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CONGRUENCE RELATIONS ON FINITARY MODELS

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1. Introduction. The theory of congruence lattices of universal algebras is one of the most rich and developed parts of contemporary algebra. Unfortunately, the rather special and purely internal definition of the congruence relation does not extend directly to structures other than sets with operations; in fact it even does not apply to the slightly more general models (sets with operations and relations). A way out is to simply omit the relations (cf. [8, p. 44]); however, this is not quite satisfactory: for example take the linearly ordered additive group $\mathbb{Z}$ of integers and note that the quotient modulo a non-trivial congruence is a finite group $\mathbb{Z}_n$ which cannot be linearly ordered.

The same difficulties typically occur in non-algebraic structures, e.g. what is a congruence on an ordered set, graph or topological space? A certain solution is provided by categories, where congruences are defined via kernel pairs of morphisms ([2], [7, p. 154]). Of course, this definition is external and relative because the congruences depend not only on the structure but also on the choice of the category.

This paper presents a compromise between the overly general, categorial approach and the quite special universal algebraic one.

2. Conventions. The set of finite cardinals is denoted ω and $E(S)$ is the complete lattice of equivalence relations on a set S, ordered by $\subseteq$. The *kernel* of a mapping $f\colon S \to T$ is the equivalence

$$\ker f := \{\, \langle x, y\rangle \in S^2\,;\, f(x) = f(y)\,\}.$$

$\mathcal{L}$ is a first-order language with identity determined by two disjoint sets $\mathbf{O}$ (of operation symbols) and $\mathbf{R}$ (of relation symbols), and ar : $(\mathbf{O} \cup \mathbf{R}) \to \omega$ (arity function) where $\operatorname{ar}(r) > 0$ for each $r \in \mathbf{R}$. An $\mathcal{L}$-*model* A is given by

$$\begin{array}{ll} \text{a non-empty set } A' \text{ (the universe of } A), & \\ \text{operations } o^A : (A')^{\operatorname{ar}(o)} \to A' & (o \in \mathbf{O}), \text{ and} \\ \text{relations } r^A \subseteq (A')^{\operatorname{ar}(r)} & (r \in \mathbf{R}). \end{array}$$

As usual, a nullary (or zero-ary) operations just fixes an element of A'.

Let A, B be $\mathcal{L}$-models. An $\mathcal{L}$-*homomorphism* is a mapping $f\colon A' \to B'$ satisfying

$$f(o^A(x_1, \dots, x_{\operatorname{ar}(o)})) = o^B(f(x_1), \dots, f(x_{\operatorname{ar}(o)})),$$
$$\langle x_1, \dots, x_{\operatorname{ar}(r)} \rangle \in r^A \implies \langle f(x_1), \dots, f(x_{\operatorname{ar}(r)}) \rangle \in r^B$$

for all $o \in \mathbf{O}$, $r \in \mathbf{R}$ and $x_1, x_2, \ldots \in A'$. L is the category, corresponding to $\mathcal{L}$, whose class L^{Ob} (of L-objects) consists of all $\mathcal{L}$-models, and whose class L^{Mo} (of L-morphisms) is the class of all $\mathcal{L}$-homomorphism. In the sequel K denotes a subcategory of L and A a K-object.

3. Definition. An equivalence $\sigma \in E(A')$ is a K-*congruence* on a K-object A if $\sigma = \ker h$ for some K-morphism $h\colon A \to B$. The set of all K-congruences on A is denoted by $\operatorname{Con}_K A$.

4. Remarks. a. Our definition is a special case of the categorial definition from [2, p. 385] and [7, p. 154]. V. A. Gorbunov and V. P. Tumanov [4, p. 17] define congruences on models in a different way.

For an $\mathcal{L}$-model A, the set $\operatorname{Con}_L A$ is the set of all congruences of the algebra $\langle A', \langle o^A ; o \in \mathbf{O} \rangle\rangle$ in the usual, universal-algebraic sense (or in the sense of [8, p. 44]).

Let $\mathbf{O} = \emptyset$, $\mathbf{R} = \{\leqslant\}$ and $\operatorname{ar}(\leqslant) = 2$. Further, let K be the full subcategory of L whose objects are the non-empty ordered sets. The corresponding $\operatorname{Con}_K A$ were studied by the second author [12]. Congruences on relational structures determined by abstract orthogonality were investigated by Z. Rozenský in [10]. See also Section 16.

b. The subset $\operatorname{Con}_K A$ of $E(A')$ is naturally ordered by inclusion. It has always a least element, the diagonal, $\delta_{A'} := \{ \langle x, x \rangle : x \in A' \}$. Clearly $\delta_{A'}$ is the kernel of the identity map on A' which is always a K-morphism. In fact, it is only general order property of $\operatorname{Con}_K A$.

A subposet $\langle S, \leqslant \rangle$ of a lattice A is a *strong subposet* of A if every existing supremum (infimum) in $\langle S, \leqslant \rangle$ of a finite non-empty subset X of S is also the supremum (infimum) of X in A. Similarly, a subposet $\langle S, \leqslant \rangle$ of a complete meet-semilattice A is a ***complete strong subposet*** if every existing infimum in $\langle S, \leqslant \rangle$ is also the infimum in A.

5. Theorem. *An ordered set P is isomorphic to* $\operatorname{Con}_K A$ *for some subcategory K of L and for some K-object A if and only if P has a least element.*

If P has least element then such K and A may be chosen so that either

(a) $\operatorname{Con}_K A$ *is a strong subposet of the lattice $E(A')$ where, moreover, A' is finite provided P' is finite; or*

(b) $\operatorname{Con}_K A$ *is a complete strong subposet of* $\langle E(P'), \cap\rangle$.

Proof. The part "only if" hods by Remark 4.b. Let P have a least element 0. We give the following two constructions:

(a) Let $\alpha\colon P \to Q$ denote the McNeile completion of P (cf. [6, p. 98]). Here Q is a lattice which is finite if P is finite. It is well known that α preserves the existing suprema and infima and so im α is a strong subposet of Q. By the well-known Whitman's result, Q has a lattice embedding β into an equivalence lattice $E(S')$ (cf. [6, p. 194]) where S' is finite if Q is finite (P. Pudlák and J. Tůma [9]). Put $\gamma := \beta\alpha$. It is easy to see that we may assume $\gamma(0) = \delta_{S'}$ (if not, take $S'/\gamma(0)$).

(b) Define $\gamma\colon P' \to E(P')$ by setting $\gamma(x) := \{y \in P';\, y \leqslant^P x\}^2 \cup \delta_{P'}$ for all $x \in P'$ (cf. J. Adámek [1, Ex. 1H.h, p. 56]). It is easy to see that γ is an embedding of P into $E(P')$ preserving the existing infima such that $\gamma(0) = \delta_{P'}$. Put $N := \{\gamma(x);\, x \in P'\}$. Clearly P is isomorphic to the complete strong subposet $\langle N, \subseteq\rangle$ of $E(S')$ where $S' := P'$.

Recall the following standard notation, see e.g. [3, pp. 15 and 17]. For a non-empty set M and $\sigma \in E(M)$ the natural surjection $\operatorname{nat}\sigma\colon M \to M/\sigma$ assigns to $x \in M$ the equivalence class of σ containing x. For ϱ, $\sigma \in E(M)$, $\varrho \subseteq \sigma$, put

$$\sigma/\varrho := \{\, \langle X, Y\rangle \,;\, X,\ Y \in M/\varrho \text{ and } X \times Y \subseteq \sigma\}.$$

Clearly $\sigma/\varrho \in E(M/\varrho)$.

Now we construct a subcategory K of L and its K-object A such that $N \cong \operatorname{Con}_K A$. For $\sigma \in N$ define S/σ as the $\mathcal{L}$-model with the universe S'/σ, whose operations and relations are defined as follows. Fix an element $a \in S'$ and for all $o \in \mathbf{O}$, $r \in \mathbf{R}$ and $x_1, x_2, \ldots \in S'/\sigma$ put

$$o^{S/\sigma}(x_1, \ldots, x_{\operatorname{ar}(o)}) := \operatorname{nat}\sigma(a) \quad \text{and} \quad r^{S/\sigma} := (S/\sigma)^{\operatorname{ar}(r)}.$$

For ϱ, $\sigma \in N$ with $\varrho \subseteq \sigma$ the map $\operatorname{nat}(\sigma/\varrho)\colon S/\varrho \to S/\sigma$ is an L-morphism with kernel σ/ϱ. The category K is given by

$$K^{\mathrm{Ob}} := \{S/\sigma;\, \sigma \in N\} \text{ and}$$

$$\operatorname{Hom}_K(S/\varrho, S/\sigma) := \begin{cases} \{\operatorname{nat}(\sigma/\varrho)\} & \text{if } \varrho,\ \sigma \in N \text{ and } \varrho \subseteq \sigma, \\ \emptyset & \text{else.} \end{cases}$$

Since we have $\ker \operatorname{nat}(\sigma/\delta_{S'}) = \sigma/\delta_{S'}$ for all $\sigma \in N$, the set N ordered by inclusion is isomorphic to the congruence poset $\mathrm{Con}_K(S/\delta_{S'})$. □

Sections 6–13 give conditions for congruence posets to be complete and algebraic lattices.

6. Lemma. *Let K be a full subcategory of L. If K is closed under the formation of cartesian products, then for each object A of K the poset $\mathrm{Con}_K A$ is a closure system on $E(A')$, and as such a complete lattice.*

P r o o f. Take a non-empty subset Σ of $\mathrm{Con}_K A$. For every $\sigma \in \Sigma$ we have $\sigma = \ker f_\sigma$ for some K-morphism $f_\sigma \colon A \to B_\sigma$. Since the cartesian product $B := \prod \langle B_\sigma \,;\, \sigma \in \Sigma \rangle$ belongs to K^{Ob}, there exists a unique $\mathcal{L}$-homomorphism $f \colon A \to B$ such that for all $\sigma \in \Sigma$ we have $\pi_\sigma f = f_\sigma$ where $\pi_\sigma \colon B \to B_\sigma$ denotes the σ-th projection. As for all $x, y \in A$ we have $f(x) = f(y)$ if and only if $\pi_\sigma f(x) = \pi_\sigma f(y)$ holds for all $\sigma \in \Sigma$, we have $\bigcap \Sigma = \ker f$. Since K is a full subcategory of L and $A, B \in K^{\mathrm{Ob}}$, we have $f \colon A \to B \in K^{\mathrm{Mo}}$ and therefore $\mathrm{Con}_K A$ is closed under the set-theoretical intersection of its non-empty subsets.

We have also the trivial terminal object (the product over the empty set) in K, and so $A' \times A'$ is a K-congruence on A. □

7. Theorem. *Let K be a full subcategory of L. The following conditions are equivalent:*

(i) *For every $A \in K^{\mathrm{Ob}}$ the set $\mathrm{Con}_K A$ is a closure system on the complete lattice $E(A')$.*

(ii) *K is a subcategory of a full subcategory N of L closed under the formation of cartesian products, and such that $\mathrm{Con}_K A = \mathrm{Con}_N A$ for every $A \in K^{\mathrm{Ob}}$.*

P r o o f. (i) $\Longrightarrow$ (ii): Let N be the least full subcategory of L closed under cartesian products and such that $K^{\mathrm{Ob}} \subseteq N^{\mathrm{Ob}}$. Then K is a subcategory of N, hence $\mathrm{Con}_K A \subseteq \mathrm{Con}_N A$ for every $A \in K^{\mathrm{Ob}}$.

To prove the converse inclusion, take $A \in K^{\mathrm{Ob}}$ and $\varrho \in \mathrm{Con}_N A$. Note that $A' \times A' \in \mathrm{Con}_K A$ because $\mathrm{Con}_K A$ is a closure system on $E(A')$. Thus assume $\varrho \neq A' \times A'$. Then $\varrho = \ker f$ for some N-morphism $f \colon A \to C$. By the definition of N, there is an $\mathcal{L}$-isomorphism φ of C onto $B := \prod \langle B_\sigma \,;\, \sigma \in \Sigma \rangle$ for some non-void family $\langle B_\sigma \,;\, \sigma \in \Sigma \rangle$ of K-objects. As K is a full subcategory of L, for every $\sigma \in \Sigma$ the $\mathcal{L}$-homomorphism $f_\sigma := \pi_\sigma \varphi f \colon A \to B_\sigma$ is K-morphism. In particular, $\{\ker f_\sigma \,;\, \sigma \in \Sigma\}$ is a subset of $\mathrm{Con}_K A$. As in the proof of Lemma 6, we have $\varrho = \bigcap \{\ker f_\sigma \,;\, \sigma \in \Sigma\}$. Consequently $\varrho \in \mathrm{Con}_K A$ because $\mathrm{Con}_K A$ is closed under intersection of its non-empty subsets.

(ii) $\Longrightarrow$ (i) is a consequence of Lemma 6. □

8. Theorem. *Let K be a full subcategory of L and A be a K object. If K is closed under ultraproducts, then $\mathrm{Con}_K\, A$ is closed under the set-theoretical unions of non-empty chains (i.e., under their suprema in $E(A')$).*

Proof. Let Σ be a non-empty chain in $\mathrm{Con}_K\, A$. For every $\sigma \in \Sigma$ we have a K-morphism $f_\sigma : A \to B_\sigma$ such that $\sigma = \ker f_\sigma$. Take an ultrafilter F over Σ containing $\{\varrho \in \Sigma;\ \sigma \subseteq \varrho\}$ for every $\sigma \in \Sigma$ (it exists because $\langle \Sigma, \subseteq \rangle$ is a non-empty chain). Put $B := \prod \langle B_\sigma;\ \sigma \in \Sigma \rangle$ and $C := B/F$ (the ultraproduct $\mathrm{mod}\, f$), (cf. [5, p. 145]). As it is well known, the elements of C' are the equivalence classes of $\approx$ where for b, $c \in B$ we put $b \approx c$ whenever $\{\sigma;\ b(\sigma) = c(\sigma)\} \in F$. By assumption, C is a K-object. For $x \in A'$ denote by $f(x)$ the element of B satisfying $f(x)(\sigma) := f_\sigma(x)$ for all $\sigma \in \Sigma$. Next put $g(x) := \{y \in B;\ y \approx f(x)\}$. It is a well-known fact that $g(x) \in C'$ and that $g: A \to C$ is an $\mathcal{L}$-homomorphism (cf. [5, §39] or [8, Chap. IV. 8]). Because K is full in L, we have $g: A \to C \in K^{\mathrm{Mo}}$.

We prove $\bigcup \Sigma = \ker g$. Indeed, for x, $y \in A$, we have

$$\begin{aligned} g(x) = g(y) &\iff f(x) = f(y) \\ &\iff \{\sigma \in \Sigma;\ f_\sigma(x) = f_\sigma(y)\} \in F \\ &\iff \langle x, y \rangle \in \bigcup \Sigma \end{aligned}$$

by our choice of F. Hence, $\bigcup \Sigma \in \mathrm{Con}_K\, A$ because $g: A \to C \in K^{\mathrm{Mo}}$ and $\bigcup \Sigma = \ker g$. □

9. Corollary. *If K is a full subcategory of L closed under cartesian products and ultraproducts, then $\mathrm{Con}_K\, A$ is an algebraic closure system on the complete lattice $E(A')$, and hence $\mathrm{Con}_K\, A$ is an algebraic lattice.*

Proof. The first part follows from Lemma 6 and Theorem 8. An algebraic closure system on the algebraic lattice $E(A')$ is also an algebraic lattice (cf. [5, p. 25, Theorem 5] or [14, Theorem 4.3]). □

10. Remarks. a. None of the conditions of Lemma 6, Theorem 8 and Corollary 9 is neccesary. Take $\mathcal{L}$ with $\mathbf{O} = \mathbf{R} = \emptyset$, and the category K with $K^{\mathrm{Ob}} = \{\omega\}$ (where ω is the set of finite cardinals) and $K^{\mathrm{Mo}} = \{f;\ f: \omega \to \omega\}$. Then K is a full subcategory of L (the objects of L are all the sets and the L-morphisms are all the mappings between them). We have $\mathrm{Con}_K\, \omega = E(\omega)$. Nevertheless, K is closed neither under cartesian products nor under ultraproducts (as ω is an infinite set, its cartesian powers and ultrapowers can be uncountable: in the case of ultrapowers, it is a consequence of Frayne, Morel and Scott's theorem, cf. [5, p. 246, Theorem 2] or [8, p. 176]).

b. Let Φ be a set of $\mathcal{L}$-sentences. Denote by Mod Φ the full subcategory of L whose objects are all models of Φ. References for some well-known relevant notions are: *Horn formulae:* [5, p. 285], [8, p. 145]; *universal formulae:* [5, p. 233], [8, p. 130]; *positive formulae:* [5, p. 280], [8, p. 143]; and *quasi-identities:* [5, p. 339 and §63], [8, p. 149].

For a set Φ of $\mathcal{L}$-quasiidentities the category Mod Φ is called an $\mathcal{L}$-*quasivariety* (cf. [5, p. 339] or [8, Chap. V]).

11. Corollary. *Let Φ be a set of $\mathcal{L}$-sentences preserved under the formation of arbitrary non-empty cartesian products* (cf. [5, §§46, 47]). *If A is a model of Φ then $\{A' \times A'\} \cup \mathrm{Con}_{\mathrm{Mod}\,\Phi} A$ is an algebraic closure system on $E(A')$.*

P r o o f. By our assumption about Φ the category Mod Φ is closed under cartesian products of non-empty families and by Loś'theorem (cf. [5, p. 241 Theorem 1]) it is also closed under ultraproducts. □

12. Corollary. *Let $\Gamma = \Phi \cup \Psi$ whre Φ is a set of universal Horn $\mathcal{L}$-sentences, and Ψ is a set of positive $\mathcal{L}$-sentences. If A is a model of Γ, then $\{A' \times A'\} \cup \mathrm{Con}_{\mathrm{Mod}\,\Gamma} A$ is an algebraic closure system on $E(A')$.*

P r o o f. Mod Φ and Mod Γ are full subcategories of L, and Mod Γ is a subcategory of Mod Φ. Since elements of Φ are Horn $\mathcal{L}$-senteces, Mod Φ is closed under cartesian products of non-empty families (cf. [5, p. 285, Corollary 1] or [8, pp. 146 and 148]). Let $A \in (\mathrm{Mod}\,\Gamma)^{\mathrm{Ob}}$ and let $f\colon A \to B$ be a Mod Φ-morphism. Then $f(A')$ is the universe of a Φ-model because sentences of Φ are universal ones (cf. [8, p. 130]). Denote this model by $f(A)$. As sentences of Ψ are positive and A is a Ψ-model, clearly $f(A)$ is a model of Ψ also (because of Lyndon's theorem cf. [5, p. 281] or [8, p. 143]). Therefore $f\colon A \to f(A)$ is an Mod Γ-morphism. Now apply Theorem 7. Note that the category Mod Γ is closed under ultraproducts in view of Loś' Theorem. Now apply Corollary 9. □

13. Corollary. *If K is a quasivariety then $\mathrm{Con}_K A$ is an algebraic closure system on $E(A')$.*

P r o o f. By [8, p. 214, Corollary 3] K is closed under ultraproducts and cartesian products (and contains trivial terminal objects). □

14. Remark. In the last part of the paper we study a possibility of internal characterisation of K congruences.

Our definition of K-congruences on A depends both on internal properties of A and on the chosen category K. In some special cases we can internally characterize

$\mathrm{Con}_K\, A$. For example, we show that $\sigma \in E(A')$ is an L-congruence on A if and only if it is compatible with the operations o^A for all $o \in \mathbf{O}$ of positive arity. Other examples will be given below.

Let $\sigma \in \mathrm{Con}_L\, A$. For $o \in \mathbf{O}$ with $\mathrm{ar}(o) = m$ define the operation $o^{A/\sigma} : (A'/\sigma)^m \to A/\sigma$

$$o^{A/\sigma}(X_1, \dots, X_m) := (\mathrm{nat}\,\sigma)\big(o^A(x_1, \dots, x_m)\big)$$

for all $X_1, \dots, X_m \in A'/\sigma$ and $x_1 \in X_1, \dots, x_m \in X_m$.

For each $r \in \mathbf{R}$ with $\mathrm{ar}(r) = n$, $\varrho \in E(A')$ and $X_1, \dots, X_n \in A'/\varrho$, put $\langle X_1, \dots, X_n\rangle \in r^A_\varrho$ if $\langle x_1, \dots, x_n\rangle \in r^A$ for some $x_1 \in X_1, \dots, x_n \in X_n$.

Further for $\sigma \in \mathrm{Con}_L\, A$ let $A(\sigma)$ denote the set of $\mathcal{L}$-models B such that

(i) $B' := A'/\sigma$,

(ii) $o^B := o^{A/\sigma}$ for all $o \in \mathbf{O}$, and

(iii) $r^A_\sigma \subseteq r^B$ for all $r \in \mathbf{R}$.

The following definition is a standart one:

An $\mathcal{L}$-model A is a *strong subobject* of an $\mathcal{L}$-model B if $\langle A', \langle o^A ; o \in \mathbf{O}\rangle\rangle$ is a subalgebra of $\langle B', \langle o^B ; o \in \mathbf{O}\rangle\rangle$, and for every $r \in \mathbf{R}$ we have $r^A = r^B \cap (A')^{\mathrm{ar}(r)}$.

We omit the easy proof of Remark 15.

15. Remark. *Let K be a full subcategory of L, $A \in K^{\mathrm{Ob}}$ and $\sigma \in E(A')$. Then $\sigma \in \mathrm{Con}_K\, A$ if and only if*

(i) *$\sigma \in \mathrm{Con}_L\, A$, and*

(ii) *Some $B \in A(\sigma)$ is isomorphic (in L) to a strong subobject of some $Y \in K^{\mathrm{Ob}}$.*

If moreover, K is closed under the formation of strong subobjects and $\mathcal{L}$-isomorphic images, then $\sigma \in \mathrm{Con}_K\, A$ if and only if

$$\sigma \in \mathrm{Con}_L\, A \quad \text{and} \quad A(\sigma) \cap K^{\mathrm{Ob}} \neq \emptyset.$$

□

Both results still depend on K. However, for $K := \mathrm{Mod}\,\Phi$ for a set Φ of $\mathcal{L}$-sentences, the following three examples show that $\mathrm{Con}_K\, A$ may have a purely internal description.

16. Examples. a. (see [10]). Let $\mathcal{L}$ be given by $\mathbf{O} = \{O\}$ and $\mathbf{R} = \{\perp\}$ where $\mathrm{ar}(\perp) = 2$ and $\mathrm{ar}(O) = 0$. An $\mathcal{L}$-model A is a *set with orthogonality* if for all x, $y \in A'$

$$x \perp^A y \Longrightarrow y \perp^A x, \quad x \perp^A O^A, \quad \text{and } x \perp^A x \Longrightarrow x = O^A$$

(i.e. $\perp^A$ is a symetric relation, or a graph, with a star and a single loop at O^A).

Let K be the full subcategory of L whose objects are the sets with orthogonality. Since K is a quasivariety, $\mathrm{Con}_K\, A$ is an algebraic lattice for each $A \in K^{\mathrm{Ob}}$ (cf. Z. Rozenský [10, Satz 3.9]).

From Remark 15 we obtain the following internal description of the K-congruences on A (cf. [10, Satz 2.3]):

$\sigma \in E(A')$ is a K-congruence on A if and only if for all x, $y \in A'$ we have

$$x\sigma y \quad \& \quad x \perp^A y \quad \Longrightarrow \quad x\sigma O^A .$$

b. (see [12]). Let $\mathcal{L}$ be determined by $\mathbf{O} = \emptyset$ and $\mathbf{R} = \{\leqslant\}$ with $\operatorname{ar}(\leqslant) = 2$. Let Ord be the full subcategory of L, whose objects are the ordered sets. Remark 15 gives the following internal characterization of Ord-congruences:

Let A be an ordered set and $\sigma \in E(A')$. Then $\sigma \in \mathrm{Con}_{\mathrm{Ord}}\, A$ if and only if $\leqslant_\sigma^A$ is acyclic (i.e. the transitive closure of $\leqslant_\sigma^A$ is an antisymmetric relation on A'/σ).

Ord is an $\mathcal{L}$-quasivariety, hence $\mathrm{Con}_{\mathrm{Ord}}\, A$ is an algebraic closure system of $E(A')$. The above characterisation of $\mathrm{Con}_{\mathrm{Ord}}\, A$ leads to the following closure opetator on $E(A')$. For $\tau \in E(A')$, denote by $\preceq$ the transitive closure of $\tau \cup \leqslant^A$. Define $g(\tau)$ by

$$\langle x, y\rangle \in g(\tau) \quad \text{if} \quad x \preceq y \preceq x .$$

Then $g\colon E(A') \to E(A')$ is a closure operator on $E(A')$ such that $\mathrm{Con}_{\mathrm{Ord}}\, A$ is the set of g-closed elements of $E(A')$. (cf. [12, Sätze 17, 19,22', 49]).

c. (see [13] and [11]). For an ordered set A call $\sigma \in E(A')$ *convex* if every σ-equivalence class C is a convex subset of $\leqslant^A$ is the usual sense (i.e. $z \in C$ whenever $x < z < y$ for some x, $y \in C$). Denote by $\mathrm{Ce}\, A$ the set of all convex equivalences on A.

The second author studied the following inverse problem: Are there $\mathcal{L}$ and K such that $\langle \mathrm{Ce}\, A, \subseteq\rangle$ is order isomorphic to $\mathrm{Con}_K\, B$ for some $B \in K^{\mathrm{Ob}}$ with $A' = B'$? We sketch a solution. Let $\mathcal{L}$ be determined by $\mathbf{O} = \emptyset$ and $\mathbf{R} = \{t\}$ with $\operatorname{ar}(t) = 3$, and let K be the full subcategory of L whose objects are the models of the following quasiidentity φ:

$$t(x, y, x) \quad \Longrightarrow \quad x = y.$$

Then from Remark 15, we get an internal characterization:

Let $B \in K^{\mathrm{Ob}}$ and $\sigma \in E(B')$. Then $\sigma \in \mathrm{Con}_K\, B$ if and only if $\langle B'/\sigma, t_\sigma^B\rangle$ is a model of φ.

Returning to convex equivalences, define for an ordered set A,

$$t^{A^*} := \{\, \langle x, y, z\rangle \in A'^3 ;\, x \leqslant^A y \leqslant^A z \}$$

and set $A^* := \langle A', t^{A^*}\rangle$. Then * defines a (covariant) functor of Ord to K, satisfying

$$\mathrm{Hom}_{\mathrm{Ord}}(A, B) = \mathrm{Hom}_K(A^*, B^*)$$

for all A, $B \in \mathrm{Ord}^{\mathrm{Ob}}$. It follows from our characterization of K-congruences, that $\mathrm{Ce}\, A = \mathrm{Con}_K\, A^*$ for every ordered set A.

There is another result in this connection: *Let M be a full subcategory of the category of binary ralational structures. If* Ord *is a subcategory of M, the there is an ordered set A such that* $\mathrm{Ce}\, A \neq \mathrm{Con}_M\, A$ ([13, Theorem 4]).

17. Remark. A category of models is a special case of a *construct* in J. Adámek's sense, cf. [1, pp. 5–6]. Our Definition 3 of K-congruences makes sense even if K is a general construct. Since our proof of Lemma 6 is of a purely set-theoretical character, the result is also true for constructs (cf. [1, p. 69] for the definition of a construct which is closed under the formation of cartesian products).

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