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MODULAR BASES IN A HILBERT $A$-MODULE

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Summary. Following Ozawa [4] we introduce the concept of a modular base in a Hilbert $A$-module and prove that the cardinalities of any two such bases are the same.

Keywords: $H^*$-algebra, primitive projection, projection base, Hilbert $A$-module, modular base, modular dimension

AMS classification: 46H25

INTRODUCTION

Throughout this paper $A$ denotes a proper $H^*$-algebra with an inner product and norm $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively ([1]). A nonzero selfadjoint idempotent in $A$ is called a projection. If a projection cannot be expressed as a sum of two pairwise orthogonal projections, then it is said to be primitive. A maximal family of pairwise orthogonal primitive projections is called a projection base. Denote by $\tau(A)$ the trace class of $A$, i.e. let $\tau(A) = \{ xy: x, y \in A \}$ and let $\text{tr}$ be the trace functional on $\tau(A)$. $\text{tr}$ has the following properties: $\text{tr} xy = \langle y, x^* \rangle = \langle x, y^* \rangle = \text{tr} yx$ ($x, y \in A$). For each $a \in A$ there exists a unique positive element $[a] \in A$ (i.e. such that $\langle [a]x, x \rangle \geq 0$ ($x \in A$)) such that $[a]^2 = a^*a$, moreover $a \in \tau(A)$ if and only if $[a] \in \tau(A)$. Then a norm can be defined on $\tau(A)$ by setting $\tau(a) = \text{tr}[a]$ ($a \in \tau(A)$), for which the following relations hold: $|\text{tr}(\cdot)| \leq \tau(\cdot), \| \cdot \| \leq \tau(\cdot)$ and $\tau(xy) \leq \|x\| \|y\|$ ($x, y \in A$) ([6]). It was shown in [7] that $\tau(A)$ is a Banach $^*$-algebra. In [8] Smith proved that every nonzero positive element $a \in A$ has a unique spectral representation $a = \sum_n \lambda_n e_n$, where the $\lambda_n$-s are positive real numbers with $\lambda_i > \lambda_j$ if $i < j$, and the $e_n$-s are mutually orthogonal projections.
Now let \( H \) be a (right) \( A \)-module on which there is a generalized inner product \([,] \), i.e. \([,] : H \times H \to \tau(A)\) such that

1. \([f, f] \geq 0\) and \([f, f] = 0\) if and only if \( f = 0\);
2. \([f, g + h] = [f, g] + [f, h]\);
3. \([f, ga] = [f, g]a\);
4. \([f, g]^* = [g, f]\)

holds for every \( f, g, h \in H \) and \( a \in A \). \([,]\) satisfies the so called strong Schwartz inequality, i.e.

\[
(\tau[f, g])^2 \leq \tau[f, f] \tau[g, g] \quad (f, g \in H).
\]

For a more general statement cf. [3].

In the rest of the paper let \( H \) be a Hilbert \( A \)-module, i.e. suppose that \( H \) is complete in the metric \( d \) defined by

\[
d(f, g) = \sqrt{\tau[f - g, f - g]} \quad (f, g \in H).
\]

As Saworotnow showed in [5], on \( H \) a linear structure can be introduced such that \( \lambda(fa) = (\lambda f)a = f(\lambda a) \) (\( \lambda \in \mathbb{C}, a \in A, f \in H \)) and

\[
\langle f, g \rangle = \text{tr}[g, f] \quad (f, g \in H)
\]

defines an inner product on \( H \). Denote by \( || \cdot || \) the norm corresponding to this inner product.

It is easy to see that \( A \) is a Hilbert \( A \)-module if we define the generalized inner product by \([x, y] = x^*y \) (\( x, y \in A \)). Similar considerations can be performed for every \( eA \), where \( e \in A \) is a projection. The norms arising from these generalized inner products are equal to the original one.

If \( H_1 \) and \( H_2 \) are Hilbert \( A \)-modules, then a mapping \( U : H_1 \to H_2 \) is called an \( A \)-unitary operator if it is surjective and

1. \( U(f + g) = Uf + Ug \),
2. \( U(fa) = (Uf)a \),
3. \( [Uf, Ug] = [f, g] \)

for every \( f, g \in H_1 \) and \( a \in A \). In this case \( U \) is a unitary operator between the Hilbert spaces \( H_1 \) and \( H_2 \). Finally, it was also proved in [4] that

\[
f = \sum_{\alpha} fe_{\alpha}
\]

holds for every \( f \in H \) and projection base \( \{e_{\alpha}\}_{\alpha \in A} \).
Results

We begin with the following basic lemma.

**Lemma 1.** Let $f \in H$ be such that $[f, f]$ is a projection. Then the submodule $fA$ is isomorphic an isometric to $[f, f]A$, consequently $fA$ is closed. Moreover, we have $f[f, f] = f$.

**Proof.** Let $f \in H$ and consider the function $T(fa) = [f, fa] = [f, f]a$ ($a \in A$). Then $T$ is a linear operator preserving the module operation with the range $[f, f]A$. Since

$$[fa, fa] = a^*[f, f]^*[f, f]a = ([f, f]a, [f, f]a) \quad (a \in A),$$

taking traces we get that $T$ is an isometry. Since $[f, f]A$ is closed so is $fA$. Now let $[f, f] = e_1 + \ldots + e_n$ be the decomposition of $[f, f]$ into pairwise orthogonal primitive projections (cf. [1, Theorem 3.2]). Extend the set $\{e_1, \ldots, e_n\}$ by $\{e'_\alpha\}_{\alpha \in \Lambda}$ to a projection base. Then

$$f = f[f, f] + \sum_{\alpha} f e'_\alpha.$$

Since $[fe'_\alpha, fe'_\alpha] = e'_\alpha[f, f]e'_\alpha = 0$ ($\alpha \in \Lambda$), it follows that $f[f, f] = f$. \hfill \Box

**Definition.** The family $\{f_\alpha\}_{\alpha \in \Lambda} \subset H$ is said to be modular orthonormal if

1. $[f_\alpha, f_\beta] = 0$ if $\alpha \neq \beta$;
2. $[f_\alpha, f_\alpha]$ is primitive projection in $A$ for every $\alpha \in \Lambda$.

A maximal modular orthonormal family is called a modular base.

**Remark 1.** If $\{f_\alpha\}_{\alpha \in \Lambda} \subset H$ is a modular orthonormal family, $a_\alpha \in A$ ($\alpha \in \Lambda$) and $F \subset \Lambda$ is a finite set, then, using the above lemma, simple calculation shows that $[f - \sum\limits_{\alpha \in F} f_\alpha a_\alpha, f - \sum\limits_{\alpha \in F} f_\alpha a_\alpha]$ equals

$$[f, f] + \sum\limits_{\alpha \in F} ([f_\alpha, f] - [f_\alpha, f_\alpha] a_\alpha)^*([f_\alpha, f] - [f_\alpha, f_\alpha] a_\alpha) - \sum\limits_{\alpha \in F} [f, f_\alpha][f_\alpha, f].$$

As a consequence we have

$$[f, f] \geq \sum\limits_{\alpha \in F} [f, f_\alpha][f_\alpha, f].$$

**Theorem 1.** Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a modular orthonormal family in $H$. Then the following assertions are equivalent:

(i) $\{f_\alpha\}_{\alpha \in \Lambda}$ is a modular base.
(ii) If $f \in H$ is such that $[f_\alpha, f] = 0$ ($\alpha \in \Lambda$), then $f = 0$. 

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(iii) The orthogonal sum (in the Hilbert space sense) of the closed subspaces $H_\alpha = f_\alpha A$ ($\alpha \in \Lambda$) is $H$.

(iv) $f = \sum_\alpha f_\alpha [f_\alpha, f]$ for every $f \in H$.

(v) $[f, g] = \sum_\alpha [f, f_\alpha][f_\alpha, g]$ holds for any $f, g \in H$, where the sum is unconditionally convergent in the norm $\tau$.

(vi) $||f||^2 = \sum_\alpha ||[f_\alpha, f_\alpha]||^2$ for every $f \in H$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $f \in H$ and $[f_\alpha, f] = 0$ ($\alpha \in \Lambda$). If $f \neq 0$, then let $[f, f] = \sum_\alpha \lambda_n e_n$ be the spectral representation of $[f, f]$. Now for $f' = \frac{1}{\sqrt{\lambda_1}} f e_1$ we have $[f', f'] = e_1$ and $[f_\alpha, f'] = 0$ ($\alpha \in \Lambda$), which is a contradiction.

(ii) $\Rightarrow$ (iii). By the previous lemma $H_\alpha$ is a closed submodule which is a subspace as well ($\alpha \in \Lambda$). Now the implication follows from [5, Lemma 3].

(iii) $\Rightarrow$ (iv). If $f \in H$, then for every $\alpha \in \Lambda$ there exists an $a_\alpha \in A$ such that $f = \sum_\alpha f_\alpha a_\alpha$. This implies that

$$[f_\alpha, f] = [f_\alpha, f_\alpha] a_\alpha \quad (\alpha \in \Lambda).$$

Since $f_\alpha [f_\alpha, f_\alpha] = f_\alpha$ ($\alpha \in \Lambda$), we have (iv).

(iv) $\Rightarrow$ (v). We have to prove only the unconditional convergence. By the properties of the norm $\tau$ we have

$$\tau([f, f_\alpha][f_\alpha, g]) \leq ||[f_\alpha, f]|| ||[f_\alpha, g]|| \quad (\alpha \in \Lambda).$$

But from the proof of Lemma 1 we know that

$$||[f, f_\alpha]||^2 = ||f_\alpha [f_\alpha, f]||^2 \quad \text{and} \quad ||[f_\alpha, g]||^2 = ||f_\alpha [f_\alpha, g]||^2 \quad (\alpha \in \Lambda).$$

Now (v) follows.

(v) $\Rightarrow$ (vi). Let $f \in H$. Then

$$[f, f] = \sum_\alpha [f, f_\alpha][f_\alpha, f].$$

By the above remark, using the fact that $\tau$ is additive on the positive elements of $\tau(A)$, we have

$$\tau([f, f] - \sum_\alpha [f, f_\alpha][f_\alpha, f]) = \tau([f, f] - \sum_{\alpha \in F} [f_\alpha, f_\alpha] [f_\alpha, f]) = ||f||^2 - \sum_{\alpha \in F} ||[f_\alpha, f_\alpha]||^2$$

for every $F \subset \Lambda$, which implies (vi).

The implications (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. $\square$
Remark 2. In Corollary 1 below which can be called a generalized Bessel inequality we need the following simple statement.

If \((e_\varepsilon)_{\varepsilon \in \mathcal{E}}\) is a net of selfadjoint elements of \(\tau(A)\) converging in the norm \(\tau\) to an \(a \in \tau(A)\) such that there is an \(x \in A\) for which \(x = x^*\) and

\[ a_\varepsilon \leq x \quad (\varepsilon \in \mathcal{E}), \]

then \(a \leq x\).

To prove it we note that the convergence in \(\tau\) implies the convergence in \(||\cdot||\).

**Corollary 1.** Let \(\{f_\alpha\}_{\alpha \in \Lambda}\) be a modular orthonormal family in \(H\). Then

\[ [f, f] \geq \sum_\alpha [f, f_\alpha][f_\alpha, f], \]

where the sum is unconditionally convergent in \(\tau(A)\).

**Proof.** By Theorem 1 (vi) we have

\[ \sum_\alpha \tau([f, f_\alpha][f_\alpha, f]) = \sum_\alpha ||[f_\alpha, f]||^2 < \infty. \]

Now the statement follows from Remarks 1 and 2. \(\square\)

In the proof of our main theorem we use

**Lemma 2.** Let \(n, m \in \mathbb{N}\) be such that \(n \neq m\). Suppose that \(e_1, \ldots, e_{n+m}\) are primitive projections in \(A\). Then

\[ e_1 + \ldots + e_n \neq e_{n+1} + \ldots + e_{n+m}. \]

**Proof.** Using the second structure theorem for \(H^*\)-algebras ([1, Theorem 4.2 and 4.3]) \(A\) can be identified with the direct sum of Hilbert-Schmidt operator algebras \(\bigoplus_{\gamma \in \Gamma} \text{HS}(\mathcal{H}_\gamma)\), where the \(\mathcal{H}_\gamma\)-s are suitably chosen Hilbert spaces and the inner product on \(\text{HS}(\mathcal{H}_\gamma)\) may differ from the standard one at most by a real constant which is not less than 1. In this representation every \(e_j\) can be considered as a vector \((P^j_\gamma)_{\gamma \in \Gamma}\) such that there is exactly one \(\gamma \in \Gamma\) for which \(P^j_\gamma \neq 0\) and for this \(\gamma\) \(P^j_\gamma\) is one dimensional projection on \(\mathcal{H}_\gamma\). Now suppose that \(e_1 + \ldots + e_n = e_{n+1} + \ldots + e_{n+m}\). It is easy to see that there is a \(\gamma_0 \in \Gamma\) such that

\[ \text{card}\{k \in \{1, \ldots, n\}: P^k_{\gamma_0} \neq 0\} \neq \text{card}\{l \in \{n+1, \ldots, n+m\}: P^l_{\gamma_0} \neq 0\}. \]

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If we take the trace corresponding to the Hilbert space $\mathcal{H}_{\gamma_0}$ in the equation

$$\sum_{k=1}^{n} P_{\gamma_0}^k = \sum_{l=n+1}^{n+m} P_{\gamma_0}^l,$$

we arrive at a contradiction. \hfill \Box

**Theorem 2.** If $\{f_{\alpha}\}_{\alpha \in \Lambda}$ and $\{g_i\}_{i \in I}$ are modular bases in $H$, then $\text{card } \Lambda = \text{card } I$.

**Proof.** If $\Lambda$ and $I$ are infinite sets, then the proof is standard. In fact, for every $\alpha \in \Lambda$ consider the set

$$S_{\alpha} = \{i \in I: [f_{\alpha}, g_i] \neq 0\}.$$

By Theorem 1 (vi) $S_{\alpha}$ is countable. (ii) of the same theorem implies that every $i \in I$ belongs to at least one set $S_{\alpha}$ ($\alpha \in \Lambda$). Then we have

$$\text{card } I \leq \text{card } \Lambda \cdot \aleph_0 = \text{card } \Lambda.$$

Changing the role of $\Lambda$ and $I$ we get the other inequality.

Now we prove that if one of these bases is finite, then so is the other. To this end suppose that $\Lambda$ is finite and $I$ is infinite. Since $|\text{tr}(.)| \leq \tau(.)$, thus, by Theorem 1 (v), we have

$$\infty > \text{tr} \sum_{\alpha} [f_{\alpha}, f_{\alpha}] = \text{tr} \sum_{\alpha} \sum_{i} [f_{\alpha}, g_i][g_i, f_{\alpha}]$$

$$= \sum_{\alpha} \sum_{i} \text{tr}[f_{\alpha}, g_i][g_i, f_{\alpha}]$$

$$= \sum_{i} \sum_{\alpha} \text{tr}[g_i, f_{\alpha}][f_{\alpha}, g_i]$$

$$= \sum_{i} \text{tr}[g_i, g_i] = \infty,$$

where we have used the fact that the trace of a projection is not less than 1.

Finally, assume that $\Lambda$ and $I$ are finite. Then we have

$$\sum_{\alpha} [f_{\alpha}, f_{\alpha}] = \sum_{i} [g_i, g_i]$$

and Lemma 2 implies that $\text{card } \Lambda = \text{card } I$. \hfill \Box
As a consequence we can state

**Corollary 2.** All projection bases in $A$ have the same cardinality.

**Proof.** Consider $A$ as a Hilbert $A$-module. The only thing which has to be proved is that every projection base $\{e_\alpha\}_{\alpha \in A}$ is a modular base in $A$. By Theorem 1 (ii) we have to show that $e_\alpha x = 0$ ($\alpha \in A$) implies that $x = 0$. But this follows from the first structure theorem for $H^*$-algebras ([1, Theorem 4.1]).

**Remark 3.** By the second structure theorem for $H^*$-algebras it is to see that the relation between $\text{Dim} A$ and $\text{dim} A$ (the Hilbert space dimension of $A$) is quite complicated. However, it is easy to see that $\text{Dim} A < \infty$ if and only if $\text{dim} A < \infty$.

Just as in [4], $\text{card} A$ occurring in Theorem 2 is called the *modular dimension* of $H$ and denoted by $\text{Dim} H$.

**Remark 4.** It is natural to ask whether any two Hilbert $A$-modules $H_1$ and $H_2$ are $A$-unitarily equivalent (i.e. there is an $A$-unitary operator between $H_1$ and $H_2$) if and only if $\text{Dim} H_1 = \text{Dim} H_2$. The “only if” part is obvious while the “if” part does not hold in general. To show it let $A = C \oplus C \oplus M_{2 \times 2}(C)$ (where $M_{2 \times 2}(C)$ is the algebra of $2 \times 2$-type complex matrices) with the natural operations and inner product. Let

\[
e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

where $I \in M_{2 \times 2}(C)$ is the identity matrix. Then $H_1 = e_1 A$ and $H_2 = e_2 A$ can be considered Hilbert $A$-modules. It is trivial that $\text{Dim} H_1 = \text{Dim} H_2 = 2$, but, if $H_1$ and $H_2$ were $A$-unitarily equivalent, then they would be unitarily equivalent Hilbert spaces as well which is a contradiction.

As for our final result we need the following lemma which shows that the topological simplicity of $A$ is a necessary and sufficient condition of the validity of the statement formulated in the above remark.

**Lemma 3.** The minimal right ideals of $A$ are $A$-unitarily equivalent if and only if $A$ is topologically simple.

**Proof.** In the proof we use [2, Proposition 7 and Theorem 8 on pp. 47–48].

To prove the necessity let $I_1 = Ae_1 A$, $I_2 = Ae_2 A$ be two different minimal closed ideals of $A$, where $e_1, e_2 \in A$ are primitive projections. Then $R_1 = e_1 A \subset I_1$ and $R_2 = e_2 A \subset I_2$ are minimal right ideals for which $R_1^* R_1 \subset I_1$, $R_2^* R_2 \subset I_2$ since $I_1$, $I_2$ are selfadjoint. But $I_1 \neq I_2$ implies that $I_1 \perp I_2$, consequently we get that there
is no $A$-unitary operator between $R_1$ and $R_2$. Now it follows that $A$ is topologically simple.

To prove the sufficiency we may assume that $A = \text{HS}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and the inner product on $\text{HS}(\mathcal{H})$ is the standard one. Let $P_1$ and $P_2$ be one dimensional projections on $\mathcal{H}$. Suppose that $\varphi_1$ and $\varphi_2$ are vectors from $\mathcal{H}$ of norm 1 generating the range of $P_1$ and $P_2$, respectively. If $S$ is the operator defined by $Sx = (x, \varphi_1) \varphi_2$ ($x \in \mathcal{H}$), then let

$$U(P_1T) = SP_1T \quad (T \in \text{HS}(\mathcal{H})).$$

Simple calculation shows that $U$ is an $\text{HS}(\mathcal{H})$-unitary operator from $P_1 \text{HS}(\mathcal{H})$ onto $P_2 \text{HS}(\mathcal{H})$. \quad \Box

From this lemma, by Lemma 1 and Theorem 1 (iii) and (iv), we have

**Theorem 3.** Let $A$ be topologically simple. If $H_1$ and $H_2$ are Hilbert $A$-modules, then $H_1$ and $H_2$ are $A$-unitarily equivalent if and only if $\dim H_1 = \dim H_2$.

**References**


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