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PARALLEL METHODS IN IMAGE RECOVERY BY PROJECTIONS ONTO CONVEX SETS

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1. INTRODUCTION

In a recent paper [2] we paid attention to the problem of using parallelism in image recovery in a Hilbert space setting. The problem of image recovery may be stated as follows: the original unknown image f is known a priori to belong to the intersection C_0 of r well-defined closed convex sets $C_1, \ldots, C_r$ in a complex Hilbert space H, i.e., $f \in C_0 = \bigcap_{i=1}^{r} C_i$; given only the projection operators P_i onto the individual sets C_i $(i = 1, \ldots, r)$, recover f (i.e., find a point in C_0) by an iterative scheme.

The usual method [1; 6] to solve this problem is as follows: from each projection P_i an operator $T_i = \mathbf{1} + \lambda_i(P_i - \mathbf{1})$ is formed with $\mathbf{1}$ the identity operator on the Hilbert space H, and λ_i a positive relaxation parameter; then the operator $T = T_r T_{r-1} \ldots T_2 T_1$ is constructed and it is shown that, starting from an element x in H, under suitable conditions the sequence $\{T^n x\}_{n=0}^{\infty}$ is weakly convergent to an element of C_0.

The sequential manner in which T is constructed from the different T_i ($T_1 x$ has to be calculated before T_2 can be working, and so on) may give rise to a rather long computing time. In [2] we presented a method which may speed up the recovery process if some type of parallel computer is available; it was shown [2, theorem 3] that, when T has the form $T = \alpha_0 \mathbf{1} + \sum_{i=1}^{r} \alpha_i T_i$ with $\alpha_0 > 0, \alpha_j > 0$ for $1 \leqslant j \leqslant r$, $\sum_{j=0}^{r} \alpha_j = 1$, $T_i = \mathbf{1} + \lambda_i(P_i - \mathbf{1})$ and $0 < \lambda_i < 2$ for all i, then $\{T^n x\}_{n=0}^{\infty}$ is weakly convergent to an element of C_0.

In this paper we continue our investigation on parallel methods by considering the operator S which is a convex combination solely of the operators T_i, i.e.,

$$S = \sum_{i=1}^{r} \alpha_i T_i, \ \alpha_i > 0 \quad \text{for} \quad 1 \leqslant i \leqslant r, \ \sum_{i=1}^{r} \alpha_i = 1.$$

2. Mathematical preliminaries

H is a complex Hilbert space with norm $\| \ \|$ and identity operator $\mathbf{1}$; $C_1, \ldots, C_r$ are r closed convex sets in H with nonempty intersection C_0; for $i = 1, \ldots, r$, P_i is the (in general non-linear) projection operator onto the set C_i. Each such projection operator has the following properties: for $x \in H$, $z \in C_i$ we have $\operatorname{Re}\langle x - P_i x, z - P_i x\rangle \leqslant 0$, and for $x \in H$, $y \in H$ it is also true that

$$\|P_i x - P_i y\|^2 \leqslant \operatorname{Re}\langle x - y, P_i x - P_i y\rangle .$$

We refer to [6] for the following definition and for a proof of theorem 1:

Definition. A mapping $A\colon H \to H$ is said to be nonexpansive iff $\|Ax - Ay\| \leqslant \|x - y\|$ for all x, $y \in H$.

Theorem 1. *Let $A\colon H \to H$ be a nonexpansive mapping whose set of fixed points $F \subset H$ is nonempty. Then, for any $x \in H$ such that $A^n x - A^{n+1} x \to 0$ for $n \to \infty$ the sequence $\{A^n x\}_{n=0}^{\infty}$ is weakly convergent to an element of F.*

3. Main results

3.1 Proposition 1. *Let $S = \sum_{i=1}^{r} \alpha_i T_i$, where for $1 \leqslant i \leqslant r$: $T_i = \mathbf{1} + \lambda_i (P_i - \mathbf{1})$, $\lambda_i > 0$, $0 < \alpha_i < 1$, $\sum_{i=1}^{r} \alpha_i = 1$. Then the set of fixed points of S coincides with C_0.*

Proof. For $y \in C_0$ we have $P_i y = y$ for all i, and so $Sy = y$. Conversely, if for some $y \in H$ we have $Sy = y$ then $\sum_{i=1}^{r} \alpha_i \lambda_i (P_i y - y) = 0$. Putting $\alpha_i \lambda_i = a_i > 0$ we prove that $\sum_{i=1}^{r} a_i (P_i y - y) = 0$ implies $P_i y = y$ for all i. Indeed, from the assumption we first derive that $P_r y - y = -\sum_{i=1}^{r-1} \frac{a_i}{a_r} (P_i y - y)$. Taking an element z in $C_0 = \bigcap_{i=1}^{r} C_i$

we have

$$\begin{aligned}\operatorname{Re}\langle y - P_r y, z - P_r y\rangle &= \operatorname{Re}\Big\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i y - y), z - y + \sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i y - y)\Big\rangle \\ &= \Big\| \sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i y - y)\Big\|^2 + \operatorname{Re}\Big\langle \sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i y - y), z - y\Big\rangle,\end{aligned}$$

and the last term may be rewritten as

$$\begin{aligned}&\sum_{i=1}^{r-1} \operatorname{Re}\Big\langle \frac{a_i}{a_r}(P_i y - y), z - P_i y + P_i y - y)\Big\rangle \\ &= \sum_{i=1}^{r-1} \frac{a_i}{a_r}\|P_i y - y\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \operatorname{Re}\langle P_i y - y, z - P_i y\rangle .\end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{Re}\langle y - P_r y, z - P_r y\rangle = \\ &= \Big\| \sum_{i=1}^{r-1} \frac{a_i}{a_r}(P_i y - y)\Big\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r}\|P_i y - y\|^2 + \sum_{i=1}^{r-1} \frac{a_i}{a_r} \operatorname{Re}\langle P_i y - y, z - P_i y\rangle\end{aligned}$$

where on the right hand side considered as a sum of three terms both the first and the last term are nonnegative. We conclude that if at least one of the vectors $P_i y - y$ is different from zero for some i with $1 \leqslant i \leqslant r-1$, then $\operatorname{Re}\langle y - P_r y, z - P_r y\rangle$ would be strictly positive; this is clearly a contradiction due to the properties of projections. So it must be true that $P_i y - y = 0$ for all $i = 1, 2, \ldots, r-1$; but then also $P_r y - y = 0$; hence $P_i y = y$ for all i, $1 \leqslant i \leqslant r$, which proves that $y \in C_0$. □

A glance at the proof of the foregoing proposition reveals that it goes through unchanged when we replace in it the operator S by the operator T, where $T = \alpha_0 \mathbf{1} + \sum_{i=1}^{r} \alpha_i T_i$, with $T_i = \mathbf{1} + \lambda_i(P_i - \mathbf{1})$ for $1 \leqslant i \leqslant r$, $\alpha_0 > 0$, $\alpha_i > 0$ and $\lambda_i > 0$ for all i in $1 \leqslant i \leqslant r$, and $\sum_{j=0}^{r} \alpha_j = 1$. So also for the operator T the set of its fixed points is the nonempty set C_0; this result may also be stated as follows: $Ty = y$ if and only if $T_i y = y$ for all i.

In view of this result we can now state our proposition 2, which gives a reformulation of theorem 2 in [2] but under its minimal conditions. Since the proof is the same as in [2] we do not repeat it here. For a given element x in H and a positive integer n we put $x_n = T^n x = T(T^{n-1}x)$, with $x_0 = T^0 x = x$.

Proposition 2. *Let $T\colon H \to H$ be the operator given by $T = \alpha_0 \mathbf{1} + \sum\limits_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_j < 1$ for $j = 0, 1, \ldots, r$, $\sum\limits_{j=0}^{r} \alpha_j = 1$. Let u be a fixed point of T and let x be an element in H such that the following conditions are true:*

(A) $\|x - u\| \geqslant \|x_1 - u\| \geqslant \ldots \geqslant \|x_n - u\| \geqslant \ldots$

(B) $\|x_1 - x\| \geqslant \|x_2 - x_1\| \geqslant \ldots \geqslant \|x_n - x_{n-1}\| \geqslant \ldots$

(C) $\|\sum\limits_{i=1}^{r} \frac{\alpha_i}{1-\alpha_0}(T_i x_n - u)\| \leqslant \|x_n - u\|$ *for all n.*

Then $T^n x - T^{n+1} x \to 0$.

□

3.2 To obtain conclusions about the weak convergence of the sequence $\{S^n x\}_{n=0}^{\infty}$ or $\{T^n x\}_{n=0}^{\infty}$ on the base of theorem 1, we need the rather strong assumption of nonexpansivity of S or T. So we first investigate the Lipschitz properties of the operators T_i.

Proposition 3. *Let $T_i = \mathbf{1} + \lambda_i(P_i - \mathbf{1})$ with $\lambda_i > 0$. Then*

(i) *for $0 < \lambda_i \leqslant 2$, T_i is nonexpansive,*

(ii) *for $\lambda_i > 2$ we have $\|T_i x - T_i y\| \leqslant (\lambda_i - 1)\|x - y\|$.*

Proof. The fact that T_i is nonexpansive for $0 < \lambda_i \leqslant 2$ is well known (e.g., see [6, Th. 2.4.1.]). So we just prove (ii), using the properties of projections. For $\lambda_i > 2$ we then have

$$\begin{aligned}\|T_i x - T_i y\|^2 &= (1 - \lambda_i)^2 \|x - y\|^2 + 2\lambda_i(1 - \lambda_i) \operatorname{Re} \langle x - y, P_i x - P_i y\rangle + \lambda_i^2 \|P_i x - P_i y\|^2 \\ &\leqslant (1 - \lambda_i)^2 \|x - y\|^2 + \lambda_i(2 - \lambda_i)\|P_i x - P_i y\|^2 \\ &\leqslant (1 - \lambda_i)^2 \|x - y\|^2.\end{aligned}$$

□

We remark that in (ii) equality is obtained for two elements x and y such that $P_i x = P_i y$; hence for $\lambda_i > 2$, T_i is never nonexpansive.

If all T_i appearing in the expression of S or T are nonexpansive, then S and T are themselves nonexpansive. Turning first our attention to T we see that under this assumption the conditions (A), (B), (C) in proposition 2 are fulfilled, even for all fixed points u and all points x in H. So we conclude that, if $0 < \lambda_i \leqslant 2$ for all i, then for the operator T as given in proposition 2 the sequence $\{T^n x\}_{n=0}^{\infty}$ weakly converges to a point of C_0, whatever the starting point x. To obtain analogous results for the operator S, it might be tempting to consider S as a limit case of the operator T when $\alpha_0 \to 0$. However, a taking of the limit may cause difficulties (double limit theorem) in proving that for the limit operator S it is still true that $S^n x - S^{n+1}x \to 0$. Instead we prove the following result.

Proposition 4. *Let $S = \sum_{i=1}^{r} \alpha_i T_i$, with $T_i = \mathbf{1} + \lambda_i(P_i - \mathbf{1})$ for all i, $\sum_{i=1}^{r} \alpha_i = 1$ with $0 < \alpha_i < 1$. When $0 < \lambda_i \leqslant 2$ for all but one index j for which $0 < \lambda_j < 2$, then starting from an arbitrary x in H the sequence $\{S^n x\}_{n=0}^{\infty}$ is weakly convergent to a point of C_0.*

Proof. From what has been said above we know that any operator T, given by $T = \alpha_0 \mathbf{1} + \sum_{i=1}^{r} \alpha_i T_i$ with $0 < \lambda_i \leqslant 2$ for $1 \leqslant i \leqslant r$, $0 < \alpha_j < 1$ for $0 \leqslant j \leqslant r$ and $\sum_{j=0}^{r} \alpha_j = 1$, gives rise for any x in H to a sequence $\{T^n x\}_{n=0}^{\infty}$ which is weakly convergent to a fixed point of T, i.e., to a point of C_0.

Substituting $\mathbf{1} + \lambda_i(P_i - \mathbf{1})$ for T_i and expanding we find that T may also be written as $T = \left(1 - \sum_{i=1}^{r} \alpha_i \lambda_i\right) \mathbf{1} + \sum_{i=1}^{r} \alpha_i \lambda_i P_i$, where in particular $\sum_{i=1}^{r} \alpha_i < 1$.

On the other hand, the operator S in proposition 4 gives on expanding

$$S = (1 - \sum_{i=1}^{r} \alpha_i \lambda_i)\, \mathbf{1} + \sum_{i=1}^{r} \alpha_i \lambda_i P_i,$$

which is "formally" like T, except that now $\sum_{i=1}^{r} \alpha_i = 1$. We can give S the exact form of T by introducing small changements in the coefficients α_i and λ_i; in fact, it is sufficient in the expression of S to keep, e.g., $\alpha_2, \ldots, a_r, \lambda_2, \ldots, \lambda_r$, and to introduce α_1', λ_1' such that $\alpha_1 \lambda_1 = \alpha_1' \lambda_1'$ with $0 < \alpha_1' < \alpha_1$; determining then α_0 such that $\alpha_0 + \alpha_1' + \sum_{i=2}^{r} \alpha_i = 1$ and substituting in the expanded version of S we immediately see, by running through the expanding steps in reversed order, that S has exactly the same form as T. This means that, starting from an arbitrary x in H, the sequence $\{S^n x\}_{n=0}^{\infty}$ will be weakly convergent to a point of C_0 as soon as $0 < \lambda_1' \leqslant 2$ and $0 < \lambda_k \leqslant 2$ for $2 \leqslant k \leqslant r$. To finish the proof it is sufficient to show that, when in the expression of S we have $\sum_{i=1}^{r} \alpha_i = 1$ with $0 < \alpha_i < 1$, $0 < \lambda_1 < 2$ and $0 < \lambda_k \leqslant 2$ for $k = 2, \ldots, r$, we can choose α_1' and λ_1' as stated in the proof. This is rather easy; indeed, since $2 - \lambda_1 = \delta > 0$, choose an integer $N \geqslant 2$ such that $N - 1 > \frac{\lambda_1}{\delta}$ and put $\alpha_1' = \frac{N-1}{N}\alpha_1, \lambda_1' = \frac{N}{N-1}\lambda_1$; then $0 < \alpha_1' < \alpha_1$, $\lambda_1' \leqslant 2$, and $\alpha_1' \lambda_1' = \alpha_1 \lambda_1$. □

3.3 We finally comment on the nonexpansivity conditions of S (or T).

The nonexpansivity of S leads to a particular manner of convergence for the sequence $\{S^n x\}_{n=0}^{\infty}$. Indeed, for any point y of C_0, i.e., a fixed point of S, we then have

$$||S^{n+1}x - y|| \leqslant ||S^n x - y|| \leqslant \ldots \leqslant ||x - y||;$$

denoting by P_0 the projection operator onto C_0 we have in particular

$$||S^{n+1}x - P_0 x|| \leqslant ||S^n x - P_0 x|| \leqslant \ldots \leqslant ||x - P_0 x||.$$

Both properties together show intuitively that for a nonexpansive S (or T) convergence happens "by staying on one side of $P_0 x$", the point of C_0 closest to the starting point x. Moreover, there is convergence independent of the starting point.

We can however imagine that also without nonexpansivity conditions of the operators T_i and S or T, for suitable starting points x the sequence $\{S^n x\}_{n=0}^{\infty}$ (or $\{T^n x\}_{n=0}^{\infty}$) may be weakly convergent to a point of C_0. Another manner of convergence which may arise in such a case is somewhat induced by conditions (A) and (B) in proposition 2: the sequence $\{S^n x\}_{n=0}^{\infty}$ might "circle around and come closer to C_0", while the distances between successive points of the sequence are diminishing (we remark that in the limiting case of $\alpha_0 \to 0$, i.e., for the operator S, condition (C) in proposition 2 gives rise to condition (A)).

Although we are not aware of a mathematical proof that weak convergence for suitable starting points may exist when not all T_i are nonexpansive, experimental investigations show that fast convergence may result for the sequences $\{S^n x\}_{n=0}^{\infty}$ and $\{T^n x\}_{n=0}^{\infty}$ when some relaxation parameters in the operators T_i are bigger than 2. The practical applicability is a direct consequence of proposition 1: as soon as for some $N \in \mathbb{Z}^+$ we have that $S(S^N x) = S^N x$, we have reached a point of C_0.

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