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ON EQUALIZERS IN THE CATEGORY OF FRAMES WITH WEAKLY OPEN HOMOMORPHISMS

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The category of frames with weakly open homomorphisms, we will denote it by Frm\textsubscript{wo}, was introduced and investigated by B. Banaschewski and A. Pultr (cf. [1]) in connection with the study of booleanization. The term "weakly open" is motivated by the fact that a frame homomorphism associated with a continuous mapping \( f \) of a topological space possesses this property if and only if for each non-empty open set \( U \) in this space we have \( \text{int} f(U) \neq \emptyset \). As the category of frames with weakly open homomorphisms contains the category of Boolean frames as a reflective subcategory (cf. [1]), it cannot be complete. There is no obvious obstruction to completeness. The existence of products is easily seen. In this paper we investigate the structure of equalizers in the category Frm\textsubscript{wo} and show that there are couples of morphisms which fail to have them.

For the fundamental properties of frames the reader is referred to [2].

Recall that a frame is a complete lattice \( L \) in which \( a \land \bigvee \{a_i \mid i \in I\} = \bigvee \{a \land a_i \mid i \in I\} \) holds for any elements \( a \in L, a_i \in L (i \in I) \). Every frame is relatively pseudocomplemented and so pseudocomplemented. For the sake of clarity, when denoting frames we add an associated pseudocomplementation symbol, for example \( (F, \ast) \), whenever it is necessary. Here is a list of some properties of pseudocomplementation:

\begin{align*}
\text{(P1)} & \quad a^{***} = a^* \\
\text{(P2)} & \quad (a \lor b)^* = a^* \land b^* \\
\text{(P3)} & \quad (a \land b)^{**} = a^{**} \land b^{**} \\
\text{(P4)} & \quad (a \lor a^*)^{**} = 1 \\
\text{(P5)} & \quad (a \lor a^*)^* = 0 \\
\text{(P6)} & \quad a^* = b^{**} \iff a^{**} = b^* \\
\text{(P7)} & \quad a^{**} \land a^* = 0 \\
\text{(P8)} & \quad 0^* = 1, 1^* = 0, 0^{**} = 0, 1^{**} = 1
\end{align*}
A lattice homomorphism $f: E \to F$ of a frame $E$ to a frame $F$ is said to be a frame homomorphism if $f(\bigvee\{a_i \mid i \in I\}) = \bigvee\{f(a_i) \mid i \in I\}$ holds for any elements $a_i \in E$ ($i \in I$). A frame homomorphism $f: E \to F$ of a frame $(E, \cdot)$ to a frame $(F, \cdot)$ is weakly open if $f(a^{**}) \leq f(a^{**})$ for any element $a \in E$. A subframe $(A, \cdot)$ of a frame $(F, \cdot)$ is weakly open in $F$ if the canonical embedding is a weakly open homomorphism, i.e. $a^{**} \leq a^{**}$ for any element $a \in A$.

**Proposition 1.** A subframe $A$ of a frame $F$ is weakly open in $F$ if and only if for any $a \in A$ there exists $b \in A$ such that $a^* = b^{**}$.

**Proof.** Assume that $(A, \cdot)$ is weakly open in $(F, \cdot)$. Let $a$ be an element of $A$. We have $1 = (a \lor a^*)^{**} \leq (a \lor a^*)^{**}$ by (P4), hence $0 = (a \lor a^*)^* = a^* \land (a^*)^*$ by (P8), (P6) and (P2). Since $a^* \leq a^*$, we have $a^{**} \leq (a^*)^*$ according to (P9). Summing up, $a^{**} = (a^*)^*$. Put $b := a^*$ and apply (P6). Conversely, assume that for any $a \in A$ there exists $b \in A$ such that $a^* = b^{**}$. Let $a$ be an element of $A$ and $b$ the associated element such that $a^* = b^{**}$. Then $0 = a \land a^* = a \land b^{**} \geq a \land b$, hence $b \leq a^*$ and therefore $a^{**} \leq b^* \leq b^* = a^{**}$. □

For a sublattice $A$ of a frame $(F, \cdot)$ denote $M(A) := \{a \in A \mid \exists b \in A, a^* = b^{**}\}$.

**Remark.** We can rewrite Proposition 1 using $M(A)$: A subframe $A$ of a frame $F$ is weakly open in $F$ if and only if $A = M(A)$.

**Lemma 1.** Let $A$ be a sublattice of a frame $F$. Then $M(A)$ is a sublattice of $F$.

**Proof.** Let $a_1, a_2 \in M(A)$, let $b_1, b_2 \in A$ be those elements for which $a_k^* = b_k^{**}$ ($k = 1, 2$). In view of (P6) also $b_1, b_2 \in M(A)$. Then $(a_1 \land a_2)^* = (a_1 \land a_2)^{***} = (a_1^* \land a_2^*)^* = (b_1 \lor b_2)^* = (b_1 \lor b_2)^{**}$. We have used (P1), (P3) and (P2). Similarly, $(b_1 \lor b_2)^* = (a_1 \lor a_2)^{**}$. Inasmuch as $a_1 \land a_2, a_1 \lor a_2, b_1 \lor b_2, b_1 \lor b_2 \in A$, we obtain $a_1 \land a_2, a_1 \lor a_2 \in M(A)$. □

**Lemma 2.** The operator $M$ is order-preserving and idempotent on the set of all sublattices of the frame $F$ ordered by inclusion, that is $A \subseteq B \implies M(A) \subseteq M(B)$ and $M(M(A)) = M(A)$.

**Proof.** The proof is straightforward. □

As an immediate consequence of this lemma and Proposition 1 we obtain

**Lemma 3.** (a) Let $A$ be a subframe of a frame $F$. If $M(A)$ is a subframe of $F$, then it is weakly open in $F$.

(b) Let $A$ be a subframe of a frame $F$, and let $B$ be a subframe of $A$ weakly open in $F$. Then $B \subseteq M(A)$. 

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(c) Let $A$ be a finite subframe of a frame $F$. Then $M(A)$ is a weakly open subframe of $F$.

Remark. If $A$ is an infinite subframe of $F$, then $M(A)$ need not be a subframe of $F$.

**Lemma 4.** Let $A$ be a subframe of a frame $F$. Then the largest subframe of $A$ weakly open in $F$, if it exists, equals $M(A)$.

**Proof.** Let $A$ be a subframe of a frame $F$, and let $B$ be the largest subframe of $A$ weakly open in $F$. According to Lemma 3, $B \subseteq M(A)$. Now let $a \in M(A)$. By definition, there is an element $b \in A$ such that $a^* = b^*$. By (P6) we obtain $a^{**} = b^*$, and so $b \in M(A)$. $C := \{a, b, a \lor b, 0, 1\}$ is a subframe of $A$ weakly open in $F$ since $a^{**} = b^*$, $b^{**} = a^*$, $0^{**} = 1^*$, $1^{**} = 0^*$ and $(a \lor b)^{**} = (a^* \land b^*)^* = (a^* \land a^{**})^* = 0^*$ by (P8), (P2) and (P7). Hence $a \in C \subseteq B$. Consequently, $M(A) \subseteq B$.  

**Proposition 2.** Let $A$ be a subframe of a frame $F$. Then the following conditions are equivalent:

(i) there exists the largest subframe of $A$ weakly open in $F$;

(ii) $M(A)$ is a subframe of $F$;

(iii) $M(A)$ is the largest subframe of $A$ weakly open in $F$.

**Proof.** (i) $\implies$ (iii) has been just proved.

(iii) $\implies$ (ii) follows a fortiori.

(ii) $\implies$ (i) by Lemma 3.

It is well-known that substructures of an algebraic structure (subalgebras, sublattices, subframes) form a topped intersection structure. This is not the case for weakly open subframes.

**Proposition 3.** Weakly open subframes of a frame fail to form an intersection structure. Even finite intersections of finite weakly open subframes need not be weakly open.

**Proof.** Here is a counterexample:

(F, •)  (A, •)  (B, ◦)  (A \cap B, □)
\[0^{**} = 0 \leq 0 = 0^{**}, \quad 0^{oo} = 0 \leq 0 = 0^{oo},\]
\[a^{**} = a \leq a = a^{**}, \quad a^{oo} = a \leq a = a^{oo}, \quad a^{00} = 1 \nless a = a^{**},\]
\[b^{**} = b \leq d = b^{**}, \quad d^{oo} = d \leq d = d^{oo},\]
\[c^{**} = 1 \leq 1 = c^{**}, \quad e^{oo} = 1 \leq 1 = e^{oo},\]
\[1^{**} = 1 \leq 1 = 1^{**}, \quad 1^{oo} = 1 \leq 1 = 1^{oo}.\]

The problem of finding equalizers is a bit more complicated. We will take advantage of the following decomposition lemma, formulated also in [1]. As this article is not yet published, we present a proof.

**Lemma 5.** (Decomposition lemma.) Let \(h: E \to F\) be a weakly open homomorphism of a frame \(E\) to a frame \(F\). Then \(h(E)\) is a weakly open subframe in \(F\) and the induced homomorphism of \(E\) onto \(h(E)\) is weakly open.

**Proof.** Since \(h\) is a frame homomorphism, \(h(E)\) is a subframe of \(F\). First, we have to prove that \((h(E), \Box)\) is weakly open in \((F, \star)\). Let \(x \in h(E)\), for instance \(x = h(a), a \in E\). Define \(b := \bigvee \{y \in E \mid h(a \land y) = 0\}\). Clearly \(h(a) \land h(b) = h(a \land b) = 0\), hence \(h(b) \leq h(a)^{*}\), and \(h(b)^{*} \geq h(a)^{**}\) by (P9). Furthermore, we have \((a \lor b)^{*} = 0\) because \((a \lor b) \land c = 0\) implies \(a \land c = 0\), which yields \(h(a \land c) = 0\), hence \(c \leq b \leq a \lor b\), and finally \(c = 0\). Therefore, by (P8), \((a \lor b)^{**} = 1\) and so \(h(a \lor b)^{**} \geq h((a \lor b)^{*}) = h(1) = 1\). Consequently, \(0 = h(a \lor b)^{*} = (h(a) \lor h(b))^{*} = h(a)^{*} \land h(b)^{*}\) by (P8) and (P2). Hence \(h(b)^{*} \leq h(a)^{**}\). Summing up, \(h(a)^{*} = h(b)^{**}\), and also so \(h(E)\) is weakly open in \(F\). Now we need to check that \((h(a)^{**}) \leq h(a)^{\Box}\). Since \(h(a)^{\Box} \leq h(a)^{*}\), we obtain \(h(a^{**}) \land h(a)^{\Box} \leq h(a)^{**} \land h(a)^{*} = 0\), and consequently \(h(a^{**}) \leq h(a)^{\Box}\).

Let \(f: F \to G, g: F \to G\) be weakly open homomorphism of a frame \((F, \star)\) to a frame \((G, \circ)\). We define \(E(f, g) := \{x \in F \mid f(x) = g(x)\}\).

**Lemma 6.** \(E(f, g)\) is a subframe of \(F\).

**Proof.** Clearly 0, 1 \(\in E(f, g)\). Let \(x, y \in E(f, g)\). Then \(f(x \land y) = f(x) \land f(y) = g(x) \land g(y) = g(x \land y)\). Let \(x_i \in E(f, g)\) \((i \in I)\). Then \(f(\bigvee\{x_i \mid i \in I\}) = \bigvee\{f(x_i) \mid i \in I\} = \bigvee\{g(x_i) \mid i \in I\} = g(\bigvee\{x_i \mid i \in I\})\).

**Lemma 7.** Let \(h: E \to F\) be an equalizer of \(f, g\). Then \(h(E)\) is a subframe of \(E(f, g)\) weakly open in \(F\) and the canonical embedding of \(h(E)\) into \(F\) is also an equalizer of \(f, g\).

**Proof.** By Lemma 5, \(h(E)\) is a weakly open subframe in \(F\). By Lemma 6, \(E(f, g)\) is a subframe of \(F\). Since \(h\) is an equalizer of \(f, g\), we have \(f(h(e)) = g(h(e))\) for each \(e \in E\), and therefore \(h(e) \in E(f, g)\). To sum up, the subframe \(h(E)\) of \(F\) is a subset of the subframe \(E(f, g)\) of \(F\), hence \(h(E)\) is a subframe of \(E(f, g)\).
According to Lemma 5, \( h \) can be decomposed into a surjective weakly open homomorphism \( h': E \to h(E) \) and a weakly open embedding \( \tilde{h}: h(E) \to F \). We have \( h = \tilde{h} \cdot h' \). Let \( d: D \to F \) be a weakly open homomorphism such that \( fd = gd \). Inasmuch as \( h \) is an equalizer of \( f, g \), there exists a unique weakly open homomorphism \( \tilde{d}: D \to E \) such that \( d = \tilde{h}(h'd) \). Then also \( d = \hat{h}(h'd) \). Uniqueness should be proved. Supposing \( d = \hat{d}' \), we obtain \( d' = h'(\tilde{d}) \) because \( d'(x) = \hat{h}(d'(x)) = \hat{h}(h'(\tilde{d}(x))) = h'(\tilde{d}(x)) \). It follows that \( \hat{h} \) is an equalizer of \( f, g \).

**Lemma 8.** Let the canonical embedding of \( E \subseteq E(f, g) \) into \( F \) be an equalizer of \( f, g \). Then \( E = \text{M}(E(f, g)) \).

**Proof.** The assumptions imply that \( E \) is the largest subframe of \( E(f, g) \) weakly open in \( F \). By Lemma 4, \( E = \text{M}(E(f, g)) \).

**Lemma 9.** Let \( (A, \Box) \) be a subframe of a frame \( (F, *) \), let \( h: (E, *) \to F \) be a weakly open homomorphism such that \( h(E) \subseteq A \). Then the induced homomorphism \( \tilde{h}: E \to A \) is weakly open.

**Proof.** We have \( h(a^{**}) \leq h(a)^\Box \) and \( h(a)^\Box \leq h(a)^* \). Then \( h(a^{**}) \land h(a)^\Box \leq h(a)^* \land h(a)^* = 0 \), and consequently, \( \hat{h}(a^{**}) = h(a^{**}) \leq h(a)^\Box = \hat{h}(a)^\Box \).

**Proposition 4.** The following conditions are equivalent:

(i) there exists an equalizer of \( f, g \);

(ii) \( \text{M}(E(f, g)) \) is a subframe of \( F \);

(iii) the canonical embedding of \( \text{M}(E(f, g)) \) into \( F \) is an equalizer of \( f, g \).

**Proof.** (i) \( \implies \) (iii) by Lemmas 7 and 8, implications (iii) \( \implies \) (i) and (iii) \( \implies \) (ii) are obvious.

(ii) \( \implies \) (iii): Let \( \text{M}(E(f, g)) \) be a subframe of \( F \). It is weakly open in \( F \) by Lemma 3. Let \( d: D \to F \) be a weakly open homomorphism such that \( fd = gd \). Then clearly \( d(D) \subseteq E(f, g) \) and \( d(D) \) is weakly open in \( F \) by Lemma 5. According to Lemma 3, \( d(D) \subseteq \text{M}(E(f, g)) \). By Lemma 9, the induced homomorphism \( \tilde{d}: D \to \text{M}(E(f, g)) \) is weakly open.

**Corollary.** If \( E(f, g) \) is finite, the canonical embedding of \( \text{M}(E(f, g)) \) into \( F \) is an equalizer of \( f, g \).

**Proof.** The proof follows from Lemma 3.

**Remark.** We have shown that the equalizer of \( f, g \) is exactly (up to isomorphism) the canonical embedding of the largest subframe of \( E(f, g) \) weakly open in \( F \).

**Proposition 5.** The category \( \text{Frm}_{\text{wo}} \) fails to have equalizers.
Proof. Let \( \omega \) be the least infinite ordinal. Let \((P, \leq)\) and \((Q, \leq)\) be the subsets of \(\omega + 1 \times (\omega + 1)\) defined by \(P := \{[n, \omega] \mid n \in \omega\} \cup \{[n, n] \mid n \in \omega + 1\}\) and \(Q := \{[0, 0], [1, 1], [1, 0]\}\) with the induced order.

Let \(F\) and \(G\) be the sets of all down-sets in \((P, \leq)\) and \((Q, \leq)\), respectively. Then \(F\) and \(G\) are complete lattices of sets, and therefore frames with respect to the inclusion of sets. Denote them by \((F, \ast)\) and \((G, \circ)\). Notice that for any \(Y \in G, Y \neq \emptyset\) implies \(Y^{oo} = Q\). For \(c \in \{0, 1\}\) and \(X \in F\) put

\[
f_c(X) = \begin{cases} Q & \text{if } (\exists n \in \omega) [n, \omega] \in X \text{ (then of course } [\omega, \omega] \in X), \\ \{[c, c], [1, 0]\} & \text{if } [\omega, \omega] \in X \land (\forall n \in \omega) [n, \omega] \notin X, \\ \emptyset & \text{otherwise (i.e. } [\omega, \omega] \notin X). \end{cases}
\]

It is obvious that \(f_c \ (c \in \{0, 1\})\) are frame homomorphisms. Now let \(X \in F\). If \([\omega, \omega] \in X\), then \(f_c(X) \neq \emptyset\), hence \(f_c(X^{**}) \subseteq Q = f_c(X)^{oo}\). If \([\omega, \omega] \notin X\), then \([\omega, \omega] \in X^*\), thus \([\omega, \omega] \notin X^{**}\), and consequently \(f_c(X^{**}) = \emptyset \subseteq f_c(X)^{oo}\). We have just proved that \(f_c \ (c \in \{0, 1\})\) are weakly open. Now we are able to apply Proposition 4. For any \(n \in \omega\), we have \([n, n] \in M(E(f_0, f_1))\) since \([n, n] \in E(f_0, f_1)\) and \([n, n]^{**} = [n, n]\), but \((\cup \{[n, n] \mid n \in \omega\})^{**} = [n, n] \in \omega\), and the only element \(X \in F\) with \(X^{**} = [\omega, \omega]\) is \(X = [\omega, \omega] \notin E(f_0, f_1)\).

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References


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