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Olav Jordens

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ON MODULARITY IN LATTICES OF CONGRUENCES ON ORDERED SETS

OLAV JORDENS,[1] Durban

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0. INTRODUCTION AND NOTATION

The following notions are explored in [2]. An ordered triple $\mathbf{L} = \langle O, R, ar \rangle$ is called a *language*, where O and R are pairwise disjoint sets of operation and relation symbols respectively, and ar is the arity function from $O \cup R$ onto the set of finite cardinals. (It is also specified that for all $r \in R$, we have $ar(r) > 0$.) An $\mathbf{L}$-*model* is an ordered triple $A = \langle A', O^A, R^A \rangle$, where A' is a nonempty set (called the universe of A), $O^A = \langle o^A ; o \in O \rangle$, $R^A = \langle r^A ; r \in R \rangle$, and for every $o \in O$, o^A is an operation on A' of arity $ar(o)$, and similarly for every $r \in R$, r^A is a relation on A' of arity $ar(r)$.

The category L (corresponding to $\mathbf{L}$) has all $\mathbf{L}$-models as objects and L-morphisms are maps between the universes of $\mathbf{L}$-models, which preserve both operations and relations in the usual sense.

Let X and Y be sets and let $f\colon X \to Y$ be any map. Then the *kernel* of map f is defined as $\ker f = \{\langle x, y \rangle ; f(x) = f(y)\}$.

For any subcategory K of the category L (corresponding to $\mathbf{L}$) and any K-object A, the set $\mathrm{Con}_K A$ of all K-congruences on A is defined as the set of all kernels of K-morphisms from A to any other K-object B.

In this paper we consider the special case where the language $\mathbf{L}$ has no operations and only one binary relation, and consider the full subcategory K which consists of all ordered sets (i.e. sets endowed with a reflexive, antisymmetric, and transitive relation). We will always write $A = \langle A', \leqslant^A \rangle$ for a poset where A' is the underlying set of A. Thus we have:

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[1] I am indebted to Professor Teo Sturm as this paper originated from his seminar series on Algebraic Structures.

Definition. **a.** For an ordered set A, an equivalence σ on A' is called a *congruence* on A iff there exists an ordered set B and an order preserving map $f\colon A \to B$ such that $\sigma = \ker f$. The set of all congruences on A will be denoted by $\operatorname{Con} A$.

The lattice $\operatorname{Con} A$ has been extensively studied and many results about this lattice also hold true for the lattice $\operatorname{Ce} A$ of all convex equivalences on A. (A subset X of an ordered set A is called *convex* iff for any $x_1, x_2 \in X$ and $y \in A'$, if $x_1 \leqslant^A y \leqslant^A x_2$ then $y \in X$. An equivalence relation on A is called a *convex equivalence* iff every equivalence class is a convex subset of A.) Many results about $\operatorname{Ce} A$ can be found in [6].

b. It is easily seen that $\operatorname{Ce} A$ is an algebraic closure system on the lattice $E(A')$ of all equivalences on A', and the fact that $\operatorname{Con} A$ is an algebraic closure system on the same lattice follows from [2, corollary 13]. Further it is shown in [3, sec. 36] that $\operatorname{Con} A \subseteq \operatorname{Ce} A$.

For $x, y \in A'$ we use the symbols $x <^A y$ to denote $x \leqslant^A y$ but $x \neq y$, and $x \parallel^A y$ to denote that x and y are incomparable in A. The superscript will be dropped whenever the meaning is clear. Further,

$$[x, y] \overset{\text{def}}{=} \{z\,;\ x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\} \cup \{x, y\},$$
$$[x, \infty) \overset{\text{def}}{=} \{z\,;\ x \leqslant z\}, \quad \text{and} \quad (-\infty, x] \overset{\text{def}}{=} \{z\,;\ z \leqslant x\}.$$

If is easily seen that $[x, y]$ is the smallest convex subset containing both x and y.

c. The following result comes from [4, sec. 35]: $\operatorname{Con} A = \operatorname{Ce} A$ iff

(1) for every $x, y, u, v \in A$ with $x < y,\ u < v,\ x \parallel u$, and $y \parallel v$, we have $[x, v] \cap [y, u] \neq \emptyset$.

d. In the same paper (see [4, sec. 30]) it is shown that $\operatorname{Con} A = E(A')$ iff every subchain of A has at most two elements and any two subchains of A with two elements have a nonempty intersection.

e. It is also shown there (see [4, sec. 43]) that: $\operatorname{Con} A$ is a complete sublattice of $E(A')$ iff

either (i) $\operatorname{Con} A = E(A')$,

or (ii) A is isomorphic to the ordinal sum $B \oplus C \oplus D$ where C is a nonempty chain and B and D are antichains.

Further, it is shown that (see [4, sec. 37]), under the assumption that A has an at least three element subchain, A satisfies (ii) iff:

(2) for every $x \in A'$ such that there exist $u, v \in A'$ with $u < x < v$, we have that for every $w \in A'$, either $x \leqslant w$ or $w \leqslant x$.

f. The characterisation theorem given in [3, sec. 19] for congruences on ordered sets is most useful. If X,Y are nonempty subset of A', define:

$$X \leqslant^* Y \text{ iff there are } x \in X \text{ and } y \in Y \text{ such that } x \leqslant y.$$

Further

$$\leqslant^{A/\sigma} \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \left(\leqslant^* \cap (A'/\sigma \times A'/\sigma)\right)^n.$$

We then have that the following properties are equivalent for an ordered set A and an equivalence σ on A':

(3) $\sigma \in \operatorname{Con} A$.

(4) If $n \leqslant 1$ is an integer and $X_0, \ldots, X_n \in A'/\sigma$ satisfy $X_i \leqslant^* X_{i+1}$ for $i = 0, \ldots, n-1$ and $X_n \leqslant^* X_0$, then $X_0 = \ldots = X_n$.

(5) $\left\langle A'/\sigma, \leqslant^{A/\sigma} \right\rangle$ is an ordered set .

g. We need one further result from the paper [7]. Let L be a complete lattice. We say that L is *κ-modular* (for an infinite cardinal κ) iff for every set I with $|I| < \kappa$ and families $X = \{x_i\,;\, i \in I\} \subseteq L'$ and $Y = \{y_i\,;\, i \in I\} \subseteq L'$ with $y_i \leqslant^A x_j$ whenever $i, j \in I$ and $i \neq j$, we have:

$$\bigvee\{x_i \wedge y_i\,;\, i \in I\} = (\bigwedge X) \wedge (\bigvee Y).$$

L is called *completely modular* iff L is κ-modular for every infinite cardinal κ. It is shown that:

(6) a lattice is modular iff it is ω-modular, where ω is the least infinite cardinal (see [7, sec. 1]).

and also that

(7) every modular algebraic lattice is completely modular. (See [7, sec. 5]).

1. Modularity

1.a Lemma. *For ordered set A, let $B \subseteq \operatorname{Ce} A$ such that*

(i) *$\langle B, \subseteq \rangle$ is a lattice and $\sigma \cap \tau \in B$ for every $\sigma, \tau \in B$, and*

(ii) *If X is a convex subset of A, then $X^2 \cup \mathrm{id}_{\mathbf{A}} \in B$.*

Then the modularity of $\langle B, \subseteq \rangle$ implies that A satisfies the property (2) of section 0.e above.

Proof. Suppose that A does not have the property. Then there are four different elements $x, y, z, w \in A'$ such that $x < y < z$ and $y \parallel w$. Consider the following equivalences on A':

$$\varrho \stackrel{\text{def}}{=} [x, w]^2 \cup \mathrm{id}_{\mathbf{A}},$$
$$\tau \stackrel{\text{def}}{=} [w, z]^2 \cup \mathrm{id}_{\mathbf{A}},$$
$$\sigma \stackrel{\text{def}}{=} \left([y, \infty) \cup [w, \infty)\right)^2 \cup \mathrm{id}_{\mathbf{A}}.$$

It is evident that $\sigma, \tau, \varrho \in B$ and also that $\sigma \supseteq \tau$.

Firstly, we show that $\sigma \cap \varrho = \mathrm{id}_{\mathbf{A}}$. Let $\langle a, b \rangle \in \sigma \cap \varrho$ and suppose $a \neq b$. Then $a, b \in [x, w]$ and $a, b \in [y, \infty) \cup [w, \infty)$. Consider the element a. Clearly, $a \neq x$ for in the opposite case we would have $y \leqslant x$ or $w \leqslant x < y$! Now, if $a \neq w$ then we have $x < a < w$ or $w < a < x$ together with $y \leqslant a$ or $w \leqslant a$. It is readily seen that all four possibilities lead to a contradiction. Hence we must have $w = a$. A similar argument shows that $w = b$ and hence $a = b$! This contradiction shows $\sigma \cap \varrho = \mathrm{id}_{\mathbf{A}}$. Thus we deduce that

$$(\sigma \cap \varrho) \vee_B \tau = \tau.$$

Since $x \varrho w$ and $w \tau z$, we have $\langle x, z \rangle \in \varrho \vee_B \tau \in \operatorname{Ce} A$, and so $\{x, y, z, w\}^2 \subseteq \varrho \vee_B \tau$. Hence $\{y, z, w\}^2 \subseteq \sigma \cap (\varrho \vee_B \tau)$. But $y \notin [w, z]$ and thus $\sigma \cap (\varrho \vee_B \tau) \neq (\sigma \cap \varrho) \vee_B \tau$, i.e. $\langle B, \subseteq \rangle$ is not modular. □

1.b Proposition. *Let $|A'| \geqslant 4$. Then the modularity of $\operatorname{Con} A$ or the modularity of $\operatorname{Ce} A$ implies that A has an at least three element subchain.*

Proof. Notice that $E(A')$ is modular iff $|A'| \leqslant 3$. Suppose now that every subchain of A has at most two elements. Then evidently $\operatorname{Ce} A = E(A')$ (see [4, sec. 27] for a characterisation of this equality), and so, since $|A'| \geqslant 4$ we have that $\operatorname{Ce} A$ is nonmodular. We show that $\operatorname{Con} A$ is not modular either. We can decompose A' into a disjoint union of sets as follows: $A' = R \cup S \cup T$, where $R = \{a \in A';\ (\exists b \in A')\ b < a\}$ and $S = \{a \in A';\ (\exists b \in A')\ a < b\}$ and $T = A \setminus (R \cup S)$. If either $|R| \leqslant 1$ or $|S| \leqslant 1$ then by section 0.d, we have $\operatorname{Con} A = E(A')$ and again $\operatorname{Con} A$ is not modular.

Thus we suppose $|R| \geqslant 2$ and $|S| \geqslant 2$. It is then easy to see that there are four different elements $x, y, z, w \in A'$ such that $x < y$ and $z < w$. Define the following equivalences on A':

$$\varrho \stackrel{\text{def}}{=} \{x, w\}^2 \cup \mathrm{id}_A,$$
$$\tau \stackrel{\text{def}}{=} \{y, z\}^2 \cup \mathrm{id}_A,$$
$$\sigma \stackrel{\text{def}}{=} \{y, z, w\}^2 \cup \mathrm{id}_A .$$

Evidently $\varrho, \sigma, \tau \in \operatorname{Con} A$ and $\sigma \supset \tau$. Now $\sigma \cap \varrho = \mathrm{id}_A$ and so $(\sigma \cap \varrho) \vee \tau = \tau$, where $\vee$ denotes supremum in $\operatorname{Con} A$. In $\varrho \vee \tau$ there are blocks W and W' such that $\{x, w\} \subseteq W$ and $\{y, z\} \subseteq W'$. Since $W \leqslant^* W' \leqslant^* W$ we deduce from the characterisation theorem of section 0.f, property (4), that $W = W'$. Hence $\{x, y, z, w\}^2 \subseteq \varrho \vee \tau$ and so $\sigma \cap (\varrho \vee \tau) = \sigma$. Thus $\operatorname{Con} A$ is not modular. □

1.c Lemma. *If X is a convex subset of A then $\operatorname{Con}\langle X, \leqslant^A \cap X^2\rangle$ is embeddable into $\operatorname{Con} A$ in such a way that all nonempty infima and suprema are preserved. If $\sigma \in \operatorname{Con} A$, then $\operatorname{Con}\langle A'/\sigma, \leqslant^{A/\sigma}\rangle$ is isomorphic to the principal filter in $\operatorname{Con} A$ generated by σ.*

P r o o f. To prove the first statement define $f\colon \operatorname{Con} X \to \operatorname{Con} A$ by $f(\sigma) = \sigma \cup \mathrm{id}_A$. It is easily verified that f is the required mapping.

For the second statement, denote the principal filter by $[\sigma)$ and let $A/\sigma = \langle A'/\sigma, \leqslant^{A/\sigma}\rangle$. Define two maps

$$g\colon \operatorname{Con}(A/\sigma) \to [\sigma) \quad \text{and} \quad h\colon [\sigma) \to \operatorname{Con}(A/\sigma)$$

by: for $\varrho \in \operatorname{Con}(A/\sigma)$, $g(\varrho) \stackrel{\text{def}}{=} \{\langle x, y\rangle \,;\, \langle x/\sigma, y/\sigma\rangle \in \varrho\}$, and for $\tau \in [\sigma)$, $h(\tau) \stackrel{\text{def}}{=} \{\langle x/\sigma, y/\sigma\rangle \,;\, \langle x, y\rangle \in \tau\}$. It is easily verified that g and h are mutually inverse isomorphisms. □

1.d Theorem. *Let A be an ordered set with $|A'| \geqslant 4$. The following properties are equivalent:*

(i) $\operatorname{Con} A$ *is modular.*

(ii) $\operatorname{Con} A$ *is completely modular.*

(iii) $\operatorname{Ce} A$ *is modular.*

(iv) $\operatorname{Ce} A$ *is completely modular.*

(v) *A has an at least three element subchain and A is isomorphic to the ordinal sum $B \oplus C \oplus D$ where C is nonempty chain and B and D are antichains of at most two elements.*

P r o o f. Since $\operatorname{Con} A$ and $\operatorname{Ce} A$ are know to be algebraic lattices, we see by section 0.g, property (7), that (i) implies (ii), and also that (iii) implies (iv). Also, as complete modularity implies modularity by section 0.g, property (6), and using lemma 1.a and proposition 1.b, we see that from either assumption (ii) or from assumption (iv) we can deduce that A satisfies the property (2) of section 0.e, and also that A has an at least three element subchain. As mentioned in section 0.e, property (2), under the assumption that A has an at least three element subchain, is equivalent to A being isomorphic to an ordinal sum $B \oplus C \oplus D$ where C is nonempty chain and B and D are antichains. It remains to show that $|B'| \leqslant 2$ and $|D'| \leqslant 2$. Also note that by section 0.c we have that $\mathrm{Con}\, A = \mathrm{Ce}\, A$ and so we refer only to $\mathrm{Con}\, A$. Since C' is nonempty, pick $c \in C'$, and let $X = (-\infty, c]$ and $Y = X \cap C'$. Then both X and Y are convex subsets of A, and so by lemma 1.c, we have that $\mathrm{Con}\left\langle X, \leqslant^A \cap X^2 \right\rangle$ is modular also. Define $\sigma = Y^2 \cup \mathrm{id}_A$. Then $\sigma \in \mathrm{Con}\left\langle X, \leqslant^A \cap X^2 \right\rangle$ and hence $\mathrm{Con}(X/\sigma)$ is modular by lemma 1.c. Now in view of proposition 1.b we have immediately that $\mathrm{Con}(X/\sigma) = E(X/\sigma)$ and $|X/\sigma| \leqslant 3$. Thus $|B'| \leqslant 2$. Similarly we show $|D'| \leqslant 2$. □

All that remains is to prove that (v) implies both (i) and (iii). However, under condition (v), we see from section 0.c that $\mathrm{Con}\, A = \mathrm{Ce}\, A$ and hence condition (i) and (iii) are the same. We will therefore only mention $\mathrm{Con}\, A$. So assume (v) and let $\sigma, \varrho, \tau \in \mathrm{Con}\, A$ with $\sigma \supseteq \tau$. We show that $(\sigma \cap \varrho) \vee \tau \supseteq \sigma \cap (\varrho \vee \tau)$ where $\vee$ denotes the supremum in $\mathrm{Con}\, A$. Take $x, y \in A'$ with $x \neq y$ and $\langle x, y \rangle \in \sigma \cap (\varrho \vee \tau)$. Then $x \sigma y$ and by section 0.e we have that $\mathrm{Con}\, A$ is a complete sublattice of $E(A')$ and so there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ of not necessarily distinct elements of A' such that $x = x_0 \varrho x_1 \tau \ldots \varrho x_n = y$. Suppose that the sequence is as short as possible. There are two possibilities:

CASE 1. $x \parallel y$: By (v) we have in this case that either $B' = \{x, y\}$ or $D' = \{x, y\}$. Assume the former, a similar argument holding for the latter. Let x_k be the first element of the sequence different from x, and let x_l be the last element of the sequence different from y. (From the assumption that the sequence is as short as possible, we have that $k = 1$ or $k = 2$, and also that $l = n - 2$ or $l = n - 1$.) If $x_k = y$ or if $x_l = x$ then, as $x \sigma y$, it is readily seen that $\langle x, y \rangle \in (\sigma \cap \varrho) \vee \tau$. So we suppose $x_k \neq y$ end $x_l \neq x$. Since, by definition $x_k \neq x$ and $x_l \neq y$ we have $x_k, x_l \in C' \cup D'$. If both $x_k, x_l \in D'$ then by the convexity of the congruence classes and by the nonemptiness of C', we have that:

(*) There exists a $c \in C'$ such that $x(\varrho \cup \tau)c(\varrho \cup \tau)y$.

The other possibility is that x_k and x_l are comparable, but then the smaller of the two serves as a $c \in C'$ as required above. Hence in either case (*) must hold and so we have 4 possibilities:

- $x \varrho c \varrho y$ implies $x \varrho y$ and so $\langle x, y \rangle \in \sigma \cap \varrho \subseteq (\sigma \cap \varrho) \vee \tau$.

- $x\tau c\tau y$ implies $\langle x, y\rangle \in \tau \subseteq (\sigma \cap \varrho) \vee \tau$.
- $x\varrho c\tau y$ together with $\tau \subseteq \sigma$ and $x\sigma y$ yields $c\sigma y\sigma x$ and so $x\sigma c$ and $\langle x, y\rangle \in (\sigma \cap \varrho) \vee \tau$.
- $x\tau c\varrho y$ together with $\tau \subseteq \sigma$ and $y\sigma x$ yields $y\sigma x\sigma c$ and so $c\sigma y$ and $\langle x, y\rangle \in (\sigma \cap \varrho) \vee \tau$.

CASE 2. $x < y$ or $y < x$: We assume the former, a similar argument holding for the latter. If $x \in B'$ then for every x_i we have $x \leqslant x_i$ or $x \parallel x_i$. If $x \in C'$ then let x_k be the last element of the sequence with $x_k \leqslant x$. Then $x_{k+1} > x$ and by the convexity of the congruence class containing x_k and x_{k+1} we see that $x(\varrho \cup \tau)x_{k+1}$. However, the sequence was assumed to be as short as possible and so we may assume that in either case $x \leqslant x_i$ or $x \parallel x_i$ for all $i = 0, \ldots, n$. Similarly we may assume that for all $i = 0, \ldots, n$ we have $x_i \leqslant y$ or $x_i \parallel y$. Now for all the x_i such that $x \leqslant x_i \leqslant y$ we have by the convexity of the σ-equivalence classes that $x\sigma x_i \sigma y$. If $x_k \parallel x$ then $x \neq x_k \neq y$, and $x_{k-1}\varrho x_k \tau x_{k+1}$ or else $x_{k-1}\tau x_k \varrho x_{k+1}$. Since $\tau \subseteq \sigma$, and the sequence is as short as possible, we can deduce that there is an x_l with $x_k \neq x_l$ such that $x_k \sigma x_l$. If $x_l \neq x$ then either $x \leqslant x_l \leqslant y$ or $y \parallel x_l$. In the first case we have immediately that $x\sigma x_l$, but in the second case we use the fact that $C' \neq \emptyset$ and so, picking $c \in C'$, we see $x_l \sigma c \sigma x$. Thus in all cases we have $x_l \sigma x$, and so $x_k \sigma x$. A similar argument applies to any x_k which is incomparable to y. Thus we have that $\{x_0, \ldots, x_n\}^2 \subseteq \sigma$. Hence it follows that $\langle x, y\rangle \in (\sigma \cap \varrho) \vee \tau$.

Thus under the assumption (v) we have shown that $\operatorname{Con} A = \operatorname{Ce} A$ is modular and this completes the proof of the theorem.

2. n-PERMUTABILITY

In the light of B. Jónson's result that every 3-permutable sublattice of an equivalence lattice is modular (see [1, theorem 4.67]), the following theorem is quite surprising:

2.a Theorem. *For any ordered set $A = \langle A', \leqslant\rangle$ and for any natural number $n \geqslant 2$, the following conditions are equivalent:*

(i) $\operatorname{Con} A$ *in n-permutable.*

(ii) $\operatorname{Ce} A$ *is n-permutable.*

(iii) $E(A')$ *is n-permutable.*

(iv) $|A'| \leqslant n$.

Proof. We first define our notation. For equivalences σ and τ, define $(\sigma, \tau)^1 = \sigma$. Then we define recursively:

$$(\sigma, \tau)^{2n} = (\sigma, \tau)^{2n-1} \circ \tau, \quad \text{for } n \geqslant 1, \text{ a natural number, and}$$
$$(\sigma, \tau)^{2n+1} = (\sigma, \tau)^{2n} \circ \sigma, \quad \text{for } n \geqslant 1, \text{ a natural number.}$$

Then the condition that σ and τ are n-permutable can be expressed as $(\sigma, \tau)^n = (\tau, \sigma)^n$.

(iv) $\Rightarrow$ (iii): Let $\sigma, \tau \in E(A')$ and suppose $\langle x, y\rangle \in (\sigma, \tau)^n$. Then there exist $n+1$ elements $z_0, \ldots, z_n$ such that $x = z_0\sigma z_1\tau \ldots \sigma z_n = y$ or $x = z_0\sigma z_1\tau \ldots \tau z_n = y$, depending on whether n is even or odd. However, we have assumed that $|A'| \leqslant n$ and so for some $j \neq k$ we have $z_j = z_k$. Hence, either $\langle x, y\rangle \in (\sigma, \tau)^l$ or $\langle x, y\rangle \in (\tau, \sigma)^l$ for some $l < n$, and so using the reflexivity we see that $\langle x, y\rangle \in (\tau, \sigma)^n$.

(iii) $\Rightarrow$ (ii): Immediate as $\mathrm{Ce}\, A \subseteq E(A')$.

(ii) $\Rightarrow$ (i): Immediate as $\mathrm{Con}\, A \subseteq \mathrm{Ce}\, A$.

(i) $\Rightarrow$ (iv): For ease of notation we consider the case where n is odd. The case for n even follows *mutatis mutandis*. Suppose to the contrary that $|A'| \geqslant n + 1$ and let $n + 1 = 2m$. Let $a_1, \ldots, a_m, b_1, \ldots, b_m$ denote $n + 1$ different elements of A'. By Szpilrajn's result (see [8]), there exists a linear order $\preceq$ which extends the original order of $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$ as a subordered set of A. We may as well assume that $a_1 \prec b_1 \prec a_2 \prec b_2 \prec \ldots \prec a_m \prec b_m$.

Define the following relations on A':

$$\sigma \overset{\text{def}}{=} [a_1, b_1]^2 \cup [a_2, b_2]^2 \cup \ldots \cup [a_m, b_m]^2 \cup \mathrm{id}_{\mathbf{A}},$$
$$\tau \overset{\text{def}}{=} [b_1, a_2]^2 \cup [b_2, a_3]^2 \cup \ldots \cup [b_{m-1}, a_m]^2 \cup \mathrm{id}_{\mathbf{A}},$$

where the intervals $[a_i, b_i]$ and $[b_j, a_{j+1}]$ are taken in the ordered set A. □

O b s e r v a t i o n s.

(a) $a_i \in [a_j, b_j]$ implies $i = j$; $b_i \in [a_j, b_j]$ implies $i = j$;
$a_i \in [b_j, a_{j+1}]$ implies $i = j + 1$; $b_i \in [b_j, a_{j+1}]$ implies $i = j$.

We prove the first statement, the other three follow analogously. Suppose $i \neq j$. Then as a_i, a_j, b_j are different elements, we have $[a_j, b_j] \neq \{a_j, b_j\}$. Hence, $a_j \leqslant a_i \leqslant b_j$ or $b_j \leqslant a_i \leqslant a_j$. By definition of $\preceq$ we have $a_j \preceq a_i \preceq b_j$ or $b_j \preceq a_i \preceq a_j$. In the first case we have $j \leqslant i \leqslant j$, i.e. $i = j$, and in the second case we have $j + 1 \leqslant i \leqslant j$! In both cases we contradict our original assumption and hence we conclude $i = j$.

(b) For $i \neq j$ we have $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ and $[b_i, a_{i+1}] \cap [b_j, a_{j+1}] = \emptyset$, and for $i \neq j$ and $i \neq j + 1$ we have $[a_i, b_i] \cap [b_j, a_{j+1}] = \emptyset$.

Again, we prove only the first statement; the other two follow analogously. Suppose to the contrary that there exists $y \in [a_i, b_i] \cap [a_j, b_j]$ for $i \neq j$. Then by observation (a) above we see that $y \notin \{a_i, b_i, a_j, b_j\}$. Thus we must have $a_i \leqslant y \leqslant b_i$ and $a_j \leqslant y \leqslant b_j$. (Notice that $b_i \leqslant y \leqslant a_i$ and $b_j \leqslant y \leqslant a_j$ are both impossible as they imply $b_i \preceq a_i$ and $b_j \preceq a_j$ respectively.) Hence we have $a_i \preceq b_j$ and $a_j \preceq b_i$ which implies $i \leqslant j \leqslant i$ i.e. $i = j$! This contradiction proves the result.

(c) For each i we have $[a_i, b_i] \cap [b_i, a_{i+1}] = \{b_i\}$ and also $[b_i, a_{i+1}] \cap [a_{i+1}, b_{i+1}] = \{a_{i+1}\}$.

We show the first statement. By observation (a) we have that $a_i \notin [b_i, a_{i+1}]$ and $a_{i+1} \notin [a_i, b_i]$. Suppose that $y \in [a_i, b_i] \cap [b_i, a_{i+1}]$ and that $y \neq b_i$. Then $a_i \leqslant y \leqslant b_i$ and $b_i \leqslant y \leqslant a_{i+1}$. (Again the other possibilites are excluded as in observation (b) above.) Thus $b_i \leqslant y \leqslant b_i$ i.e. $y = b_i$! This contradiction gives the result.

Claim. $\sigma, \tau \in \operatorname{Con} A$.

It follows from observation (b) that $\sigma, \tau \in E(A')$. We show that $\sigma \in \operatorname{Con} A$, a similar argument holds for τ. Suppose $\sigma \notin \operatorname{Con} A$. Then, by the characterisation theorem of section 0.f, property (4), there exists a sequence $X_1, \ldots, X_k$ of distinct elements of A'/σ with $k \geqslant 2$ such that $X_1 \leqslant^* X_2 \leqslant^* \ldots \leqslant^* X_k \leqslant^* X_1$. Lets assume that k is the shortest length of such a sequence. If $k = 2$, then neither of the X_i are singletons by the convexity of the σ-equivalence classes. If $k > 2$ then there is also no singleton class, as its removal would yield a similar sequence of shorter length. Thus all the X_i's are nontrivial equivalence classes. Hence, let $X_i = [a_{l_i}, b_{l_i}]$ for $i = 1, \ldots, k$. Now take $i, j \in \{1, \ldots, k\}$ and $i \neq j$, and suppose $X_i \leqslant^* X_j$. Then there exist $u \in X_i$ and $v \in X_j$ with $u \leqslant^A v$. There are four possibilities as ($u \in \{a_{l_i}, b_{l_i}\}$ or $a_{l_i} \leqslant^A u \leqslant^A b_{l_i}$) and ($v \in \{a_{l_j}, b_{l_j}\}$ or $a_{l_j} \leqslant^A v \leqslant^A b_{l_j}$). It is a simple verification to show that all four possibilities yield $l_i \leqslant l_j$. But then we have $l_1 \leqslant l_2 \leqslant \ldots \leqslant l_k \leqslant l_1$! This shows that no such cycle can exist and hence $\sigma \in \operatorname{Con} A$.

We now complete the proof of the theorem. By the definition of σ we have

$$a_1 \sigma b_1 \tau a_2 \sigma b_2 \tau \ldots \sigma b_{m-1} \tau a_m \sigma b_m,$$

and hence $\langle a_1, b_m \rangle \in (\sigma, \tau)^{2m-1} = (\sigma, \tau)^n$. The proof is complete once we have shown that $\langle a_1, b_m \rangle \notin (\tau, \sigma)^n$. Suppose to the contrary. Then there exists a sequence $x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}$ of elements of A' such that

$$a_1 \tau x_1 \sigma y_1 \tau x_2 \sigma y_2 \tau \ldots \tau x_{m-1} \sigma y_{m-1} \tau b_m.$$

If any $x_i = y_i$, then by the transitivity of τ we can form a shorter sequence by the removal of both x_i and y_i which yields $\langle a_1, b_m \rangle \in (\tau, \sigma)^{n-2}$. Similarly, if any $y_i = x_{i+1}$, then by the transitivity of σ we can shorten the sequence by the removal of both y_i and x_{i+1}. Repeating this process we can transform the above sequence to obtain

$$a_1 \tau u_1 \sigma w_1 \tau u_2 \sigma w_2 \tau \dots \tau u_k \sigma w_k \tau b_m$$

where $k \leqslant m - 1$ and for $i = 1, \dots, k$ we have $u_i \neq w_i$ and for $i = 1, \dots, k-1$ we have $w_i \neq u_{i+1}$. (Obviously $k \geqslant 1$).

We now arrive at a contradiction. By observation (a), we see immediately that $a_1/\tau = \{a_1\}$. Hence $u_1 = a_1$. Now $w_1 \in (a_1/\sigma) \cap (u_2/\tau)$ and since $w_1 \neq u_2$, we see that $u_2/\tau \neq \{u_2\}$, and so $w_1 \in [a_1, b_1] \cap [b_l, a_{l+1}]$ for some $l = 1, \dots, k-1$. By observations (b) and (c) we have $l = 1$ and $w_1 = b_1$. Proceed by induction: Let $1 \leqslant j < k$ and suppose we have shown that $u_j = a_j$ and $w_j = b_j$. Then consider the subsequence

$$\dots \tau a_j \sigma b_j \tau u_{j+1} \sigma w_{j+1} \tau \dots$$

of the above sequence. Then $u_{j+1} \in (b_j/\tau) \cap (w_{j+1}/\sigma)$, and as $u_{j+1} \neq w_{j+1}$ we have $w_{j+1}/\sigma \neq \{w_{j+1}\}$. Thus $u_{j+1} \in [b_j, a_{j+1}] \cap [a_l, b_l]$ for some $l = 1, \dots, k$. Observations (b) and (c) give us that either $u_{j+1} = b_j$, or $u_{j+1} = a_{j+1}$. However, the former is not possible by our restrictions on the sequence. Hence we conclude that $u_{j+1} = a_{j+1}$ and similar argument shows that $w_{j+1} = b_{j+1}$. Thus by induction we have that for all $i = 1, \dots, k$ we must have $u_i = a_i$ and $w_i = b_i$. Especially, we see that $b_k \tau b_m$ where $k \leqslant m - 1$, i.e. $b_m \in [b_k, a_{k+1}]$ for $k \leqslant m - 1$. However, observation (a) shows that this is impossible!

Thus, finally, $\langle a_1, b_m \rangle \notin (\tau, \sigma)^n$ and hence we have shown that $|A'| \geqslant n + 1$ implies $\operatorname{Con} A$ is not n-permutable. This completes the proof of the theorem.

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Author's address: Department of Mathematics and Applied Mathematics, University of Natal, King George V Avenue, Durban 4001, South Africa.