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Ram U. Verma

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NUMERICAL RANGE AND RELATED NONLINEAR FUNCTIONAL EQUATIONS

RAM U. VERMA, Orlando

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1. INTRODUCTION

Although Browder and Gupta [1] and Minty [3] have contributed enormously to the solvability of nonlinear functional equations in reflexive Banach spaces, it seems that Zarantonello [7] was the first to apply the concept of numerical range of nonlinear operators to the solvability of nonlinear functional equations in a Hilbert space setting. The aim of this paper is to extend some of the results of Zarantonello to nonlinear Banach space operators, and relate them to approximation-solvability [4].

Let us consider an approximation scheme $\pi_0 = \{X_n, E_n, R_n, Y_n, Q_n\}$, represented by a diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ R_n \downarrow \uparrow E_n & & \downarrow Q_n \\ X_n & \xrightarrow{A_n} & Y_n \end{array} \tag{1}$$

where $T\colon X \to Y$ from an infinite-dimensional normed linear space X to another infinite-dimensional linear space Y is a nonlinear mapping corresponding to the equation

$$Tx = b \quad \text{for} \quad x \in X,\ b \in Y, \tag{2}$$

where all $A_n = Q_n T E_n$ are continuous. Here X_n and Y_n are normed spaces with $\dim X_n = \dim Y_n < \infty$ and, the operators $E_n\colon X_n \to X$ and $Q_n\colon Y \to Y_n$ are continuous and linear with

$$\sup \|E_n\| < \infty \quad \text{and} \quad \sup \|Q_n\| < \infty.$$

The operator $R_n\colon X \to X_n$ is a restriction operator.

As far as the solvability of the equation (2) is concerned, we consider not just the usual solvability—the existence of a solution of the equation (2) is somehow established, but an approximation-solvability—a solution of the equation is obtained as a limit (or at least one limit point) of solutions x_n of simpler finite-dimensional problems

$$A_n x_n = Q_n b \quad \text{for} \quad x_n \in X_n,\ Q_n b \in Y_n. \tag{3}$$

At this point, we are faced with the problem: For what type of a linear or nonlinear mapping T, is it possible to construct a solution of the equation (2) as a strong limit of solutions x_n of the equations (3)? Browder and Petryshyn [2] came up with the answer—A-proper mappings. The notion of the A-proper mappings is closely connected with the approximation-solvability of the equation (2), and further does extend and unify results concerning the Galerkin type methods for linear and nonlinear equations in the theory of strongly monotone and accretive operators, operators of the type (S), P_γ-compact, ball condensing and other mappings.

The concept of A-proper mappings extends also to the case of the stability of the projectional method in the sense of Mikhlin, and relates rather naturally to the solvability of elliptic partial differential equations.

Next, we consider an approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ in reflexive Banach spaces. The symbol $\mathcal{K}$ is used to denote either the field real or the field complex.

We consider the operator equation

$$Tx = b, \quad x \in X, \tag{4}$$

and related approximate equations

$$E_n^* T E_n x_n = E_n^* b \tag{5}$$

for $x_n \in X_n$, $n = 1, 2, \ldots$, under the following approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$:

$$\begin{array}{ccc} X & \xrightarrow{T} & X^* \\ R_n \downarrow \uparrow E_n & & \downarrow E_n^* \\ X_n & \xrightarrow{A_n} & X_n^* \end{array} \tag{6}$$

where $A_n = E_n^* T E_n$. We make the following assumptions corresponding to approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$, represented by the diagram (6):

(A1) X is a separable reflexive Banach space over field $\mathcal{K}$ with $\dim X = \infty$. Let (X_n) be a Galerkin scheme in X with

$$X_n = \{e_{1n}, \ldots, e_{n'n}\}, \quad n = 1, 2, \ldots.$$

(A2) Let $E_n\colon X_n \to X$ be the embedding operator such that $X_n \subset X$. The operator $R_n\colon X \to X_n$ is defined as follows. For each $x \in X$, there exists at least one element $R_n x \in X_n$ such that

$$\|x - R_n x\| = \text{dist}(x, X_n).$$

For $n = 1, 2, \ldots$, the approximate equations (5) are equivalent to the Galerkin equations

$$[Tx_n, e_{jn}] = [b, e_{jn}],$$

where $[\cdot, \cdot]$ is a pairing between X_n^* and X_n, and $j = 1, 2, \ldots, n'$.

(A3) The operator $T\colon X \to X^*$ is pseudo-monotone and continuous. That means, T is pseudo-monotone if there exists a $d > 0$ such that

$$\big|[Tx - Ty, x - y]\big| \geqslant d\|x - y\|^2 \quad \text{for all } x, y \in X, \tag{7}$$

or,

$$\big|[Tx - Ty, x - y]\big| \geqslant d\big|\|x\| - \|y\|\big|\,\|x - y\| \tag{7'}$$

for all $x, y \in X$.

Let us recall some of the definitions closely related to the present investigation.

Definition 1.1 (Compatibility). An approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is said to be *compatible* if

$$\lim_{n\to\infty} \|E_n R_n x - x\|_X = 0 \quad \text{for all } x \in X. \tag{8}$$

Definition 1.2 (Admissible Inner Approximation). The approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ represented by the diagram (6) is an *admissible inner approximation* iff

(i) X and X^* are infinite-dimensional normed spaces over field $\mathcal{K}$;

(ii) X_n and X_n^* are normed spaces over $\mathcal{K}$ with $\dim X_n = \dim X_n^* < \infty$ for all n;

(iii) for all n, the operator $E_n\colon X_n \to X$ and $E_n^*\colon X^* \to X_n^*$ are linear and continuous with $\sup\|E_n\| < \infty$ and $\sup\|E_n^*\| < \infty$. The operator $R_n\colon X \to X_n$ is called a *restriction operator*; and

(iv) the compatibility condition is satisfied.

We note that under the assumptions (A1)–(A3), the diagram (6) represents an admissible inner approximation scheme in the sense of the above definition.

Definition 1.3 (Consistency). An approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is said to be *consistent* if, for all $x \in X$, we have

$$\lim_{n\to\infty} \|E_n^* Tx - A_n R_n x\|_{X_n^*} = 0. \tag{9}$$

Definition 1.4 (Stability). An approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is called *stable* if there exists an n_0 such that, for $d > 0$,

$$\|A_n x - A_n y\|_{X_n^*} \geqslant d\|x - y\|_{X_n} \tag{10}$$

for all $x, y \in X_n$ and all $n \geqslant n_0$.

Definition 1.5 (Approximation-Solvability). The equation (4) is said to be *uniquely approximation-solvable*, if, for each $b \in X^*$,

(i) equation $Tx = b$, $x \in X$, has a unique solution;

(ii) for each $n \geqslant n_0$, the approximation equation $E_n^* T E_n x_n = E_n^* b$, $x_n \in X_n$, has a unique solution; and

(iii) the sequence (x_n) converges to the solution x of the equation $Tx = b$ in the sense that

$$\lim_{n\to\infty} \|E_n x_n - x\|_X = 0$$

Definition 1.6 (A-Properness). The operator $T\colon X \to X^*$ is said to be *A-proper with respect to approximation scheme* $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ if the following holds. Let (n') be any subsequence of the sequence of natural numbers. If $(x_{n'})$ is a sequence with $x_{n'} \in X_{n'}$ for all n' and if

$$\lim_{n\to\infty} \|A_{n'} x_{n'} - E_n^* b\|_{X_{n'}^*} = 0 \quad \text{for some } b \in X^*$$

and $\sup \|x_{n'}\|_{X_{n'}} < \infty$, then there exists a subsequence $(x_{n''})$ such that, for $x \in X$,

$$\lim_{n\to\infty} \|E_{n''} x_{n''} - x\|_X = 0 \quad \text{and} \quad Tx = b.$$

In what follows, the symbols "$\to$" and "$\xrightarrow{w}$" above shall denote strong and weak convergence, respectively.

Definition 1.7 (Duality Mapping). We recall that a continuous function μ: $\mathbf{R}^+ = \{t : t \geqslant 0\} \to \mathbf{R}^+$ is called a *gauge function* if $\mu(0) = 0$, and μ is strictly increasing. Let X be a reflexive Banach space over $\mathbf{R}$ and X^* its dual. A mapping

$J: X \to X^*$ is said to be a *duality mapping between X and X^* with respect to gauge function μ* if

$$[Jx, x] = \mu(\|x\|)\|x\|, \text{ and } \|Jx\| = \mu(\|x\|) \quad \text{for } x \in X.$$

Note that if $\mu(t) = t$, J is called a *'normalized duality' mapping.* If X^* is strictly convex, then J is uniquely determined by μ, and if X is also reflexive, then J is a single-valued demicontinuous mapping of X onto X^*, which is bounded and positively homogeneous; furthermore, J is monotone and satisfies the property

$$(11) \qquad [Jx - Jy, x - y] = [Jx, x - y] - [Jy, x - y] \geqslant |\mu(\|x\|) - \mu(\|y\|)| \, \|x - y\|$$

for all $x, y \in X$.

For J a normalized duality, (11) reduces to

$$(12) \qquad [Jx - Jy, x - y] \geqslant |\|x\| - \|y\|| \, \|x - y\|$$

for all $x, y \in X$.

In addition, if X is strictly convex, then the operator $J: X \to X^*$ is strictly monotone and bijective. The inverse operator

$$J^{-1}: X^* \to X$$

equals the duality mapping of the dual space X^* provided that X is reflexive.

Furthermore, it follows from

$$(13) \qquad [Jx_n - Jx, x_n - x] \to 0 \text{ as } n \to \infty,$$

that $x_n \xrightarrow{w} x \in X$ as $n \to \infty$. If, in addition, X is locally uniformly convex, then (13) implies that $x_n \to x$ as $n \to \infty$, that is, J satisfies Condition (S).

To show that Condition (13) implies that $x_n \xrightarrow{w} x$ as $n \to \infty$, if we write

$$[Jx_n - Jx, x_n - x] = (\|x_n\| - \|x\|)^2 + (\|x_n\| \|x\| - [Jx_n, x]) + (\|x_n\| \|x\| - [Jx, x_n]),$$

then, since each of the three terms on the right hand side is non-negative, we have

$$\|x_n\| \to \|x\| \quad \text{and} \quad [Jx, x_n] \to \|x\|^2 \quad \text{as} \quad n \to \infty.$$

Since X is reflexive, there is a subsequence, again denoted by (x_n), such that

$$(14) \qquad x_n \xrightarrow{w} y \quad \text{as} \quad n \to \infty.$$

It can be easily shown that $y = x$.

If, in addition, X is locally uniformly convex, then it follows from

$$x_n \xrightarrow{w} x \quad \text{and} \quad \|x_n\| \to \|x\| \quad \text{as} \quad n \to \infty$$

that $x_n \to x$ as $n \to \infty$.

$J: X \to X^*$ is continuous when X^* is locally uniformly convex.

Definition 1.8 (Numerical Range). Let X be a reflexive Banach space and X^* its dual. The numerical range of an operator $A: X \to X^*$, denoted by $V[A]$, is defined to be the set

$$V[A] = \left\{ \frac{[Ax - Ay, x - y]}{[Jx - Jy, x - y]} : x, y \in X, x \neq y \right\},$$

where $[\cdot, \cdot]$ is the pairing between X^* and X. Here $J: X \to X^*$ is strictly monotone normalized duality. Clearly, $V[A]$ is a subset of the field $\mathcal{K}$, and $V[A]$ coincides with the Zarantonello numerical range [7] when X is a Hilbert space. The Zarantonello numerical range of A, denoted by $N[A]$, is defined to be the set

$$N[A] = \left\{ \frac{\langle Ax - Ay, x - y \rangle}{\|x - y\|^2} : x, y \in X, x \neq y \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on X. Furthermore, $V[A]$ coincides with the usual numerical range when A is linear.

Next, we state the following result, crucial to the approximation-solvability.

Lemma 1.9 ([8], Theor. 34 A). *Let all operators $A_n: X_n \to X_n^*$ be continuous. If the approximation scheme represented by diagram (6) is an admissible inner approximation with consistency and stability, then the following conditions are equivalent:*

(C1) *Solvability.*

(C2) *Unique approximation-solvability.*

(C3) *A-properness.*

That means, if the approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is consistent and stable, then the equation $Tx = b$, $x \in X$, is uniquely approximation-solvable iff the operator T is A-proper.

2. Main results

This section deals with the results on the solvability and approximation-solvability. Before proceeding to the main results on the solvability (approximation-solvability), we discuss some results relating to the elementary properties of the numerical range $V[A]$.

Theorem 2.1. *Let $A, B\colon X \to X^*$ be mappings from a reflexive Banach space X to its dual X^*, and $\lambda \in \mathcal{K}$ (field). Then*

(i) $V[\lambda A] = \lambda V[A]$;

(ii) $V[A+B] \subseteq V[A] + V[B]$; *and*

(iii) $V[A - \lambda J] = V[A] - \{\lambda\}$,

where $J\colon X \to X^*$ is strictly monotone normalized duality.

Proof. The proof follows from the definition. □

Theorem 2.2. *Suppose that the operator $A\colon X \to X^*$ is continuous from a separable reflexive complex Banach space X to its dual X^*. If X and X^* are locally uniformly convex, $\lambda \in \mathcal{K}$ (field) has a positive distance from the numerical range $V[A]$ of A, i.e.,*

$$d = \inf\{|\lambda - \mu| : \mu \in V[A]\} > 0,$$

and $J\colon X \to X^*$ is normalized duality, then the equation

$$Ax - \lambda Jx = b$$

has a unique solution for every $b \in X^*$.

If, in addition, $\dim X = \infty$, then the equation

$$Ax - \lambda Jx = b$$

is uniquely approximation-solvable for each $b \in X^*$.

Proof. Since $J\colon X \to X^*$ is strictly monotone, we obtain the key inequality, for all $x, y \in X$ with $x \neq y$,

$$\begin{aligned}\big|[(A-\lambda J)x - (A-\lambda J)y, x-y]\big| &= \big|[Ax - Ay, x-y] - \lambda[Jx - Jy, x-y]\big| \\ &= \left|\frac{[Ax-Ay, x-y]}{[Jx-Jy, x-y]} - \lambda\right| \big|[Jx-Jy, x-y]\big| \\ &\geqslant d\ \mathrm{Re}[Jx-Jy, x-y].\end{aligned}$$

This, in turn, implies that

$$\big|[(A - \lambda J)x - (A - \lambda J)y, x - y]\big| \geqslant d\,\big|\|x\| - \|y\|\big|\,\|x - y\|, \tag{15}$$

and consequently,

$$\|(A - \lambda J)x - (A - \lambda J)y\| \geqslant d\big|\|x\| - \|y\|\big| \quad \text{for all } x, y \in X. \tag{16}$$

Let us first consider the case when $\dim X < \infty$. By inequality (16), it is immediate that $(A - \lambda J)$ is one-to-one. Let us take $d(r) = dr - \|(A - \lambda J)(0)\|$. Then, for $x \in X$, we find

$$\begin{aligned}\big|[(A - \lambda J)x, x]\big| &\geqslant \big|[(A - \lambda J)x - (A - \lambda J)(0), x]\big| - \big|[(A - \lambda J)(0), x]\big|\\ &\geqslant d\|x\|^2 - \|(A - \lambda J)(0)\|\,\|x\|\\ &= d(\|x\|)\|x\|,\end{aligned}$$

so that $\|(A - \lambda J)x\| \geqslant d(\|x\|)$ for $x \neq 0$. For each $M > 0$, therefore, there exists $k(M)$ such that if $\|(A - \lambda J)x\| \leqslant M$ then $\|x\| \leqslant k(M)$. Thus, $(A - \lambda J)^{-1}$ carries bounded subsets of $R(A - \lambda J)$ into bounded subsets of X, and is continuous from $R(A - \lambda J)$ to X. By Brouwer theorem on invariance of domain, $R(A - \lambda J)$ is open. Now, it only remains to show that $R(A - \lambda J)$ is closed. To this end, let $(A - \lambda J)x_m \to b$ as $m \to \infty$. Thus, $((A - \lambda J)x_m)$ is a Cauchy sequence, and it is immediate that, for some $x \in X$,

$$(A - \lambda J)x_m - (A - \lambda J)x \to b - (A - \lambda J)x \quad \text{as} \quad m \to \infty.$$

Since X is reflexive, there exists a subsequence, again denoted by (x_m), such that, for some $x \in X$,

$$x_m \xrightarrow{w} x \quad \text{as} \quad m \to \infty.$$

It follows from the inequality (15) and above arguments that, as $m \to \infty$,

$$\big|\|x_m\| - \|x\|\big|\,\|x_m - x\| \leqslant d^{-1}\big|[(A - \lambda J)x_m - (A - \lambda J)x, x_m - x]\big| \to 0,$$

and thus, $\|x_m\| \to \|x\|$ as $m \to \infty$.

Since X is locally uniformly convex, $x_m \xrightarrow{w} x$ and $\|x_m\| \to \|x\|$ as $m \to \infty$ implies that $x_m \to x$ as $m \to \infty$. It follows from the continuity of A (and hence $A - \lambda J$) that $(A - \lambda J)x = b$ and, consequently, $b \in R(A - \lambda J)$.

Thus, the non-empty set $R(A - \lambda J)$ is both open and closed in X^*, and hence $R(A - \lambda J) = X^*$, and $A - \lambda J$ is bijective. This completes the proof of the first part when X is finite-dimensional. Next, consider the case when $\dim X = \infty$. We need to show first that diagram (6) represents an admissible inner approximation scheme. Since $\|E_n\| = 1$, this implies that $\|E_n^*\| = 1$ for all n, and since (X_n) is a Galerkin scheme, we have $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for all $x \in X$. Thus, $\|R_n x - x\| \to 0$ as $n \to \infty$, and the compatibility condition is satisfied.

Since A (and hence $(A-\lambda J)$) is continuous, the consistency condition is as follows: Since $\|(A - \lambda J)E_n R_n x - (A - \lambda J)x\| \to 0$ and $\|E_n^*\| < \infty$, we arrive at the consistency condition,

$$\begin{aligned}\|E_n^*(A - \lambda J)x - A_n R_n x\| &= \|E_n^*(A - \lambda J)x - E_n^*(A - \lambda J)E_n R_n x\| \\ &\leqslant \|E_n^*\| \, \|(A - \lambda J)x - (A - \lambda J)E_n R_n x\| \to 0\end{aligned}$$

as $n \to \infty$.

The stability condition follows from the inequality (15), for if x, $y \in X_n$, we have

$$\begin{aligned}\|A_n x - A_n y\| \, \|x - y\| &\geqslant \big|[A_n x - A_n y, x - y]\big| \\ &= \big|[E_n^*(A - \lambda J)E_n x - E_n^*(A - \lambda J)E_n y, x - y]\big| \\ &= \big|[(A - \lambda J)x - (A - \lambda J)y, E_n x - E_n y]\big| \\ &= \big|[(A - \lambda J)x - (A - \lambda J)y, x - y]\big| \\ &\geqslant d\|x - y\| \, \big|\|x\| - \|y\|\big|\end{aligned}$$

and so

$$\|A_n x - A_n y\| \geqslant d\big| \, \|x\| - \|y\| \, \big| \quad \text{for all } x, y \in X_n.$$

Finally, we need to show that $A-\lambda J$ is A-proper with respect to the approximation scheme $\pi_1 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$, represented by the diagram (6). Let $\sup \|x_n\| < \infty$ for some $x_n \in X_n$ such that

$$\|A_n x_n - E_n^* b\| = \|E_n^*(A - \lambda J)x_n - E_n^* b\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since X is reflexive and separable, there exists a subsequence, again denoted by (x_n), such that, for some $x \in X$,

$$x_n \xrightarrow{w} x \text{ in } X \quad \text{as} \quad n \to \infty.$$

We also have $\|R_n x - x\| \to 0$ as $n \to \infty$, that is, $R_n x \to x$, and so $x_n \xrightarrow{w} x$ as $n \to \infty$ implies that

$$x_n - R_n x \xrightarrow{w} 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

Thus, as $n \to \infty$,

$$
\begin{aligned}
&E_n^*((A-\lambda J)x_n - (A-\lambda J)R_n x) \\
\text{(18)} \qquad &= \left(E_n^*(A-\lambda J)x_n - E_n^* b\right) + \left(E_n^* b - E^*(A-\lambda J)R_n x\right) \\
&\to E_n^* b - E_n^*(A-\lambda J)x.
\end{aligned}
$$

It would suffice to show that $x_n \to x \in X$ as $n \to \infty$, and $(A-\lambda J)x = b$. From (17) and (18), it follows, for some $x_n \in X_n$ as above, that, as $n \to \infty$,

$$
\begin{aligned}
d\big|\|x_n\| - \|R_n x\|\big| \, \|x_n - R_n x\| &\leqslant \big|[A_n x_n - A_n R_n x, x_n - R_n x]\big| \\
&= \big|[E_n^*(A-\lambda J)x_n - E_n^*(A-\lambda J)R_n x, x_n - R_n x]\big| \to 0.
\end{aligned}
$$

This implies that either $\big|\|x_n\| - \|R_n x\|\big| \to 0$ or $\|x_n - R_n x\| \to 0$ as $n \to \infty$. As the second case is trivial, we consider the first one. Since $\|R_n x - x\| \to 0$ as $n \to \infty$, it follows from

$$\big|\|x_n\| - \|R_n x\|\big| \to 0 \text{ as } n \to \infty$$

that $\|x_n\| \to \|x\|$ as $n \to \infty$. Since X is reflexive and locally uniformly convex, and $\|x_n\| \to \|x\|$ as $n \to \infty$, this implies that $x_n \to x$ as $n \to \infty$. Hence, $(A - \lambda J)x = b$ by the continuity of A (and hence $A - \lambda J$), and the theorem follows from Lemma 1.9. □

Corollary 2.3. *If X is Hilbert space, Theorem 2.2 reduces to the following result ([8], Theorem 34C):*

Suppose $A\colon X \to X$ is continuous on the separable Hilbert space X over $\mathcal{K}$. If the λ in $\mathcal{K}$ has a positive distance from the numerical range $N[A]$ of A, i.e.,

$$d = \operatorname{dist}(\lambda, N[A]) > 0,$$

then the equation

$$Ax - \lambda x = b$$

has a unique solution for every $b \in X$.

If, in addition, $\dim X = \infty$, then equation $Ax - \lambda x = b$ is uniquely approximation-solvable for each $b \in X$.

Remark 2.4. If we drop the separability for space X in Theorem 2.2., it still holds by proving the convergence of the Galerkin method by $M - S$ sequences as follows. Let $\Lambda = \{G\}$ be the system of all finite-dimensional subspaces G of X. We define order relation $G \leqslant H$ iff $G \subseteq H$. Then Λ is a directed set, and (x_G) is a $M-S$ sequence which is bounded in the reflexive Banach space X. Since each closed ball in X is weakly compact, there exists a M-S subsequence $(x_{G'})$ such that

$$x_{G'} \xrightarrow{w} x.$$

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Author's address: University of Central Florida, Department of Mathematics, Orlando, Florida 328 16, U.S.A.