Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

CHARACTERIZATIONS OF HAMILTONIAN ALGEBRAS

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(Received January 21, 1991)

A group is *Hamiltonian* if every its subgroup is normal. This concept was generalized for algebras in [4]: an algebra A is Hamiltonian if every its subalgebra is a class (block) of some congruence on A. A variety $\mathcal{V}$ is *Hamiltonian* if each $A \in \mathcal{V}$ has this property.

Hamiltonian algebras were characterized in [5]:

Lemma 1 (see Lemma 3 in [5]). *An algebra A is Hamiltonian if and only if for every unary algebraic function φ over A and each x, y of A there exists a ternary polynomial p such that*

$$\varphi(x) = p\big(y, \varphi(y), x\big). \tag{$*$}$$

The same characterization is also used for Hamiltonian varieties in [4] (only the unary algebraic function is substituted by an $(n+1)$-ary polynomial in $(*)$). However, all examples of Hamiltonian algebras occuring in [4] are members of varieties of loops or modules, i.e. of congruence-permutable varieties with one nullary operation. The aim of this short note is to show that for such varieties the characterization from Lemma 1 can be simplified using only a binary polynomial in $(*)$.

An algebra A is called "*with* 0" if 0 is a nullary operation of A. A variety $\mathcal{V}$ is "*with* 0" if 0 is a nullary operation in the type of $\mathcal{V}$.

Theorem 1. *Let A be an algebra with 0. A is Hamiltonian if and only if for every unary algebraic function φ over A and each x of A there exist binary polynomials p, r such that*

$$\varphi(x) = p\big(x, \varphi(0)\big), \quad \varphi(0) = r\big(x, \varphi(x)\big). \tag{$**$}$$

Proof. Let A be Hamiltonian. Putting $y = 0$ in $(*)$ we obtain $\varphi(x) = p(x, \varphi(0))$ for some binary polynomial p. Putting $x = 0$ (and replacing y by x) in $(*)$, we obtain the second equation in $(**)$. Conversely, let A satisfy $(**)$. Then

$$\varphi(x) = p(x, \varphi(0)) = p(x, r(y, \varphi(y))),$$

whence $(*)$ is evident. □

A variety is *n-permutable* if

$$\Theta \circ \Phi \circ \Theta \circ \cdots = \Phi \circ \Theta \circ \Phi \circ \cdots$$

for each $A \in \mathcal{V}$ and every $\Theta, \Phi \in \operatorname{Con} A$, where there are n factors on both sides of the equality. Denote by $\Theta_A(a, b)$ the least congruence on A containing $\langle a, b\rangle$.

Now we proceed to show that for n-permutable varieties the first equation of $(**)$ is satisfied.

Lemma 2. *Let $\mathcal{V}$ be an n-permutable variety with 0, $A \in \mathcal{V}$ and $0 \in B \subseteq A$. The following conditions are equivalent:*

(i) *B is a block of some $\Theta \in \operatorname{Con} A$;*

(ii) *B is a block of $\Theta = \bigvee\{\Theta_A(0, x);\ x \in B\}$;*

(iii) *for every algebraic function φ,*

$$\varphi(0) \in B \text{ implies } \varphi(B) \subseteq B.$$

Proof. (i) ⇔ (ii) is evident and (i) ⇒ (iii) is a direct consequence of Theorem 5 in [6]. Let us prove (iii) ⇒ (ii): Let $b \in B$, $a \in A$ and $\langle a, b\rangle \in \Theta = \bigvee\{\Theta_A(0, x);\ x \in B\}$. Then $b \in B$ implies $\langle b, 0\rangle \in \Theta$. Transitivity of Θ gives $\langle a, 0\rangle \in \Theta$. Since $\mathcal{V}$ is n-permutable, congruences on A concide with *compatible quasiorders* on A (i.e. reflexive and transitive relations satisfiyng the Substitution Condition with respect to all operations of A), see e.g. [2] or [3]. Thus

$$\Theta = Q = \bigvee_{Q}\{Q(0, x);\ x \in B\},$$

where $\bigvee_{Q}$ is the join in the lattice of all quasiorders on A and $Q(0, x)$ is the quasiorder on A generated by the pair $\langle 0, x\rangle$, see [1], [2] for details. By [1], there exist unary algebraic functions $\varphi_0, \ldots, \varphi_n$ and elements $x_0, \ldots, x_n \in B$ such that

$$0 = \varphi_0(0),\ \varphi_0(x_0) = \varphi_1(0),\ \ldots,\ \varphi_i(x_i) = \varphi_{i+1}(0),\ \ldots,\ \varphi_n(x_n) = a.$$

Since $0 \in B$, we have $\varphi_0(0) \in B$. By (iii) also $\varphi_0(x_0) \in B$, i.e. $\varphi_1(0) \in B$. Similarly, this yields $\varphi_1(x_1) \in B$, etc. After n steps we obtain $a \in B$. By Theorem 5 in [6], (ii) is evident. □

Theorem 2. *Let $\mathcal{V}$ be an n-permutable variety with 0. An algebra $a \in \mathcal{V}$ is Hamiltonian if and only if for every unary algebraic function φ there exists a binary polynomial p such that*

$$(***) \qquad \varphi(x) = p(x, \varphi(0)).$$

Proof. Let $\mathcal{V}$ be an n-permutable veriety with 0. Let $A \in \mathcal{V}$ satisfy (***) and let B be a subalgebra of A. Let $b \in B$ and let φ be a unary algebraic function over A. If $\varphi(0) \in B$, then (***) also implies $\varphi(B) \subseteq B$. By Lemma 2, B is a block of some $\Theta \in \operatorname{Con} A$. The converse implication is a consequence of Theorem 1. □

Remark. Results of Theorem 1 and Theorem 2 can be easily formulated for varieties with 0 in the same way as in [4] using $(n+1)$-ary polynomials instead of the unary algebraic function in the conditions (**), (***).

Example. Any variety $\mathcal{V}$ of loops has 0 and is permutable, hence n-permutable (for each $n \geqslant 2$). If $A \in \mathcal{V}$ is an abelian group (additive notation), then every unary algebraic function $\varphi(x)$ can be written in the form $\varphi(x) = n \cdot x + z$, where $n \in N$, $z \in A$. Choose $p(x, y) = n \cdot x + y$. Then

$$p(x, \varphi(0)) = n \cdot x + \varphi(0) = n \cdot x + n \cdot 0 + z = n \cdot x + z = \varphi(x),$$

i.e. (***) of Theorem 2 is satisfied.

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