_{i\lambda} u_n = P_u A_{i\lambda} u
\]

and (29) is proved.

Finally, let us deal with the case \( n_i(u) \neq n_j(u) \). We set

\[
  a = \left[ -\frac{\partial f_1}{\partial x_1}(x), -\frac{\partial f_1}{\partial x_2}(x), 1 \right], \\
  b = \left[ -\frac{\partial f_j}{\partial x_1}(x), -\frac{\partial f_j}{\partial x_2}(x), 1 \right], \\
  c = [x_2, -x_1, 0],
\]

where \( u = x_1 \bar{u} + x_2 \bar{v} + x_3 \bar{w} \). (Note that \( a, b \) are normals to \( \partial K_i, \partial K_j \) with respect to \( \langle \cdot, \cdot \rangle \).) Assume for a moment that \( (a, c) = (b, c) \). Then \( (a - b, c) = 0 \) and it follows from the properties of the functions \( f_1, f_j \) that \( (a - b, x) = 0, (a - b, [0, 0, 1]) = 0 \). Thus the vector \( a - b \) would be orthogonal to three independent vectors and therefore would equal zero. However, the assumption \( n_i(u) \neq n_j(u) \) implies \( a \neq b \). Hence \( (a, c) \neq (b, c) \). We can assume \( (a, c) < (b, c) \) and write this inequality as

\[
  -\frac{\partial f_1}{\partial x_1} \sin(\varphi_0 - \varphi(t_1)) + \frac{\partial f_1}{\partial x_2} \cos(\varphi_0 - \varphi(t_1)) \\
  < -\frac{\partial f_j}{\partial x_1} \sin(\varphi_0 - \varphi(t_1)) + \frac{\partial f_j}{\partial x_2} \cos(\varphi_0 - \varphi(t_1)),
\]

where \( x_1 = \varphi \cos(\varphi_0 - \varphi(t_1)), x_2 = \varphi \sin(\varphi_0 - \varphi(t_1)) \). Hence we obtain

\[
  \frac{d}{d\varphi} f_i(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi)) > \frac{d}{d\varphi} f_j(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi))
\]
at the point \( \varphi = \varphi(t_1) \). Consequently,

\[
(32) \quad f_i(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi)) > f_j(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi))
\]

whenever \( \varphi > \varphi(t_1) \) and \( \varphi \) is sufficiently close to \( \varphi(t_1) \). It follows from (27), (28) that the function \( \varphi(t) \) attains its strict local minimum at the point \( t = t_1 \). Taking (4) into account we obtain from (32) an \( \epsilon > 0 \) satisfying

\[
(33) \quad U(t) \in \text{int } K_j, \ t \in (t_1 - \epsilon, t_1) \cup (t_1, t_1 + \epsilon).
\]

Consequently, \( T_K(U(t)) = T_i(U(t)) \) and

\[
(34) \quad P_{U(t)} A_{\lambda} U(t) = P_{U(t)}^i A_{\lambda} U(t) \text{ a.e. on } (t_1 - \epsilon, t_1 + \epsilon).
\]

By Remark 7 we conclude that the function \( U(t) \) on \( (t_1 - \epsilon, t_1 + \epsilon) \) is a solution of the inequality (3) with \( K \) replaced by \( K_i \). Remark 8 implies that formula (34) is valid everywhere on \( (t_1 - \epsilon, t_1 + \epsilon) \). In particular, \( P_u A_{\lambda} u = P_{U(t)}^i A_{\lambda} u \). Moreover, as we have noted above, \( U(t) \) belongs to \( \partial K \) for \( t \in (t_1 - \epsilon, t_1) \). Thus it follows from (33) that \( U(t) \in \partial K_i \) for \( t \in (t_1 - \epsilon, t_1) \) and therefore

\[
\lim_{t \to t_1^-} P_{U(t)} A_{\lambda} U(t) = \lim_{t \to t_1^-} P_{U(t)}^i A_{\lambda} U(t) = P_{U(t)}^i A_{\lambda} U(t_1)
= P_{U(t)} A_{\lambda} U(t) = P_u A_{\lambda} u = P_{U(t)} A_{\lambda} U(t_1).
\]

Thus (29) follows from Remark 8 and the proof of (IV) is complete.

(V) The assertion follows immediately from the definition of \( T_0(\lambda, v) \) and from (IV).

\[\square\]

**Lemma 3.** Let \( \alpha(\lambda) + \nu(\lambda) > 0 \) for all \( \lambda \in [\lambda_1, \lambda_2] \). Then for any \( T > 0 \) there exists \( R > 0 \) such that the following implications hold for any \( u \in K \setminus S \):

\[
|u| \leq R, \ g(u) \leq \tau \implies g(U_{\lambda}(t, u)) \leq \tau \text{ for all } \lambda \in [\lambda_1, \lambda_2], \ t \in [0, T],
\]

\[
g(u) \leq \tau \implies g(U_{\lambda, 0}(t, u)) \leq \tau \text{ for } \lambda \in [\lambda_1, \lambda_2], \ t \in [0, +\infty).
\]

**Proof.** First of all we realize (see Remark 1) that if \( |u| \) is small enough the solution \( U_{\lambda}(t, u) \) exists on \( [0, T] \) for all \( [\lambda_1, \lambda_2] \). We shall prove

\[
|u| \leq R, \ g(u) \leq \tau \implies U_{\lambda}(t, u) \notin S \text{ for all } \lambda \in [\lambda_1, \lambda_2], \ t \in [0, T].
\]

Indeed, suppose \( U_{\lambda_n}(t_n, u_n) \in S, g(u_n) \leq \tau \) for some \( u_n \to 0, \ t_n \in [0, T], \lambda_n \in [\lambda_1, \lambda_2] \). We may suppose \( \lambda_n \to \lambda, \ t_n \to t \) and \( \frac{u_n}{|u_n|} \to w \). Then \( w \in K \setminus S \) and by Lemma 1, (iv)

\[
\frac{U_{\lambda_n}(t_n, u_n)}{|u_n|} \to U_{\lambda, 0}(t, w).
\]

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Hence $U_{\lambda_0}(t, w) \in S$, which contradicts Lemma 2,(1).

Now if the first implication of the lemma were false there would necessarily exist sequences $u_n \in K$, $u_n \to 0$, $t_n \to t$, $\lambda_n \to \lambda \in [\lambda_1, \lambda_2]$, $\varepsilon_n > 0$ such that

\begin{align}
(35) & \quad g(U_{\lambda_n}(t_n, u_n)) = \tau, \\
(36) & \quad g(U_{\lambda_n}(t, u_n)) > \tau \text{ for } t \in (t_n, t_n + \varepsilon_n), \ n = 1, 2, \ldots.
\end{align}

Recalling Remark 5,(ii) we can see from (36) that $U_{\lambda_n}(t, u_n) \in \text{int} \ K$ for $t \in (t_n, t_n + \varepsilon_n)$, $n = 1, 2, \ldots$. Therefore the equation

$$
\dot{U}_{\lambda_n}(t, u_n) = A_{\lambda_n} U_{\lambda_n}(t, u_n) + G(\lambda_n, U_{\lambda_n}(t, u_n))
$$

is valid on $(t_n, t_n + \varepsilon_n)$. Particularly, Remark 8 gives

$$
\dot{U}_{\lambda_n}(t_n, u_n) = A_{\lambda_n} U_{\lambda_n}(t_n, u_n) + G(\lambda_n, U_{\lambda_n}(t_n, u_n)).
$$

As a result of (35), (36) the right derivative of the function $g(U_{\lambda_n}(\cdot, u_n))$ is nonnegative at the point $t_n$. Setting $v_n = U_{\lambda_n}(t_n, u_n)$ we get

$$
0 \leq (\text{grad } g(U_{\lambda_n}(t_n, u_n)), \dot{U}_{\lambda_n}(t_n, u_n)) = (\text{grad } g(v_n), A_{\lambda_n} v_n + G(\lambda_n, v_n)).
$$

Lemma 1,(i) implies $v_n \to 0$ and, since $v_n \neq 0$, we may suppose $v_n / |v_n| \to w$. By passing to the limit in the inequality

$$
0 \leq \left(\text{grad } g\left(\frac{v_n}{|v_n|}\right), A_{\lambda_n} \frac{v_n}{|v_n|} + \frac{G(\lambda_n, v_n)}{|v_n|}\right)
$$

we obtain from (2)

\begin{equation}
(37) \quad 0 \leq (\text{grad } g(w), A_{\lambda} w).
\end{equation}

We set

$$
\tilde{g}(x) = \frac{x_3}{\sqrt{x_1^2 + x_2^2}}, \quad x \in \mathbb{R}^3 \setminus S
$$

and obtain

$$
\text{grad } g(w)L = \text{grad } \tilde{g}(x),
$$

where $Lx = w$. We have $\tilde{g}(x) = \tau$ and simple calculation yields

$$
\text{grad } \tilde{g}(x) = \left[-\frac{\tau x_1}{x_1^2 + x_2^2}, -\frac{\tau x_2}{x_1^2 + x_2^2}, \frac{1}{\sqrt{x_1^2 + x_2^2}}\right].
$$

Consequently,

$$(\text{grad } g(w), A_{\lambda} w) = (\text{grad } g(w), LB_{\lambda} x) = (\text{grad } g(w)L, B_{\lambda} x) = (\text{grad } \tilde{g}(x), B_{\lambda} x) = -\tau(\alpha(\lambda) + \nu(\lambda)).$$

By virtue of (5) we have $\tau > 0$ and therefore by our assumption $(\text{grad } g(w), A_{\lambda} w) < 0$, which contradicts (37).

The second implication of the lemma is an easy consequence of the first one. \[\square\]
Proof of Theorem 1. We shall successively prove that the following assertions (I)-(VII) hold with some $R > 0$ sufficiently small.

(I) $\hat{\varphi}_\lambda(0, rv) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$, $r \in (0, R)$. In particular, $\hat{\varphi}_{\lambda, 0}(0, v) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$.

The second inequality (the linearized case) follows directly from the assumption (9) and Lemma 2.(V). Suppose that there exist sequences $\lambda_n \to \lambda$, $r_n \to 0$ such that

\begin{equation}
\hat{\varphi}_\lambda(0, r_n v) \leq 0, \ n = 1, 2, \ldots
\end{equation}

Remark 8 together with (2) and the fact that the cones $T_K(v), T_K(r_n v)$ coincide, imply

\begin{equation}
\frac{1}{r_n} \hat{U}_{\lambda_n}(0, r_n v) = \frac{1}{r_n} P_{r_n v}(A_{\lambda_n} r_n v + G(\lambda_n, r_n v)) \nonumber
\end{equation}

\begin{equation}
= P_v \left( A_{\lambda_n} v + \frac{1}{r_n} G(\lambda_n, r_n v) \right) - P_v A_v = \hat{U}_{\lambda, 0}(0, v).
\end{equation}

Now Remark 2 implies $\hat{\varphi}_{\lambda_n}(0, r_n v) \to \hat{\varphi}_{\lambda, 0}(0, v)$ and therefore (38) yields $\hat{\varphi}_{\lambda, 0}(0, v) \leq 0$, which is impossible by the second inequality.

(II) $\hat{\varphi}_{\lambda, 0}(T_0(\lambda, v), v) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$.

Since $g(v) = \tau$ it follows from the assumption (10) and from Lemma 3 that $g(U_{\lambda, 0}(t, v)) \leq \tau$ for all $t > 0$. Therefore Remark 5.(i) yields

\begin{equation}
U_{\lambda, 0}(T_0(\lambda, v), v) = k(\lambda) v, \ \lambda \in [\lambda_1, \lambda_2],
\end{equation}

where $k(\lambda) > 0$. By (I) we get

\begin{equation}
\hat{\varphi}_{\lambda, 0}(T_0(\lambda, v), v) = \hat{\varphi}_{\lambda, 0}(0, k(\lambda) v) = \hat{\varphi}_{\lambda, 0}(0, v) > 0.
\end{equation}

(III) There exists $T > 0$ such that $T(\lambda, rv) < T$ for all $r \in (0, R)$, $\lambda \in [\lambda_1, \lambda_2]$.

We use (II) and Lemma 1.(vi) to find that $T(\lambda_n, r_n v) \to T_0(\lambda, v)$ whenever $\lambda_n \to \lambda$, $r_n \to 0$. As a result, any such sequence $T(\lambda_n, r_n v)$ is bounded.

(IV) For any $v \in (0, R)$ and $\lambda \in [\lambda_1, \lambda_2]$ there exists a unique $k(\lambda, r) > 0$ such that $U_{\lambda}(T(\lambda, rv), rv) = k(\lambda, r) v$.

We use Lemma 3 together with (III) to obtain

\begin{equation}
g(U_{\lambda}(T(\lambda, rv), rv)) \leq \tau, \ \lambda \in [\lambda_1, \lambda_2], \ r \in (0, R).
\end{equation}

Since $rv \in \partial K$ and $g(rv) = \tau$, the statement is a direct consequence of Remark 5.(i).
Suppose
\begin{equation}
\varphi_{\lambda_n}(T(\lambda_n, rv), rv) \leq 0, \ n = 1, 2, \ldots
\end{equation}
where $\lambda_n \to \lambda$, $r_n \to 0$. It follows from (2) that $G(\lambda, 0) = 0$ and therefore $U_\lambda(t, 0) = 0$, $t \geq 0$. Lemma 1(i) together with (III) implies $U_{\lambda_n}(T(\lambda_n, r_n v), r_n v) \to 0$. We use (IV) to write $U_{\lambda_n}(T(\lambda_n, r_n v), r_n v) = k(\lambda_n, r_n) v$, $n = 1, 2, \ldots$ and so (40) yields
\[0 \geq \varphi_{\lambda_n}(T(\lambda_n, r_n v), r_n v) = \varphi_{\lambda_n}(0, k(\lambda_n, r_n) v).
\]
Since $k(\lambda_n, r_n) \to 0$, this contradicts (I).

(VI) The function $\lambda \to k(\lambda, r)$ is continuous on $[\lambda_1, \lambda_2]$ for each $r \in (0, R)$.

It follows from (III), (V) by Lemma 1,(iii) that $T(\lambda_n, rv) \to T(\lambda, rv)$ whenever $\lambda_n \to \lambda$, $\lambda_n \in [\lambda_1, \lambda_2]$ and $r \in (0, R)$ is fixed. Recalling (IV), we obtain from Lemma 1,(i) that
\[k(\lambda_n, r) v = U_{\lambda_n}(T(\lambda_n, rv), rv) \to U_\lambda(T(\lambda, rv), rv) = k(\lambda, r) v
\]
and consequently, $k(\lambda_n, r) \to k(\lambda, r)$.

(VII) We have $k(\lambda_1, r) < r < k(\lambda_2, r)$ for all $r \in (0, R)$.
Suppose $k(\lambda_1, r_n) \geq r_n > 0$, $r_n \to 0$. As in (III) we find $T(\lambda_1, r_n v) \to T_0(\lambda_1, v)$ and therefore by Lemma 1,(iv)
\[\frac{k(\lambda_1, r_n) v}{r_n} = \frac{U_{\lambda_1}(T(\lambda_1, r_n v), r_n v)}{r_n} \to U_{\lambda_1, 0}(T_0(\lambda_1, v), v).
\]
Finally,
\[|v| \leq \frac{k(\lambda_1, r_n)|v|}{r_n} - |U_{\lambda_1, 0}(T_0(\lambda_1, v), v)|,
\]
which contradicts (12).

Analogously, the assumption $k(\lambda_2, r_n) \leq r_n$, $r_n \to 0$ leads to a contradiction with (13).

It follows from (IV), (VI) and (VII) that for any $v$ satisfying (8) and for each $r \in (0, R)$ there exists a value $\lambda \in [\lambda_1, \lambda_2]$ satisfying $U_{\lambda}(T(\lambda, rv), rv) = rv$, which completes the proof. \qed

**Lemma 4.** Let $0 \neq v \in \partial K$ and let $\lambda \in \mathbb{R}$ be such that
\[0 \neq u \in \partial K \implies A_\lambda u \notin T_K(u).
\]
Then $0 \neq U_{\lambda, 0}(t, v) \in \partial K$ for $t \geq 0$. 354
Proof. Set \( U(t) = U_{\lambda,0}(t, v) \). Since \( v \notin S \) we obtain from Lemma 2,(I) that \( U(t) \neq 0 \) for all \( t \geq 0 \). Now if the statement were false there would exist \( t_0 \geq 0 \) and a sequence \( t_n \to t_0^+ \) satisfying

\[
0 \neq U(t_0) \in \partial K, \quad U(t_n) \in \text{int} K, \; n = 1, 2, \ldots
\]

We get \( T_K(U(t_n)) = \mathbb{R}^3 \) and by Remark 8 we obtain \( \dot{U}(t_n) = A_{\lambda}U(t_n) \). By the same remark we get \( P_{U(t_0)}A_{\lambda}U(t_0) = \dot{U}(t_0) = \lim_{n \to +\infty} A_{\lambda}U(t_n) = A_{\lambda}U(t_0) \) and therefore \( A_{\lambda}U(t_0) \in T_K(U(t_0)) \). This contradicts our assumption. \( \square \)

Proof of Theorem 2 is based on Theorem 1. We take an arbitrary fixed element \( v \) satisfying (8) (see Remark 3) and verify the assumptions of Theorem 1 for an interval \([\lambda_1, \lambda_2] \subset [\lambda_1, \lambda_2] \).

We set

\[
(41) \quad \delta = \sup \{ \lambda \in [\lambda_1, \lambda_2] ; T_0(\lambda, v) < +\infty \text{ for all } \lambda \in [\lambda_1, \lambda_2] \}
\]

and prove successively the following assertions (i)-(vii).

(i) We have \( \Lambda_1 < \delta \).

Let \( U(t) = X_1(t)\bar{u} + X_2(t)\bar{v} + X_3(t)\bar{w} \) be the solution of the equation \( \dot{U}(t) = A_{\lambda}U(t) \) with the initial condition \( U(0) = v \) for \( \lambda = \Lambda_1 \). Using the formulas (19) we get

\[
g(U(t)) = \frac{X_3(0)}{\sqrt{X_1^2(0) + X_2^2(0)}} e^{-(\alpha(\Lambda_1) + \nu(\Lambda_1))t} = g(v)e^{-(\alpha(\Lambda_1) + \nu(\Lambda_1))t}, \; t \geq 0,
\]

where \( \lambda = \Lambda_1 \). By virtue of (8) and (14) the last relation becomes \( g(U(t)) = \tau, \; t \geq 0 \) and from Remark 5,(ii) we conclude that \( U(t) \in K \) for all \( t \geq 0 \). Therefore \( U(t) = U_{\lambda,0}(t, v), \; t \geq 0 \) and we have

\[
(42) \quad \dot{U}_{\lambda,0}(t, v) = A_{\lambda}U_{\lambda,0}(t, v) \text{ for } \lambda = \Lambda_1, \; t \geq 0.
\]

Remark 10 implies

\[
(43) \quad \dot{\varphi}_{\lambda,0}(t, v) = \beta(\lambda) \text{ for } \lambda = \Lambda_1, \; t \geq 0.
\]

Consequently,

\[
(44) \quad T_0(\Lambda_1, v) < +\infty, \; \dot{\varphi}_{\Lambda_1,0}(T_0(\Lambda_1, v), v) > 0.
\]

Lemma 1,(iii) implies \( T_0(\lambda_n, v) \to T_0(\Lambda_1, v) \) whenever \( \lambda_n \to \Lambda_1 \). Therefore \( T_0(\lambda, v) < +\infty \) for all \( \lambda \) sufficiently close to \( \Lambda_1 \) and (41) implies (i).
(ii) $|U_{\lambda,0}(T_0(\lambda, v), v)| < |v|$ for all $\lambda$ sufficiently close to $A_1$.

We use (43) to obtain $T_0(A_1, v) = 2\pi/\beta(A_1)$ and (14) together with Remark 4 to find $U_{\lambda,0}(T_0(\lambda, v), v) = e^{2\pi A(\lambda)} v$ for $\lambda = A_1$. By the assumptions (14), (16) we get $|U_{\lambda,0}(T_0(\lambda, v), v)| = e^{2\pi A(\lambda)} |v| < |v|$ provided $\lambda = A_1$. The statement now follows from (44) and from Lemma 1, (i), (iii).

(iii) If $T_0(\delta, v) < +\infty$ then $|U_{\delta,0}(t, v)| > |v|$ for $t > 0$.

We shall first prove $\delta = \Lambda_2$.

Because of (i) and (15) we have $\alpha(\delta) + \nu(\delta) > 0$ and Lemma 3 implies $g(U_{\delta,0}(t, v)) \leq \tau$ for $t \geq 0$. Consequently, by Remark 5,(i)

$$U_{\delta,0}(T_0(\delta, v), v) = kv \text{ with some } k > 0.$$ 

Hence

$$\dot{\varphi}_{\delta,0}(T_0(\delta, v), v) = \varphi_{\delta,0}(0, kv) = \varphi_{\delta,0}(0, v).$$

According to Lemma 2,(V) the assumption $T_0(\delta, v) < +\infty$ implies $\dot{\varphi}_{\delta,0}(0, v) > 0$. Thus $\dot{\varphi}_{\delta,0}(T_0(\delta, v), v) > 0$ and from Lemma 1,(iii) we obtain that $T_0(\lambda, v) < +\infty$ for all $\lambda$ sufficiently close to $\delta$. Thus (41) implies $\delta = \Lambda_2$.

Furthermore, by virtue of (17) we can use Lemma 4 to obtain

(45) 

Thus we can use (18) together with Remark 11 to obtain

$$|U_{\delta,0}(t, v)|^2 - |v|^2 = |U_{\delta,0}(t, v)|^2 - |U_{\delta,0}(0, v)|^2$$

$$= \int_0^t 2(U_{\delta,0}(s, v), U_{\delta,0}(s, v))ds$$

$$= \int_0^t 2(A_\delta U_{\delta,0}(s, v), U_{\delta,0}(s, v))ds > 0, \ t > 0.$$ 

(iv) There exists a real constant $B$ such that

$$\frac{\dot{U}_{\lambda,0}(t, v), U_{\lambda,0}(t, v))}{\dot{\varphi}_{\lambda,0}(t, v)} \geq B |U_{\lambda,0}(t, v)|^2$$

for all $\lambda \in [A_1, \delta], \ t \in [0, T_0(\lambda, v))$.

Assume that, on the contrary, there exist sequences $\lambda_n \in [A_1, \delta], \ t_n \in [0, T_0(\lambda_n, v))$ satisfying

(46) 

$$\frac{\dot{U}_{\lambda_n,0}(t_n, v), U_{\lambda_n,0}(t_n, v))}{\dot{\varphi}_{\lambda_n,0}(t_n, v)} \leq -n|U_{\lambda_n,0}(t_n, v)|^2, \ n = 1, 2, \ldots$$
Since $U_{\lambda_n,0}(t_n, v) \neq 0$ (see Lemma 2,(1)) we can rewrite (46) as

$$\frac{\dot{U}_{\lambda_n,0}(0, u_n), u_n}{\varphi_{\lambda_n,0}(0, u_n)} \leq -n, \ n = 1, 2, \ldots$$

(47)

where

$$u_n = \frac{U_{\lambda_n,0}(t_n, v)}{|U_{\lambda_n,0}(t_n, v)|}.$$  

We may assume $u_n \rightarrow u \in K$, $\lambda_n \rightarrow \lambda \in [A_1, \delta]$ and, since $|P_{u_n}A_{\lambda_n}u_n| \leq |A_{\lambda_n}u_n| \leq C$, also

$$\dot{U}_{\lambda_n,0}(0, u_n) = P_{u_n}A_{\lambda_n}u_n \rightarrow w \in \mathbb{R}^3$$

(48)

(see Remark 8). Moreover, Remark 11 yields

$$\dot{U}_{\lambda_n,0}(0, u_n) = (A_{\lambda_n}u_n, u_n) \rightarrow (A_{\lambda}u, u).$$

(49)

On the other hand, considering (41) we obtain from Lemma 2,(V)

$$\dot{\varphi}_{\lambda,0}(t, v) > 0 \text{ for all } \lambda \in [A_1, \delta], \ t \in [0, T_0(\lambda, v)).$$

Hence

$$\dot{\varphi}_{\lambda_n,0}(0, u_n) = \dot{\varphi}_{\lambda_n,0}(t_n, v) > 0.$$  

(50)

On the other hand, (47), (49) imply

$$\dot{\varphi}_{\lambda_n,0}(0, u_n) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$  

(51)

Using Remark 2 we get

$$\langle \dot{U}_{\lambda_n,0}(0, u_n), x_{n2}\bar{u} - x_{n1}\bar{v} \rangle = (x_{n1}^2 + x_{n2}^2)\varphi_{\lambda_n,0}(0, u_n) \rightarrow 0,$$

where $Lx_n = u_n$, $Lx = u$, and consequently, (48) yields

$$\langle w, x_{2}\bar{u} - x_{1}\bar{v} \rangle = 0.$$  

(52)

Furthermore, we have

$$P_{u_n}A_{\lambda_n}u_n \in \partial T_K(u_n) \text{ for all } n \text{ sufficiently large.}$$

(53)

Indeed, if $P_{u_n}A_{\lambda_n} \in \text{int} \ T_K(u_n)$ we would get by Observation 4 that $\dot{U}_{\lambda_n,0}(0, u_n) = P_{u_n}A_{\lambda_n}u_n = A_{\lambda_n}u_n$ and by Remark 10 $\dot{\varphi}_{\lambda_n,0}(0, u_n) = \beta(\lambda_n)$. But (52) would
imply \( \beta(\lambda) = 0 \) for some \( \lambda \) in \([A_1, A_2]\), which contradicts (16). By Observation 3,(i) we conclude that (54), (48) imply \( w \in \partial T_i(u) \) for some \( i, 1 \leq i \leq N \). Recalling Observation 1 we obtain from (53) \( w = \mu u, \mu \in \mathbb{R} \). Remark 8 and (48) yield

\[
0 \leq (U_{\lambda_n, 0}(0, u_n) - A_{\lambda_n} u_n, v - u_n) - (\mu u - A_{\lambda} u, v - u) \text{ for all } v \in K
\]

and therefore \( u \) is an eigenvector of (7). Moreover, \( u \in \partial K \) because \( u \in \text{int } K \) would imply \( \phi_{\lambda_n, 0}(0, u_n) = \beta(\lambda_n) \to \beta(\lambda) > 0 \) (see Remark 10), which would contradict (52). By the assumption of Theorem 2 the eigenvalue \( \mu \) is positive. Finally, recalling Remark 9 we have \( (A_{\lambda} u, u) = \mu |u|^2 > 0 \) and therefore (49) yields \( (U_{\lambda_n, 0}(0, u_n), u_n) > 0 \) for \( n \) large. This inequality together with (51) contradicts (47).

(v) The function \( \varphi_{\delta, 0}(t, v) \) is nondecreasing on \([0, T_0(\delta, v))\).

Assume there exist \( 0 \leq t_1 < t_2 < T_0(\delta, v) \) such that \( \varphi_{\delta, 0}(t_1, v) > \varphi_{\delta, 0}(t_2, v) \). By Lemma 1

\[
\varphi_{\lambda, 0}(t_1, v) > \varphi_{\lambda, 0}(t_2, v), \quad 0 \leq t_1 < t_2 < T_0(\lambda, v)
\]

for all \( \lambda \) sufficiently close to \( \delta \). As we have proved in (i) the interval \([A_1, \delta]\) is nonempty and therefore we conclude from (55) that \( \phi_{\lambda_0, 0}(t_0, v) \leq 0 \) for some \( \lambda_0 \in [A_1, \delta] \) and \( t_0 \in [0, T_0(\lambda_0, v)) \). This contradicts (50) and (v) is proved.

(vi) If \( T_0(\delta, v) = +\infty \) then \( \lim_{t \to +\infty} |U_{\delta, 0}(t, v)| = +\infty \).

Lemma 2,(I) implies \( U_{\delta, 0}(t, v) \notin S \) for all \( t > 0 \). Thus we get from the definition of \( T_0(\delta, v) \) that \( \varphi_{\delta, 0}(t, v) < 2\pi \) for all \( t > 0 \). It follows from (v) that the function \( \varphi_{\delta, 0}(t, v) \) has a proper limit as \( t \to +\infty \). Set \( U(t) = U_{\delta, 0}(t, v) \). Then Lemma 2,(III) yields

\[
\frac{U(t)}{|U(t)|} \to u \text{ as } t \to +\infty,
\]

where \( u \in \partial K \) is an eigenvector of (7). By Remark 11 we have

\[
\frac{d}{dt} |U(t)|^2 = 2(U(t), U(t)) = 2(A_{\lambda} U(t), U(t)) = 2|U(t)|^2 \left( A_{\lambda} \frac{U(t)}{|U(t)|}, \frac{U(t)}{|U(t)|} \right),
\]

and by (56)

\[
\left( A_{\lambda} \frac{U(t)}{|U(t)|}, \frac{U(t)}{|U(t)|} \right) \to (A_{\lambda} u, u) \text{ as } t \to +\infty.
\]

Let \( \mu \) be the eigenvalue of (7) corresponding to \( u \). By the last assumption of Theorem 2, \( \mu \) is positive and Remark 9 yields \( (A_{\lambda} u, u) = \mu |u|^2 > 0 \). Consequently, (vi) follows from (57) and (58).
If \( T_0(\delta, v) = +\infty \) then \( |U_{\lambda_n, 0}(T_0(\lambda_n, v), v)| \to +\infty \) for a sequence \( \lambda_n \to \delta \).

Since \( T_0(\delta, v) = +\infty \) we use Lemma 1.1(i),(ii) to conclude from (i) and (vi) that there exist sequences \( \lambda_n \to \delta \), \( t_n \in [0, T_0(\lambda_n, v)) \) satisfying

\[
|U_{\lambda_n, 0}(t_n, v)| \to +\infty, \ n \to +\infty.
\]

To prove \( |U_{\lambda_n, 0}(T_0(\lambda_n, v), v)| \to +\infty \) we define for each \( \lambda \in [\Lambda_1, \delta) \) a function \( V_{\lambda}: [0, 2\pi] \to K \) as follows:

\[
V_{\lambda}(\varphi) = U_{\lambda, 0}(t, v) \text{ for } \varphi = \varphi_{\lambda, 0}(t, v), \ t \in [0, T_0(\lambda, v)].
\]

It follows from (50) that \( V_{\lambda}(\varphi) \) is correctly defined. Moreover, \( V_{\lambda}(\varphi) \) is absolutely continuous and right differentiable on \([0, 2\pi)\) (see Remark 8). Thus we obtain from (iv)

\[
\frac{d}{d\varphi}|V_{\lambda}(\varphi)|^2 = 2 \left( \frac{d}{d\varphi} V_{\lambda}(\varphi), V_{\lambda}(\varphi) \right) \\
= 2 \left( U_{\lambda, 0}(t, v), U_{\lambda, 0}(t, v) \right) \geq 2B|U_{\lambda, 0}(t, v)|^2 = 2B|V_{\lambda}(\varphi)|^2
\]

for some \( B < 0 \) and all \( \varphi \in [0, 2\pi) \). Now Gronwall’s lemma yields

\[
|V_{\lambda}(2\pi)|^2 \geq |V_{\lambda}(\varphi)|^2 e^{2B(2\pi - \varphi)}, \ \varphi \in [0, 2\pi).
\]

We set \( \varphi_n = \varphi_{\lambda_n, 0}(t_n, v) \in [0, 2\pi) \) and obtain

\[
|U_{\lambda_n, 0}(T_0(\lambda_n, v), v)|^2 = |V_{\lambda_n}(2\pi)|^2 \\
\geq e^{2B(2\pi - \varphi_n)}|V_{\lambda_n}(\varphi_n)|^2 \geq e^{-4\pi|B|} |U_{\lambda_n, 0}(t_n, v)|^2.
\]

The statement now follows from (59).

We shall complete the proof of Theorem 2 by finding values \( \lambda_1 < \lambda_2 \) in the interval \([\Lambda_1, \Lambda_2]\) such that the conditions (9)-(13) are valid. To do this we need to consider two cases: \( T_0(\delta, v) < +\infty \) and \( T_0(\delta, v) = +\infty \).

When \( T_0(\delta, v) = +\infty \) we use (i), (ii) and (vii) to conclude that the conditions (12), (13) hold for some \( \Lambda_1 < \lambda_1 < \lambda_2 < \delta \). In addition, (41) implies (9) and the conditions (10), (11) are guaranteed by (15), (16).

In the case \( T_0(\delta, v) < +\infty \) we find \( \lambda_1 \in (\Lambda_1, \delta) \) satisfying (12) by (i), (ii). Further, we set \( \lambda_2 = \delta \) to obtain (13) from (iii). The conditions (9), (10), (11) are obtained as above. \( \square \)
Lemma 5. Suppose that $0 \neq u \in \partial K$, $\lambda \in \mathbb{R}$ and there is $j$ such that

\begin{equation}
P_u A\lambda u = P_u^j A\lambda u.
\end{equation}

Set $x = L^{-1}u, y = L^*n_j(u)$ (the inner normal to $L^{-1}K_j$), $z = L^{-1}n_j(u)$. If

\begin{equation}
z_3 > 0, \quad x_3 > 0, \quad \beta(\lambda) - |\nu(\lambda)| \frac{\sqrt{y_1^2 + y_2^2} \sqrt{z_1^2 + z_2^2}}{y_3 z_3} > 0, \quad \beta(\lambda) > 0
\end{equation}

and $u$ is an eigenvector of (7) then the corresponding eigenvalue $\mu$ of (7) is positive.

Proof. We can suppose without loss of generality that $x_1^2 + x_2^2 = 1$ and we shall write $n$ instead of $n_j(u)$. Realize that $0 = \langle u, n \rangle = \langle x, L^*n \rangle = \langle x, y \rangle$, i.e.

\begin{equation}
-x_3 y_3 = x_1 y_1 + x_2 y_2.
\end{equation}

We have $A\lambda u \notin T_j(u)$ because otherwise (62) would yield $P_u A\lambda u = A\lambda u$ and therefore $u$ would be an eigenvector of $A\lambda$ by Remark 9. However, $A\lambda$ has no eigenvectors on $\partial K$ under the assumption (4). Hence, formula (62) yields

\begin{equation}
P_u A\lambda u = A\lambda u - (A\lambda u, n)n
\end{equation}

and by Remark 9

\begin{equation}
\mu u = A\lambda u - (A\lambda u, n)n,
\end{equation}

which is equivalent to

\begin{equation}
\mu x = B\lambda x - (B\lambda x, y)z.
\end{equation}

Multiplying this equation successively by $[x_1, x_2, 0], [x_2, -x_1, 0]$ and using (64) we obtain

\begin{align}
\mu &= \alpha - [(\alpha + \nu)(x_1 y_1 + x_2 y_2) + \beta(x_2 y_1 - x_1 y_2)](x_1 z_1 + x_2 z_2), \\
0 &= \beta - [(\alpha + \nu)(x_1 y_1 + x_2 y_2) + \beta(x_2 y_1 - x_1 y_2)](x_2 z_1 - x_1 z_2),
\end{align}

where we write $\alpha$, $\beta$, $\nu$ instead of $\alpha(\lambda)$, $\beta(\lambda)$, $\nu(\lambda)$. Set $a = x_1 y_1 + x_2 y_2$, $b = x_2 y_1 - x_1 y_2$, $c = x_2 z_1 - x_1 z_2$, $d = x_1 z_1 + x_2 z_2$.

Let us show that

\begin{equation}
c < 0, \quad y_3 > 0, \quad \frac{a}{y_3} < 0.
\end{equation}

The first inequality can be obtained from (67) by using the inequalities $\beta > 0$, $(\alpha + \nu)a + \beta b = (B\lambda x, y) = (A\lambda u, n) < 0$ (because $A\lambda u \notin T_j(u)$). The second follows
from the assumption (4) and from the fact that \( y \) is the normal to the cone \( L^{-1} K_j \) at the point \( x \). Finally, formulas (63), (64) imply \( a/y_3 = -x_3 < 0 \). Calculating \( \alpha \) from (67) and substituting in (66) we get

\[
\alpha = \frac{\beta - \nu a c - \beta b c}{a c},
\]

(69)

\[
\mu = \beta \frac{1 - b c - a d}{a c} - \nu.
\]

Also, \( (y, z) = (L^* n, L^{-1} n) = (n, n) = 1 \) and by a simple calculation we get

\[
1 - b c - a d = 1 - y_1 z_1 - y_2 z_2 = 1 - (y, z) + y_3 z_3 = y_3 z_3.
\]

Hence, we use (68), (63) to obtain from (69)

\[
\mu = \beta \frac{y_3 z_3}{a c} - \nu = \frac{y_3 z_3}{a c} (\beta - \frac{a c}{y_3 z_3} \nu) \geq \frac{y_3 z_3}{a c} \left( \beta - |\nu| \sqrt{\frac{1}{y_1^2} + \frac{1}{y_2^2} + \frac{1}{y_3^2}} \right) > 0.
\]

Example. Consider the matrix \( A_\lambda \) and the cone \( K \) in \( \mathbb{R}^3 \) defined by

\[
A_\lambda = \frac{1}{6} \begin{pmatrix} 5\lambda + 17 & -\lambda + 17 & -\lambda - 19 \\ -2\lambda - 50 & 4\lambda - 14 & -2\lambda + 22 \\ -3\lambda + 27 & -3\lambda - 9 & 3\lambda - 9 \end{pmatrix},
\]

\( K = \{ u \in \mathbb{R}^3; u_j \geq 0, j = 1, 2, 3 \} \).

The eigenvalues \( \alpha(\lambda) \pm i\beta(\lambda) = \lambda \pm 6i, -\nu(\lambda) = -1 \) clearly satisfy (14), (15), (16) with \( \Lambda_1 = -1, \Lambda_2 > -1 \) arbitrary. The corresponding eigenvectors are \( \bar{u} \pm i\bar{v} = [1, -3, 2] \pm i[2, -1, -1], \bar{w} = [1, 2, 3] \). Hence,

\[
L = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \quad L^{-1} = \frac{1}{30} \begin{pmatrix} -1 & -7 & 5 \\ 13 & 1 & -5 \\ 5 & 5 & 5 \end{pmatrix}.
\]

Our cone can be described as

\[
K = \{ u = L x; (L x)_j \geq 0, j = 1, 2, 3 \} = \{ u = x_1 \bar{u} + x_2 \bar{v} + x_3 \bar{w}; x_3 \geq f_j(x_1, x_2), j = 1, 2, 3 \},
\]

where \( f_j \) are defined by \( x_3 - f_j(x_1, x_2) = (L x)_j \).

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Suppose that $u \in \partial K$ is an eigenvector of (7) with some $\lambda \geq -1$. We shall prove that then the corresponding eigenvalue must be positive. Consider successively points $u \in \partial K$ of two types (see the notation from Section 3):

(a) $u \in \partial K_3 \cap \operatorname{int} K_1 \cap \operatorname{int} K_2$, i.e. $u = [u_1, u_2, 0], u_1 > 0, u_2 > 0$. Then $T_K(u) = T_3(u) = K_3$ and therefore (62) holds with $j = 3$. We have $n_3(u) = [0, 0, 1], y = [2, -1, 3], z = \frac{1}{6}[1, -1, 1], x = \frac{1}{30}[-u_1 - 7u_2, 13u_1 + u_2, 5u_1 + 5u_2]$ and (63) is fulfilled. Lemma 5 implies $\mu > 0$.

(b) $u \in \partial K_3 \cap \partial K_1$; we can suppose $u = [0, 1, 0]$. Then $T_K(u) = K_3 \cap K_1$, $A_\lambda u = \frac{1}{6}[-\lambda + 17, 4\lambda - 14, -3\lambda - 9]$. If $\lambda \leq 17$ then $P_u A_\lambda u = P_u^3 A_\lambda u$ and the same argument as in (a) can be used to prove $\mu > 0$. On the other hand, we use Remark 9 to obtain $\mu = (A_\lambda u, u) = \frac{1}{6}[4\lambda - 14] > 0$ when $\lambda > 17$. The cases $u \in \partial K_1 \cap \operatorname{int} K_2 \cap \operatorname{int} K_3, u \in \partial K_2 \cap \operatorname{int} K_1 \cap \operatorname{int} K_3$ and $u \in \partial K_1 \cap \partial K_2, u \in \partial K_2 \cap \partial K_3$ can be treated as (a) and (b), respectively. Summarizing all possible cases we can see that (7) can have only positive eigenvalues corresponding to eigenvectors $u \in \partial K$ if $\lambda \geq -1 = \Lambda_1$. Furthermore, considering as above the separate regions of the cone $K$, we find that the condition (17) is fulfilled with $\Lambda_2 = 20$. For instance, in the region (a) we have $A_{20}u = \frac{1}{6}[117u_1 - 3u_2, -90u_1 + 66u_2, -33u_1 - 69u_2]$ and therefore $A_{20}u \notin T_K(u) = K_3$ because $-33u_1 - 69u_2 < 0$ for points under consideration. For the points $u$ belonging to the region (b) the condition (17) for any $\lambda > -3$ follows from the expression for $A_\lambda$ written above. The other cases can be treated similarly. The assumption (18) with $\Lambda_2 = 20$ is also satisfied. For instance in the case (a) we obtain $(A_{20}u, u) = \frac{1}{6}[117u_1^2 + 66u_2^2 - 93u_1u_2] > 0$ for all $u_1 \neq 0, u_2 \neq 0$.

References


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