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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF A CERTAIN  
NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION

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1. We are interested in oscillatory solutions of a nonlinear differential equation of the third order

$$(1) \quad u''' + q(t)u' + p(t)u^\alpha = 0,$$

where  $p(t)$ ,  $q(t)$  and  $q'(t)$  are continuous functions on the interval  $(a, \infty)$ ,  $-\infty < a$ ,  $\alpha > 1$  is a ratio of odd integers.

By a solution of (1) we mean a function  $u(t)$  defined on an interval  $(T, \infty)$ ,  $a \leq T$ , with a continuous third derivative, which satisfies equation (1). By an oscillatory solution we mean a nontrivial solution  $u$  of (1) that has infinitely many null-points with a limit point at infinity. Otherwise the solution is called nonoscillatory.

The object of generalization are results in the papers [4] and [1] concerning oscillatory solutions of equation (1) in the case  $q(t) \equiv 0$  on  $(a, \infty)$ .

In the proofs of this paper some results of the paper [3] are applied.

2. N. Parhi and S. Parhi [4] proved the following theorem:

**Theorem A.** *Let  $p(t) < 0$  for  $t \in (a, \infty)$  and let  $\int_{t_0}^{\infty} p(\tau) d\tau = -\infty$ ,  $t_0 > a$ . Then every bounded nontrivial solution  $u$  of the differential equation*

$$(1_1) \quad u''' + p(t)u^\alpha = 0$$

*defined on  $(t_0, \infty)$  is oscillatory on  $(t_0, \infty)$ .*

In the paper [1] the following theorem is proved:

**Theorem B.** *Let the assumptions of Theorem A be fulfilled. Then a necessary and sufficient condition for a solution  $u$  of (1<sub>1</sub>) to be oscillatory for  $t \geq t_0$  is that*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) < 0$$

for  $t \geq t_0$ .

I. W. Heidel [3] proved several interesting results. Some of them are formulated in the following two theorems.

**Theorem C.** *Let  $q(t) \leq 0$  and  $p(t) \leq 0$  for  $t \in (a, \infty)$ . If  $u(t)$  is a nontrivial nonoscillatory solution of (1) on  $(t_0, \infty)$ ,  $t_0 > a$ , then there is a number  $c \geq t_0$  such that either  $u(t)u'(t) > 0$  for  $t \geq c$ , or  $u(t)u'(t) \leq 0$  for  $t \geq c$ .*

**Theorem D.** *Let the supposition of Theorem C be fulfilled and let, moreover,  $\int_{t_0}^{\infty} tq(t)dt > -\infty$ , or  $-\frac{2}{t^2} \leq q(t) \leq 0$  for  $t \geq t_0$ ,  $t_0 > 0$ . If  $u(t)$  is a nontrivial nonoscillatory solution of (1) on  $(t_0, \infty)$ , then  $u(t)u'(t) > 0$  for  $t \in (t_0, \infty)$ .*

In some proofs the two following lemmas will be used. They are special cases of Lemma 4 of [5].

**Lemma A.** *Let  $u(t) \in C^2((t_0, \infty))$ . Then  $u(t) > 0$ ,  $u'(t) < 0$ ,  $u''(t) \leq 0$  cannot hold for all  $t \geq t_0$ .*

**Lemma B.** *Let  $u(t) \in C^3((t_0, \infty))$ . Then  $u(t) > 0$ ,  $u'(t) < 0$ ,  $u'''(t) \geq 0$  cannot hold for all  $t \geq t_0$ .*

3. In this section we generalize Theorem A and Theorem B for equation (1) if  $p(t) < 0$ ,  $q(t) \leq 0$  and  $q'(t) \geq 0$  for  $t \in (a, \infty)$  and prove a corollary of Theorem D for the solutions of equation (1).

**Lemma 1.** *Let  $p(t) < 0$ ,  $q(t) \leq 0$  for  $t \in (a, \infty)$  and let  $u(t)$  be a solution of (1) with the properties  $u(t_0) \geq 0$ ,  $u'(t_0) \geq 0$ ,  $u''(t_0) > 0$ ,  $t_0 > a$ . Then  $u(t) > 0$ ,  $u'(t) > 0$ ,  $u''(t) > 0$  for  $t > t_0$  and  $u(t) \rightarrow \infty$ ,  $u'(t) \rightarrow \infty$  for  $t \rightarrow \infty$ .*

This lemma can be proved in a similar manner as Theorem 1 in [1] and therefore the proof is omitted.

In the paper [2] the following theorem is presented without proof and therefore we prove it.

**Theorem 1.** *Let the coefficients of equation (1) fulfil the suppositions of Lemma 1 and let, moreover,  $\int_{t_0}^{\infty} p(\tau)d\tau = -\infty$  and  $q'(t) \geq 0$  for  $t \geq t_0$ . Then every nontrivial*

bounded solution  $u$  of (1) on  $\langle t_0, \infty \rangle$ ,  $t_0 > a$ , is either oscillatory on  $\langle t_0, \infty \rangle$ , or converges monotonically to zero for  $t \rightarrow \infty$ .

**Proof.** Without loss of generality suppose  $u(t) > 0$  and bounded on  $\langle t_0, \infty \rangle$ . We prove that this can occur only if it converges monotonically to zero for  $t \rightarrow \infty$ . By Lemma 1,  $u'(t)$  cannot have on  $\langle t_0, \infty \rangle$  more than two zeros and then it does not change the sign. Then there exists a point  $T \geq t_0$  such that for  $u'(t)$ ,  $t > T$  we have two possibilities. Let

$$(i) \quad u'(t) > 0 \quad \text{for} \quad t > T.$$

Integrating equation (1), in this case we get the identity

$$(2) \quad u''(t) + q(t)u(t) + \int_{t_0}^t [p(\tau)u(\tau)^{\alpha-1} - q(\tau)]u(\tau)d\tau = k.$$

The boundedness of  $u(t)$  and the suppositions  $q'(t) \geq 0$  for  $t \geq t_0$  and  $\int_{t_0}^{\infty} p(\tau)d\tau = -\infty$  imply that there exists a point  $T_1 \geq T$  such that

$$u''(t) > 0 \quad \text{for} \quad t > T_1.$$

From Lemma 1 we get a contradiction with the supposition that  $u(t)$  is bounded.

Let

$$(ii) \quad u'(t) < 0 \quad \text{for} \quad t > T.$$

Then  $u(t)$  is decreasing for  $t > T$ . There are two cases for  $u(t)$ . Either  $u(t) > K > 0$  and then the identity (2) implies that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , which contradicts  $u'(t) < 0$  for  $t > T$ , or  $K = 0$  and  $u(t)$  converges monotonically to zero.  $\square$

**Lemma 2.** Let the coefficients of equation (1) fulfil the assumptions of Theorem 1. Let  $u$  be a solution of (1) with the property  $u(t) > 0$  for  $t \geq t_0$ . Then there exists a point  $t_1 \geq t_0$  such that either  $u(t) > 0$ ,  $u'(t) > 0$ ,  $u''(t) > 0$  for  $t \geq t_1$ , or  $u'(t) < 0$  for  $t \geq t_1$ , and

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} \sup u'(t) = 0.$$

**Proof.** Suppose that  $u(t) > 0$  for  $t \geq t_0$ . There are three possibilities for  $u''(t)$ .

1)  $u''(t) > 0$  for  $t > t_0$ .

Then  $u'(t)$  is increasing for  $t > t_0$  and we have two cases:

(i)  $u'(t) > 0$  for  $t \geq t_1 \geq t_0$ . In this case  $u(t) > 0$ ,  $u'(t) > 0$ ,  $u''(t) > 0$  for  $t \geq t_1$ , and this is the assertion of Lemma 2.

(ii)  $u'(t) < 0$  for  $t \geq t_1 \geq t_0$  and then there exists  $\lim_{t \rightarrow \infty} u'(t) = K \leq 0$ . If  $K < 0$  then  $u(t) \leq K(t - t_1) + u(t_1)$  which is a contradiction with  $u(t) > 0$  for large  $t > t_1$ . Therefore  $\lim_{t \rightarrow \infty} u'(t) = 0$  and  $\lim_{t \rightarrow \infty} u(t) = k \geq 0$ . If  $k > 0$ , then the identity (2) implies that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , but this is a contradiction with  $u'(t) \rightarrow 0$  for  $t \rightarrow \infty$  and therefore  $\lim_{t \rightarrow \infty} u(t) = 0$ .

2)  $u''(t) < 0$  for  $t < t_0$ . By Lemma A the case  $u'(t) < 0$  cannot occur for  $t \geq t_1 \geq t_0$ . If there exists  $t_1 \geq t_0$  such that  $u'(t) > 0$  for  $t \geq t_1$ , then from the identity (2) we obtain a contradiction.

3)  $u''(t)$  has infinitely many null-points for  $t \geq t_0$  at which it changes the sign ( $u''(t)$  oscillates on  $\langle t_0, \infty \rangle$ ). For  $u'(t)$  we have three possibilities:

(i)  $u'(t) > 0$  for  $t \geq t_1 \geq t_0$ . Then  $u(t)$  is increasing and from (2) we obtain a contradiction with the oscillatoriness of  $u''(t)$ .

(ii)  $u'(t) < 0$  for  $t \geq t_1 \geq t_0$ . Then necessarily  $\limsup_{t \rightarrow \infty} u'(t) = 0$  and  $\lim_{t \rightarrow \infty} u(t) = 0$ . In the opposite case (2) implies that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$  and then it cannot oscillate.

(iii)  $u'(t)$  is oscillatory on  $\langle t_1, \infty \rangle$ ,  $t_1 \geq t_0$ . This case is in contradiction with the assertion of Theorem C.  $\square$

**Lemma 3.** *Let the supposition of Theorem 1 on the coefficients of equation (1) be fulfilled. Then for every solution  $u$  of (1) which converges monotonously to zero for  $t \rightarrow \infty$  with the property  $u(t)u'(t) < 0$  for  $t > t_0$  there exists  $T \geq t_0$  such that for  $t \geq T$  the inequality*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) < 0$$

holds.

**Proof.** Let  $u(t) > 0$  and  $u'(t) < 0$  for  $t \geq t_0$ . Let  $t_0 < t_1 < t_2 < \dots$  be an arbitrary sequence of points diverging to infinity if  $u'(t)$  is a monotone function, or such a sequence of points for which  $u'(t_i) \rightarrow 0$  if  $t_i \rightarrow \infty$ . By Lemma 2 such sequence of  $t_i$ ,  $i = 1, 2, \dots$  exists.

Multiply equation (1) by the solution  $u$  and integrate from  $t_i$  to  $t$ . We obtain the integral identity

$$\begin{aligned} (3) \quad & u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) + \int_{t_i}^t \left[ p(\tau)u(\tau)^{\alpha-1}(t) - \frac{1}{2}q'(\tau) \right] u^2(\tau) d\tau = \\ & = u(t_i)u''(t_i) - \frac{1}{2}u'^2(t_i) + \frac{1}{2}q(t_i)u^2(t_i), \quad i = 1, 2, \dots \end{aligned}$$

For a solution  $u(t)$  for which  $u(t) > 0$ ,  $u'(t) < 0$  for  $t \geq t_0$  the identity (2) and Theorem 1 yield that  $u''(t)$  must be bounded on  $(t_0, \infty)$  the integral  $\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)d\tau$  exists and  $u(t) \rightarrow 0$  for  $t \rightarrow \infty$ . Then there exists a point  $T > t_0$  such that for  $t > T$  we have

$$-\int_{t_i}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)d\tau \leq -\int_{t_i}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)d\tau.$$

If in the identity (3)  $t_i \rightarrow \infty$  for  $i \rightarrow \infty$  we obtain the relation

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) = \int_{t_i}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)d\tau < 0$$

for  $t \geq T$ . □

**Theorem 2.** Let  $p(t) < 0$ ,  $q(t) \leq 0$ ,  $q'(t) \geq 0$  for  $t \in (a, \infty)$  and let  $\int_{t_0}^{\infty} p(\tau)d\tau = -\infty$ ,  $t_0 \geq a$ . Then a necessary and sufficient condition for the solution  $u$  of (1) defined on  $(t_0, \infty)$  to be oscillatory for  $t \geq t_0$ , or to be monotonously converging to zero is

$$(4) \quad u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) < 0$$

for  $t > t_1$ ,  $t_1 \geq t_0$ .

*Proof.* Sufficient condition. Let (4) hold for  $t > t_1 \geq t_0$  and let e.g.  $u(t) > 0$  for  $t \geq t_0$ . It follows from Lemma 2 that there exists  $t_1 \geq t_0$  such that either  $u(t) > 0$ ,  $u'(t) > 0$ ,  $u''(t) > 0$  for  $t > t_1$ , or  $u(t) > 0$ ,  $u'(t) < 0$  for  $t > t_1$ . In the latter case the solution  $u$  by Lemma 2 monotonously converges to zero (and by Lemma 3 fulfils the condition (4)). In the former case, by Lemma 1  $u(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , and from the integral identity (3) for  $t_i = T \geq t_1$  and from the suppositions of Theorem 2 it follows that for large  $t$  the inequality

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) > 0$$

holds and this is a contradiction with (4).

Necessary condition. By Lemma 3 we must prove that an oscillatory solution in  $(t_0, \infty)$  fulfils the condition (4). Let  $u(t)$  be an oscillatory solution of (1) on  $(t_0, \infty)$  and let  $t_i$   $i = 1, 2, \dots$  be its null-points on  $(t_0, \infty)$ . Then the identity (3) implies that the function  $u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t)$  is increasing on  $(t_1, \infty)$  and  $u(t_i)u''(t_i) - \frac{1}{2}u'^2(t_i) + \frac{1}{2}q(t_i)u^2(t_i) < 0$  for  $i = 1, 2, \dots$ . Consequently, (4) holds for  $t \geq t_1$ . □

**Corollary 1.** *Let the suppositions of Theorem 2 be fulfilled and let, moreover, the suppositions of Theorem D be fulfilled. Then a necessary and sufficient condition for a solution  $u$  of (1) to be oscillatory on  $\langle t_0, \infty \rangle$  is that the condition (4) is fulfilled for  $t > T \geq t_0$ , where  $T$  is sufficiently large.*

4. In this section we shall study equation (1) with  $p(t) < 0$ ,  $q(t) \geq 0$ ,  $q'(t) \geq 0$  for  $t \in (a, \infty)$ .

**Theorem 3.** *Let  $p(t) < 0$ ,  $q(t) \geq 0$ ,  $q'(t) \geq 0$  for  $t \in (a, \infty)$ , let  $q(t)$  be bounded on  $(a, \infty)$  and  $\int_{t_0}^{\infty} p(t)dt = -\infty$ ,  $t_0 > a$ . Then every bounded solution of (1) on  $\langle t_0, \infty \rangle$  is oscillatory on this interval.*

*Proof.* Let e.g.  $u(t) > 0$  be bounded on  $\langle t_0, \infty \rangle$ ,  $t_0 > a$ . Three cases for its first derivative  $u'(t)$  are possible.

1)  $u'(t) > 0$  for  $t \geq T \geq t_0$ . The identity (2) for  $t \geq t_0$  implies that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$  and therefore  $u(t)$  cannot be bounded on  $\langle t_0, \infty \rangle$ , which is a contradiction.

2)  $u'(t) \leq 0$  for  $t \geq T \geq t_0$ . In this case equation (1) implies that  $u'''(t) > 0$  for  $t \geq T$  and by Lemma B this is impossible.

3)  $u'(t)$  has infinitely many null-points at which it changes the sign. If in this case  $u(t) > K > 0$  for  $t \geq t_0$ , then we obtain from (2) that  $u''(t) > 0$  for  $t \geq T \geq t_0$  and therefore  $u'(t)$  must be increasing for  $t \geq T$ , which is a contradiction with the oscillatoricity of  $u'(t)$ . Therefore  $\liminf_{t \rightarrow \infty} u(t) = 0$ . If we suppose that  $\liminf_{t \rightarrow \infty} u(t) = 0$ , we have the following two possibilities:

(i)  $\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)d\tau = \infty$ . However, in this case we obtain from (2) that  $u''(t) > 0$  for  $t \geq T \geq t_0$  and this contradiction with the oscillatory of  $u'(t)$ .

(ii)  $0 \leq -\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)d\tau < \infty$ .

In this case let  $\{t_i\}_{i=1}^{\infty}$ ,  $t_i \rightarrow \infty$  for  $i \rightarrow \infty$ , be a sequence of points at which  $u'(t_i) = 0$  and  $u''(t_i) > 0$ . Clearly  $u(t_i) \rightarrow 0$  for  $i \rightarrow \infty$ . It follows from (2) that  $\{u''(t_i)\}$  is bounded on  $\langle t_0, \infty \rangle$ .

Now if we write the identity (3) in the form

$$(5) \quad u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) + \int_{t_1}^t [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)d\tau = k,$$

where  $k = u(t_1)u''(t_1) - \frac{1}{2}u'^2(t_1) + \frac{1}{2}q(t_1)u^2(t_1) > 0$ , we obtain for  $u''(t_i)$ ,  $i = 2, 3, \dots$  the equality

$$u''(t_i) = \frac{k}{u(t_i)} - \frac{1}{2}q(t_i)u(t_i) - \frac{1}{u(t_i)} \int_{t_1}^{t_i} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)d\tau.$$

It follows from this relation for  $t_i \rightarrow \infty$  that  $u''(t_i) \rightarrow \infty$ , which is a contradiction with the boundedness of  $\{u''(t_i)\}$ .  $\square$

**Lemma 4.** *Let the supposition of Theorem 3 be fulfilled. Then for every solution  $u$  of (1) with the property  $u(t) > 0$  for  $t \geq t_0$ , there exists  $T \geq t_0$  such that for all  $t \geq T$  the inequality*

$$(6) \quad u(t)u''(t) - \frac{1}{2} u'^2(t) + q(t)u^2(t) > 0$$

holds.

**Proof.** Let  $u(t) > 0$  for  $t \geq t_0$ . Then there are three possibilities for  $u'(t)$ .

(i)  $u'(t) > 0$  for  $t \geq t_0$ . It follows from (5), where  $t_1 = t_0$ , that there exists  $T \geq t_0$  such that for all  $t \geq T$  the inequality (6) holds.

(ii)  $u'(t) \leq 0$  for  $t \geq t_0$ . From equation (1) we obtain in this case that  $u'''(t) > 0$  for  $t \geq t_0$ , but by Lemma B this is not possible.

(iii)  $u'(t)$  has on  $\langle t_0, \infty \rangle$  at least two null-points at which it changes the sign.

At one of them we have  $u''(t) \geq 0$ . Let  $t = T_1$ . It follows from (5) with  $t_1 = T_1$  that  $k = u(T_1)u''(T_1) + \frac{1}{2}q(T_1)u^2(T_1) \geq 0$  and that there exists  $T \geq T_1$  such that (6) holds for  $t > T$ .  $\square$

**Theorem 4.** *Let the suppositions of Theorem 3 be fulfilled. Then a necessary and sufficient condition for the solution  $u$  of (1) defined on  $\langle t_0, \infty \rangle$ ,  $t_0 > a$  to be oscillatory on  $\langle t_0, \infty \rangle$  is that*

$$(7) \quad u(t)u''(t) - \frac{1}{2} u'^2(t) + \frac{1}{2} q(t)u^2(t) < 0$$

for  $t \geq T \geq t_0$ .

**Proof.** Sufficient condition. Let  $u$  be a solution of (1) satisfying the condition (7) for  $t \geq T \geq t_0$ , and let e.g.  $u(t) > 0$  for  $t \geq T$ . By Lemma 4 there exists  $T_1 \geq t_0$  such that (6) holds for  $t \geq T_1$ , and this is a contradiction with (7). This proves that  $u$  must be oscillatory.

Necessary condition can be proved in the same manner as in Theorem 2.  $\square$

**Remark 1.** Let  $u$  be a solution of (1) with the property  $u(t_0) = u'(t_0) = 0$ ,  $u''(t_0) > 0$  and let the supposition of Theorem 2 or of Theorem 4 be fulfilled. Then  $u(t) > 0$  for  $t > t_0$ .

This assertion follows from the identity (5), where  $k = 0$ .

5. In this section we shall discuss two cases of suppositions on the coefficients of equation (1), in which we do not prove a necessary and sufficient condition for the oscillatoricity of solutions of equation (1).



**Theorem 5.** Let  $p(t) < 0$ ,  $q(t) \geq 0$ ,  $q'(t) \leq 0$  for  $t \in (a, \infty)$  and let  $\lim_{t \rightarrow \infty} q(t) = 0$  and  $\int_{t_0}^{\infty} p(\tau) d\tau = -\infty$ ,  $t_0 \geq a$ . Then every bounded solution  $u$  of (1) defined on  $(t_0, \infty)$  is either oscillatory on  $(t_0, \infty)$ , or  $\liminf_{t \rightarrow \infty} |u(t)| = 0$ .

**Proof.** Let  $u$  be a bounded solution of (1) defined on  $(t_0, \infty)$ ,  $t_0 \geq a$ . If we integrate (1) term by term for  $t \geq t_0$  we have

$$(8) \quad u''(t) + \int_{t_0}^t q(\tau) u'(\tau) d\tau + \int_{t_0}^t p(\tau) u^\alpha(\tau) d\tau = u''(t_0).$$

Let  $|u(t)| \leq K$ ,  $K > 0$ . The function  $q(t)$  is nonincreasing and  $q(t) \rightarrow 0$  for  $t \rightarrow \infty$ . For a given  $\varepsilon/4K > 0$  there exists  $T_0 \geq t_0$  such that  $0 \leq q(t) \leq \frac{\varepsilon}{4K}$  for  $t > T_0$ . Let  $T_1, T_2$  be such that  $T_2 > T_1 > T_0$ . By the second mean value theorem there exists  $c$ ,  $T_1 \leq c \leq T_2$  such that

$$\begin{aligned} \left| \int_{T_1}^{T_2} q(\tau) u'(\tau) d\tau \right| &= \left| q(T_1) \int_{T_1}^c u'(\tau) d\tau + q(T_2) \int_c^{T_2} u'(\tau) d\tau \right| \\ &\leq |q(T_1)| |u(c) - u(T_1)| + |q(T_2)| |u(T_2) - u(c)| \\ &\leq \frac{\varepsilon}{4K} 4K = \varepsilon. \end{aligned}$$

Then by the Cauchy-Bolzano criterion  $\int_{t_0}^{\infty} q(\tau) u'(\tau) d\tau$  converges.

Suppose now that  $u(t) > 0$  for  $t > t_0$  for  $t > t_0$  and  $u$  is bounded on  $(t_0, \infty)$ . For  $u'(t)$  there are three possibilities on  $(t_0, \infty)$ .

(i)  $u'(t) > 0$  for  $t \geq t_1 \geq t_0$ . Then it follows from (8) that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , which is a contradiction with the boundedness of  $u(t)$ .

(ii)  $u'(t) \leq 0$  for  $t \geq t_1 \geq t_0$ . In this case equation (1) implies that  $u'''(t) > 0$  for  $t \geq t_1$ , but by Lemma B this is impossible.

(iii)  $u'(t)$  changes its sign infinitely many times on  $(t_0, \infty)$ . If in this case  $u(t) > K_1 > 0$  for  $t \geq t_1 \geq t_0$ , then  $\int_{t_0}^{\infty} p(\tau) u^\alpha(\tau) d\tau = -\infty$  and (8) yields  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , which is a contradiction with the boundedness of  $u(t)$ . Therefore  $\liminf_{t \rightarrow \infty} u(t) = 0$ .  $\square$

**Theorem 6.** Let  $p(t) < 0$ ,  $q(t) \geq 0$ ,  $q'(t) \leq 0$  for  $t \in (a, \infty)$  and let  $\lim_{t \rightarrow \infty} q(t) = 0$  and  $-p(t) + \frac{1}{2} q'(t) \geq k > 0$  for  $t \in (a, \infty)$ . If  $u$  is a solution of (1) defined on  $(t_0, \infty)$ ,  $t_0 \geq a$  such that it fulfils the condition

$$(9) \quad u(t) u''(t) - \frac{1}{2} u'^2(t) + \frac{1}{2} q(t) u^2(t) < 0$$

for  $t \geq t_1 \geq t_0$ , then  $u$  is oscillatory on  $(t_0, \infty)$ .

**Proof.** The supposition  $-p(t) + \frac{1}{2}q'(t) \geq k > 0$  clearly implies the relation  $\int_{t_0}^{\infty} p(\tau)d\tau = -\infty$ .

Suppose now that a solution  $u(t)$  of (1) fulfils the condition (9) and that it is nonoscillatory. Let e.g.  $u(t) > 0$  for  $t \geq t_0$ . Then for  $u'(t)$  there are two possibilities. The possibility  $u'(t) \leq 0$  for  $t \geq t_1 \geq t_0$  is eliminated by Lemma B.

(i)  $u'(t) > 0$  for  $t \geq t_1 \geq t_0$ . If  $u(t)$  is bounded from above then it is oscillatory, or  $\liminf_{t \rightarrow \infty} u(t) = 0$  by Theorem 5. But  $\liminf_{t \rightarrow \infty} u(t) = 0$  is in contradiction with  $u(t) > 0$ ,  $u'(t) > 0$  for  $t \geq t_1$ . If  $\lim_{t \rightarrow \infty} u(t) = \infty$ , then there exists  $T_1 \geq t_1$  such that for  $t \geq T_1$  we have  $u(t) > 1$  and  $-[p(t)u^{\alpha-1}(t) - \frac{1}{2}q'(t)]u^2(t) \geq ku^2(t) > k > 0$ . The integral identity (5) implies that  $u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) > 0$  for large  $t$  and this is a contradiction with (9).

(ii)  $u'(t)$  changes its sign infinitely many times. Then there exists a sequence  $\{t_k\}_{k=1}^{\infty}$ ,  $t_k \rightarrow \infty$  for  $k \rightarrow \infty$ , such that  $u'(t_k) = 0$ ,  $u''(t_k) \geq 0$  for  $k = 1, 2, \dots$ . But at the points  $t_k$  we obtain a contradiction with (9).  $\square$

**Theorem 7.** Let  $p(t) < 0$ ,  $q(t) \leq 0$ ,  $q'(t) \leq 0$  for  $t \in (a, \infty)$ , let  $q(t)$  be bounded from below on  $(a, \infty)$  and  $\int_{t_0}^{\infty} p(\tau)d\tau = -\infty$ ,  $t_0 \geq a$ . Then every bounded solution  $u$  defined on  $(t_0, \infty)$  is either oscillatory on  $(t_0, \infty)$ , or it converges monotonously to zero for  $t \rightarrow \infty$ .

**Proof.** Let  $u$  be a solution of (1) defined and bounded on  $(t_0, \infty)$ ,  $t_0 \geq a$  and let e.g.  $u(t) > 0$  for  $t > t_0$ . Then for  $u'(t)$  we have three possibilities:

(i)  $u'(t) > 0$  for  $t \geq t_1 \geq t_0$ . In this case we obtain from (1) that  $u'''(t) > 0$  for  $t \geq t_1$ . If  $u''(t) > 0$  for  $t \geq t_1$  then Lemma 1 yields  $u(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , which is a contradiction with boundedness of  $u(t)$ .

If  $u''(t) < 0$  for  $t \geq t_1 \geq t_0$ , then after integration of (1) term by term we obtain the relation (8) where  $t_0 = t_1$ . Clearly  $\lim_{t \rightarrow \infty} u(t) = k < \infty$ ,  $\int_{t_1}^{\infty} pu^{\alpha}(\tau)d\tau = -\infty$  and if  $m \leq q(t) \leq 0$  then  $\int_{t_1}^t q(\tau)u'(\tau)d\tau \geq m \int_{t_1}^t u'(\tau)d\tau \geq m[k - u(t_1)]$ , and  $\int_{t_1}^t q(\tau)u'(\tau)d\tau \rightarrow l$ ,  $0 > l > -\infty$ .

We see now from (8) that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$  and this is a contradiction with the boundedness of  $u(t)$ .

(ii)  $u'(t) \leq 0$  for  $t \geq t_1 \geq t_0$ . Then  $u(t)$  is nonincreasing. Let  $\lim_{t \rightarrow \infty} u(t) = k \geq 0$ . If  $k > 0$  we obtain from (8) that  $u''(t) \rightarrow \infty$  for  $t \rightarrow \infty$  which is again a contradiction.

(iii)  $u'(t)$  changes its sign infinitely many times and there exists a point  $T_1 \geq t_0$  such that  $u(T_1) > 0$ ,  $u'(T_1) = 0$ ,  $u''(T_1) \geq 0$ . By Lemma 1  $u(t) \rightarrow \infty$  for  $t \rightarrow \infty$  and this is a contradiction.  $\square$

**Theorem 8.** Let  $-p(t) > k > 0$ ,  $q(t) \leq 0$ ,  $q'(t) \leq 0$  for  $t \in (a, \infty)$  and let  $q'(t) \rightarrow 0$  for  $t \rightarrow \infty$ . Let  $u(t)$  be a solution of (1) defined on  $(t_0, \infty)$ ,  $t_0 \geq a$ , which

fulfils the condition (9) for  $t \geq t_1 \geq t_0$ . Then  $u(t)$  is either oscillatory on  $(t_0, \infty)$ , or it converges monotonously to zero for  $t \rightarrow \infty$ .

**Proof.** Let  $u$  be a solution of (1) defined on  $(t_0, \infty)$  which fulfils (9) for  $t \geq t_1$ , and let  $u$  be nonoscillatory.

Let e.g.  $u(t) > 0$  for  $t \geq t_0$ .  $u'(t)$  has three possibilities:

(i)  $u'(t) \geq 0$  for  $t \geq T_1 \geq t_0$ . Then the identity (5) with  $t_1 = T_1$  contradicts (9), because  $\int_{T_1}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)d\tau = -\infty$ .

(ii)  $u'(t) < 0$  for  $t \geq T_1 \geq t_0$ . In this case, if  $u(t) > L > 0$  we obtain a contradiction as in the case (i). Therefore  $u(t)$  can converge monotonously to zero for  $t \rightarrow \infty$ .

(iii)  $u'(t)$  changes its sign infinitely many times. In this case there exists a point  $T > t_0$  at which  $u(T) > 0$ ,  $u'(T) = 0$ ,  $u''(T) \geq 0$ , and by Lemma 1 we obtain a contradiction with the property (iii) of  $u'(t)$ .  $\square$

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